results. This agreement, and the substantially equal values for the mean life measured in Al and Fe, render it plausible to assume that these values represent the mean life for spontaneous decay.

The present experiments seem to indicate a number of disintegration electrons per mesotron definitely smaller than unity. The statistical error is certainly large, and systematic errors are not to be excluded, although it seems rather unlikely that they might be large enough to account for a discrepancy by more than a factor two.

The results, however, are in agreement with the assumption that only half of the mesotrons undergo free decay. Since the analysis of mesotron tracks in a magnetic field has shown that there are about as many positive as negative mesotrons, or a small excess of positive,[11] the result found is what should be expected if only mesotrons of one sign (positive) undergo free decay. Actually, if, according to the calculations of Tomonaga and Araki, reactions with nuclear particles are much more probable than spontaneous disintegration for negative mesotrons, then we should only record an electron for each positive mesotron absorbed. The nuclear reactions produced by negative mesotrons will probably lead to excited states of nuclei and eventually give rise to electrons through processes of β-decay. It is exceedingly unlikely, however, that such particles could be emitted with sufficient energy and within a sufficiently short time to be registered in the present experiments.

[11] See H. Jones, Rev. Mod. Phys. **11**, 235 (1939), also for earlier literature.

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Theory of Nuclear Surface Energy

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The eigenvalues of a free particle in a spherical potential well of finite depth are computed and used to calculate the surface energy of nuclear systems. No assumptions are made concerning the specifically nuclear forces. For depth and radius $56mc^2$ and $\frac{1}{2}A^{\frac{1}{3}}e^2/mc^2$, respectively, the computed surface energy is about two-thirds the empirical value. A well of infinite depth yields a surface energy more than double the empirical value. One-dimensional and cubical wells are discussed for the purpose of orientation.

Introduction

The empirical packing fraction curve[1] and the phenomenon of fission[2] require the existence of a nuclear surface energy having the magnitude $26A^{\frac{2}{3}}mc^2$ within limits of perhaps ± 10 percent. There exists no adequate theoretical calculation of this quantity, although estimates have been obtained by Weizsäcker[3] and Bethe.[4] A complete theoretical discussion is not possible without the use of special assumptions about the nuclear forces, but it is clear without calculation that the specifically nuclear forces must make a positive contribution to the surface energy. If γ is written for the coefficient of $A^{\frac{2}{3}}$ in the semi-empirical formula for nuclear energies, and γ_K, γ_P for the contributions from the kinetic and potential energy operators, respectively, these remarks may be summarized in the relations

$$\gamma=\gamma_K+\gamma_P\sim 26mc^2, \quad \gamma_K<26mc^2. \qquad (1)$$

The difficulties and uncertainties barring a theoretical determination of γ_P are not present to the same degree for γ_K. It is, therefore, desirable to obtain a first approximation for γ_K

[1] A. J. Dempster, Phys. Rev. **53**, 869 (1938).
[2] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).
[3] C. F. v. Weizsäcker, Zeits. f. Physik **96**, 431 (1935).
[4] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936).

before attempting to estimate γ_P. This is done in the last section with the model of free particles in a spherical potential well of depth D and radius R. For the purpose of orientation, one-dimensional and cubical wells are also studied.

The free-particle model is useful chiefly for the calculation of the kinetic and Coulomb energies, quantities which do not depend sensitively on the exact form of the short range correlations between the positions of the heavy particles. These calculations as customarily made involve two approximations: (a) the correct single particle eigenfunctions are replaced by plane waves which vanish abruptly at the boundary of the well and (b) sums over the occupied energy levels are replaced by integrals in momentum space. For the kinetic energy one obtains

$$E_K=\frac{3}{40}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{M}V\{(N/V)^{5/3}+(Z/V)^{5/3}\}$$
$$\cong 28.64A\left\{1+\frac{5}{9}\left(\frac{N-Z}{A}\right)^2\right\}(R_0/R)^2mc^2, \quad (2)$$

where V is the volume of the sphere and $R_0=\frac{1}{2}A^{\frac{1}{3}}e^2/mc^2$. The absence of a surface term in Eq. (2) is characteristic, not of the free-particle model, but of the simplifying assumptions (a) and (b).

One-Dimensional Well of Infinite Depth

The vanishing of the single particle wave functions at $x=0$ and $x=L$ determines the eigenvalues

$$\epsilon_n=h^2n^2/8ML^2, \quad n=1,\,2,\,3,\,\cdots. \quad (3)$$

With one particle per orbit, and a total of μ particles, the kinetic energy of the degenerate gas is

$$E=(h^2/8ML^2)\textstyle\sum_1^{\mu}n^2=(h^2/24M)$$
$$\times\{L(\mu/L)^3+\tfrac{3}{2}(\mu/L)^2+(1/2L)(\mu/L)\}. \quad (4)$$

If the breadth of the well is assumed proportional to the number of particles, the first term in brackets may be interpreted as the "line" energy and the second and constant term as the "end" energy. The third term has no simple geometrical interpretation and vanishes for infinite L.

One-Dimensional Well of Depth D

The results are stated only for a well of sufficient depth to satisfy the inequality

$$(h^2/8ML^2)\mu^2<\tfrac{1}{3}D \quad \text{with} \quad \mu\gg 1.$$

Under this restriction the effect of raising the bottom of the well is to subject the occupied levels to a uniform compression. The eigenvalues are

$$\epsilon_n=-D+(h^2n^2/8ML^2)\{1-(2h^2/\pi^2MDL^2)^{\frac{1}{2}}\} \quad (5)$$

yielding the total energy

$$E=-\mu D+(h^2/24M)\{L(\mu/L)^3$$
$$+\tfrac{3}{2}(\mu/L)^2[1-(8\mu^2h^2/9\pi^2MDL^2)^{\frac{1}{2}}]+O(1/L)\}. \quad (6)$$

Thus the "line" energy is unaffected by putting a bottom on the well, while the "end" energy is reduced by the factor

$$1-(8\mu^2h^2/9\pi^2MDL^2)^{\frac{1}{2}}.$$

The artificial boundary condition that the phase of the wave function have the value $\frac{1}{4}\pi$ at $x=0$ and $\pm\frac{1}{4}\pi$ at $x=L$ determines eigenvalues

$$\epsilon_n=(h^2/8ML^2)(n-\tfrac{1}{2})^2, \quad n=1,\,2,\,3,\,\cdots. \quad (7)$$

In this case there is no "end" term in the expression for the energy of a completely degenerate gas. We note that for a well of finite depth the phases $\pm\frac{1}{4}\pi$ at the boundaries correspond to $\epsilon_n\sim\frac{1}{2}D$.

Cubical Well of Infinite Depth

The eigenvalues are

$$\epsilon_{lmn}=(h^2/8ML^2)(l^2+m^2+n^2), \quad l,\,m,\,n=1,\,2,\,3,\,\cdots. \quad (8)$$

To calculate the kinetic energy of a degenerate gas we should sum ϵ_{lmn} over all the points of lmn space included in an octant of a sphere. The calculations are greatly simplified without materially altering the results if instead of summing over an octant of a sphere we sum over a cube. With two

TABLE I. *Average energy per particle* (mc^2 *units*).

A	4	16	40	68	80	116	136	180	212	264	276
Table II	80.6	56.9	48.4	44.9	44.0	41.6	41.4	39.7	39.5	38.2	38.0
Eq. (12)	79.8	57.0	48.3	44.6	43.6	41.7	40.9	39.7	39.0	38.2	38.1

particles in each orbit, and N particles altogether, this procedure yields

$$\begin{aligned} E_N &= (3h^2/4ML^2)\{\tfrac{1}{3}(N/2)+\tfrac{1}{2}(N/2)^{\frac{2}{3}}+\tfrac{1}{6}(N/2)^{\frac{1}{3}}\}(N/2)^{\frac{2}{3}} \\ &= (h^2/2^{11/3}M)\{V(N/V)^{5/3}+2^{-5/3}S(N/V)^{4/3}+2^{-1/3}L(N/V)\}, \end{aligned} \tag{9}$$

where V and S are written for the volume and surface area, respectively, of the cube. If the volume is proportional to the number of particles, Eq. (9) represents a sum of volume, surface and edge energies. The absence of a constant term which might be interpreted as a "corner" energy indicates that the geometrical identifications must not be taken too literally.

Comparison of Eq. (9) with Eq. (2) shows that the coefficient of the volume energy is 8 percent too large because of the approximation involved in summing over a cubical region in the quantum number space. Correcting this discrepancy and interpreting V and S as the volume and surface area, respectively, of a sphere of radius R we find

$$E_K = E_N + E_Z = 28.64A\left[1+\frac{5}{9}\left(\frac{N-Z}{A}\right)^2+\frac{1.92}{A^{\frac{1}{3}}}\left\{1+\frac{2}{9}\left(\frac{N-Z}{A}\right)^2\right\}+\frac{1.26}{A^{\frac{2}{3}}}\right](R_0/R)^2mc^2. \tag{10}$$

The surface energy given by this crude calculation is about double the empirical value. The significance of the last term in brackets is very uncertain, but it is retained because a similar term is found necessary to fit the exact energies of a spherical well of infinite depth.

SPHERICAL WELL OF INFINITE DEPTH

The eigenvalues are determined by the equation

$$J_{l+\frac{1}{2}}(kR)=0, \quad k=2\pi(2M\epsilon/h^2)^{\frac{1}{2}}, \tag{11}$$

where $J_{l+\frac{1}{2}}$ is a Bessel function of half-integral order. Values of kR/π are listed in the last section of Table II. A table of values of $(kR)^2$ is given by Margenau.[5] The numerical results for the average energy per particle follow closely the law

$$E_K/A = 28.64\left[1+\frac{5}{9}\left(\frac{N-Z}{A}\right)^2+\frac{1.92}{A^{\frac{1}{3}}}\left\{1+\frac{2}{9}\left(\frac{N-Z}{A}\right)^2\right\}+\frac{1.45}{A^{\frac{2}{3}}}\right](R_0/R)^2mc^2. \tag{12}$$

Table I shows how the computed values compare with Eq. (12) for the radius $R=R_0=\frac{1}{2}A^{\frac{1}{3}}e^2/mc^2$ and $N=Z$.

SPHERICAL WELL OF FINITE DEPTH

Calculations have been made for depths and radii defined by the equations

$$D=56mc^2/\eta^2, \quad R=\eta R_0/\lambda, \tag{13}$$

with $\lambda=1.05$, 1.00, 0.95 and 0.90. No restriction is required on η. Tables of Bessel functions computed by the W.P.A. Mathematical Tables Project greatly facilitated the numerical work. Margenau's[5] form of the equation expressing the continuity of the logarithmic derivative of the wave function was found convenient for use in connection with the W.P.A. tables. Some results for $\lambda=\eta=1$ are exhibited in Table II. All values are uncertain in the third decimal place.

[5] H. Margenau, Phys. Rev. **46**, 613 (1934).

TABLE II. *Values of* $kR/\pi=[8M(D+\epsilon)/h^2]^{\frac{1}{2}}R$, $\lambda=\eta=1$.

A	$(8MD/h^2)^{\frac{1}{2}}R$	$l=0$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
4	0.832	0.689						
16	1.320	0.794	1.118					
40	1.792	0.844 1.635	1.200	1.526				
68	2.138	0.867 1.706	1.237 2.057	1.579	1.901			
80	2.257	0.873 1.724	1.245 2.089	1.591	1.919	2.232		
116	2.555	0.887 1.758	1.267 2.145	1.620 2.489	1.958	2.283		
136	2.694	0.893 1.771	1.274 2.164	1.631 2.523	1.972	2.301	2.623	
180	2.958	0.901 1.793 2.649	1.288 2.195	1.649 2.568	1.995 2.913	2.331	2.657	
212	3.124	0.907 1.804 2.673	1.296 2.210	1.659 2.589	2.007 2.945	2.346	2.676	3.004
264	3.361	0.912 1.817 2.702	1.304 2.229 3.100	1.671 2.615	2.023 2.980	2.365	2.700	3.030
276	3.411	0.914 1.821 2.708	1.307 2.233 3.108	1.673 2.620	2.026 2.988	2.369	2.704	3.033
	∞*	1.000 2.000 3.000 4.000	1.430 2.459 3.471 4.475	1.835 2.895 3.922	2.224 3.314	2.605	2.978	3.342

* Here $kR=(8\pi^2M\epsilon/h^2)^{\frac{1}{2}}R$ with ϵ the kinetic energy measured from the bottom of the well.

Table III shows the average energy per particle, measured from the bottom of the well, as a function of λ and A when $N=Z$ and $\eta=\lambda$. The generalization from $\eta=\lambda$ to arbitrary values of η follows in an obvious manner from the invariance of the quantity $RD^{\frac{1}{2}}$ under changes in η. The numerical results in Table III follow

TABLE III. *Values of* $(E+ADmc^2)/Amc^2$ *for* $N=Z$, $\eta=\lambda$.

A	$\lambda=0.90$	$\lambda=0.95$	$\lambda=1.00$	$\lambda=1.05$
4	41.1	39.5	38.3	37.1
16	36.8	35.7	34.6	33.4
40	35.1	34.4	33.6	32.8
68	34.2	33.7	33.1	32.4^5
80	33.9^5	33.3	32.7	32.1
116	33.1	32.7	32.2	31.6
136	33.3	32.7^5	32.2^5	31.7
180	32.6	32.2	31.8	31.3
212	32.6^5	32.3	31.8	31.4
264	—	—	31.4	—
276	—	—	31.3	—

closely the formula

$$E=[28.64A+\{17.8+50(1-\lambda)\}A^{\frac{2}{3}}]\times(\lambda/\eta)^2mc^2-AD. \quad (14)$$

From Eq. (14) we obtain

$$\gamma_K=(R_0/R)^2\times[17.8+50\{1-(56mc^2/D)^{\frac{1}{2}}R_0/R\}]mc^2, \quad (15)$$

in agreement with the inequality (1) for reasonable values of λ and η. Depths greater than $86mc^2$ are excluded by the inequality. The use of Eqs. (14) and (15) for values of A which do not correspond to completely filled shells implies the reasonable assumption that discontinuities in the energy due to shell effects are, to a large extent, smoothed out in actual nuclear systems. Because of the restriction of the numerical work to completely filled shells, it was not found possible to determine the dependence of energy on $N-Z$. The calculation of such second-order terms must await the development of a satisfactory method of smoothing out shell effects.