The Effect of Dislocations on Magnetization Near Saturation

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The effect of dislocations on the magnetization curve at high fields is calculated by direct use of dislocation theory. The deviation from saturation is assumed to be due to magnetostrictive forces, localized in the stress field about the dislocation rather than at the dislocation itself; their effect is more complicated than that of the simple "line concentrations of force" considered in an earlier article. Pairs of dislocations of opposite sign, separated by a short distance, contribute a term a/H to the deviation from saturation; in this respect they resemble line concentrations. Pairs separated by a long distance and surplus dislocations of one sign contribute a term b/H^2, with b theoretically not a constant but a logarithmically varying function of H. From data on the variation of the empirical a and b with plastic strain, it is possible to calculate the density of dislocations and the "block" length if a value is assumed for the distance Y between the members of a dislocation pair. The orders of magnitude obtained agree with those obtained in the theory of hardening if Y is taken $\leqq 2\times 10^{-6}$ cm. It appears that all but about 1 percent of the dislocations are members of such pairs.

THE magnetic properties of ferromagnetic materials are greatly affected by plastic strain. A theoretical analysis of such effects, developed chiefly by Kersten,[1] has been very successful in correlating a large number of experimental observations. Kersten's theory offers a rather detailed description of the effect of "internal stresses" on magnetic processes, but no description at all of the way in which such stresses arise; they are merely postulated. Meanwhile, other workers have developed a fairly successful theory of plastic strain and hardening, in which the most fruitful concept has been that of "dislocations."[2] It should, therefore, be possible to interpret Kersten's "internal stress" in terms of dislocations, or still better to describe directly the effect of dislocations on the magnetization process.

At low fields the theoretical problem is an extremely complicated one; but as saturation is approached, simplifications may be introduced. The resulting equations are solvable, and data are available from which it is possible to judge the success of such a theoretical analysis. If theory and experiment agree here, then there is some ground for hope that the dislocation concept may also prove useful in interpreting properties of practical importance at lower fields: initial and maximum permeability, hysteresis, remanence, and coercive force.

The magnetization curve at high fields can usually be fitted by an empirical formula $J=J_s-a/H-b/H^2+CH$, where J is the magnetization at a field H, and the other quantities are constants. The "spontaneous" or "intrinsic" magnetization J_s and susceptibility C are satisfactorily explained by atomic theory.[3] The constant b in annealed material can be calculated from data on crystalline and stress-induced anisotropy,[4] but in plastically strained material it has a higher value. The constant a is much more dependent on plastic strain, and quite possibly would vanish for a perfectly annealed specimen. It will be shown in this article that the orders of magnitude of a, and of the part of b not attributable to anisotropy, are consistent with the view that they are due to dislocations. The empirical formula, however, is theoretically far from exact.

1. THE STRESS PRODUCED BY DISLOCATIONS

A "positive" line dislocation, of the sort postulated by Taylor[5] to explain plastic flow, is shown in Fig. 1(a). If such a dislocation moves through the crystal from left to right, the atoms above it are displaced to the right, with respect

[1] R. Becker and W. Döring, *Ferromagnetismus* (Julius Springer, Berlin, 1939), pp. 154–167.
[2] F. Seitz and T. A. Read, J. App. Phys. **12**, 100, 170 (1941).
[3] Reference 1, pp. 25–101; T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[4] Reference 1, pp. 168–171, 175; T. Holstein and H. Primakoff, Phys. Rev. **59**, 388 (1941).
[5] G. I. Taylor, Proc. Roy. Soc. **A145**, 362 (1934).

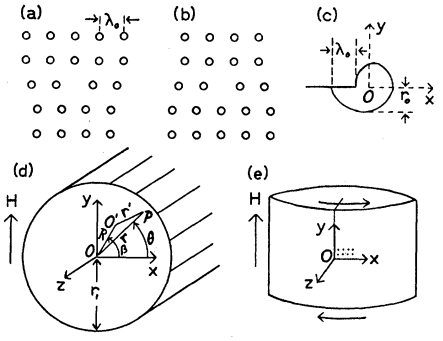

FIG. 1. (a) Approximate positions of atoms in a plane perpendicular to a dislocation line. Positive dislocation. (b) Negative dislocation. (c) The equivalent of (a) for a continuous medium: a "dislocation" in the theory of elasticity. (d) With an excess of dislocations of one sign, the specimen may no longer be treated as having infinite dimensions in the plane perpendicular to the dislocations. The cylindrical specimen shown has been chosen for detailed study. The stress produced at P by a dislocation at O' is to be expressed in terms of polar coordinates referred to the center O. (e) Directions of field, dislocation lines, and reference axes in a twisted wire, in the simplified model adopted here. In the detailed consideration of a microscopic neighborhood of a dislocation, the model is further simplified by supposing that the dislocations are all parallel to Oz and infinitely long.

to the atoms below it, by an amount λ_0. The same result is attained if a "negative" dislocation (Fig. 1(b)) moves through the crystal from right to left. A specimen after plastic strain is supposed to contain dislocations of both signs.

The lattice distortion in the neighborhood of a dislocation may be calculated by elasticity theory at distances r that are large compared with λ_0, that is, with the interatomic distance. In such a calculation the dislocation is pictured as in Fig. 1(c) and may be treated by methods developed by Timpe, Volterra, and others.[6] The internal surface is assumed free from traction; its radius r_0 is of atomic order of magnitude, but disappears in the final formulas, since in the application to Fig. 1(a) it must be supposed that $r \gg r_0$.

In a ferromagnetic material, this distortion of the lattice produces torques that tend to aline the magnetic moments of the atoms along certain directions; the directions change from point to point, since the strain is not uniform. At high fields the effect is a slight deviation of the local magnetization from the field direction, and a consequent decrease of the observed holomagnetization J below the saturation value J_s. Where the spatial variation of the strains is slow, the local direction of magnetization is determined by an equilibrium between the torques due to the lattice distortion and to the field. But in the immediate neighborhood of a dislocation, where the strains vary rapidly, the magnetization is prevented from changing equally rapidly by the interatomic coupling torques, which tend to keep the magnetic moments of neighboring atoms parallel. Thanks to this smoothing out effect, the region $r \leqq \lambda_0$, where the formulas of elasticity theory are not reliable, is precisely the region where the exact nature of the lattice distortion is least important in the magnetic problem; thus for magnetic calculations the elastic formulas may be used at *all* points without serious error.

The dislocation of Fig. 1(c) produces, in an infinite medium, stresses which may be calculated from the Airy stress function[7]

$$\chi = -(G'\lambda_0/2\pi) y \ln(x^2+y^2) = -(G'\lambda_0/\pi) r \ln r \cdot \sin\theta \quad (1)$$

by means of the formulas

$$X_x = \partial^2\chi/\partial y^2, \quad Y_y = \partial^2\chi/\partial x^2, \quad X_y = -\partial^2\chi/\partial x \partial y,$$
$$\sigma_{rr} = r^{-2}\partial^2\chi/\partial\theta^2 + r^{-1}\partial\chi/\partial r, \quad \sigma_{\theta\theta} = \partial^2\chi/\partial r^2,$$
$$\sigma_{r\theta} = -\partial(r^{-1}\partial\chi/\partial\theta)/\partial r. \quad (2)$$

A state of plane strain will be assumed; the constant G' is then equal to $\mu(\lambda+\mu)/(\lambda+2\mu)$ in the usual notation for elastic constants.[8] For the magnetic calculation the only stress required is

$$X_y = cx(x^2-y^2)/r^4 = c(\cos\theta + \cos 3\theta)/2r, \quad (3)$$

where $c = G'\lambda_0/\pi$.

[6] A. Timpe, Zeits. f. Math. Physik **52**, 348 (1905); V. Volterra, Ann. Ec. Norm [3] **24**, 401 (1907); A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge, 1934), fourth edition, pp. 221–228.

[7] A. Timpe, reference 6; L. N. G. Filon, *British Association Report* (1921), p. 305.

[8] For generalized plane stress, $G' = \mu(3\lambda+2\mu)/4(\lambda+\mu) = \frac{1}{4} \times$ Young's modulus. These stresses differ from those given by Taylor, reference 5, p. 376. The incompleteness of Taylor's formulas has been pointed out by J. M. Burgers, Proc. K. Ned. Akad. Wet. **42**, 293 (1939), footnote pp. 305–6. Taylor's Cartesian stresses could be maintained only if surface tractions were applied to the internal boundary $r = r_0$. They appear to have been derived from a preliminary displacement formula given by Volterra, reference 6, p. 463, to which Volterra later added other terms in order to satisfy the boundary conditions. Volterra's complete displacement formula, p. 465, agrees with that of Burgers (except for a rotation of axes) and leads to stress formulas in agreement with those given here. Taylor's polar stresses are inconsistent with his Cartesian; they are incorrect in the sign of σ_{rr} and in the value of the constant (Taylor has μ instead of G').

The displacement corresponding to Eq. (1) becomes logarithmically infinite at $r=\infty$. If the numbers of positive and of negative dislocations are equal, the *resultant* displacement behaves properly at infinity, and a large specimen may be considered infinite in the xy plane; but if there is an excess of dislocations of one sign, the external boundary must be taken into account. A similar situation occurs in the magnetic problem. In most of the following calculations it will be assumed that positive and negative dislocations are present in equal numbers; but one particularly simple case of unequal numbers will be investigated. In this case the boundary is assumed to be a circular cylinder of radius r_1 with center at O, illustrated in Fig. 1(d). The stress may be found as follows.

If the medium were infinite, the stress function for a positive dislocation at O' would be $\chi_1=-cy'\ln r'$. Referred to O as origin, this becomes,[9] for $r>R$,

$$\chi_1=-c\{(r\sin\theta-R\sin\beta)\ln r+C_1r^{-1}\cos\theta+C_1'r^{-1}\sin\theta+\sum_{n=2}^{\infty}[(C_nr^{-n}+D_nr^{-n+2})\cos n\theta+(C_n'r^{-n}+D_n'r^{-n+2})\sin n\theta]\},\quad(4)$$

where

$$\begin{aligned}C_n&=[\sin(n+1)\beta-(n+1)\sin(n-1)\beta]\times R^{n+1}/2n(n+1),\\C_n'&=-[\cos(n+1)\beta-(n+1)\cos(n-1)\beta]\times R^{n+1}/2n(n+1),\\D_n&=[\sin(n-1)\beta]R^{n-1}/2(n-1),\\D_n'&=-[\cos(n-1)\beta]R^{n-1}/2(n-1).\end{aligned}\quad(5)$$

The additional term χ_2 required by the boundary must correspond to a single-valued displacement, must have no singularity at O, and must be such that at $r=r_1$ the sum $\chi_1+\chi_2$ satisfies Michell's boundary conditions[10]

$$\begin{aligned}\chi&=\kappa_1r_1\cos\theta+\kappa_2r_1\sin\theta+\kappa_3,\\\partial\chi/\partial r&=\kappa_1\cos\theta+\kappa_2\sin\theta,\end{aligned}\quad(6)$$

where κ_1, κ_2, and κ_3 are independent of θ. Hence[11]

$$\chi_2=-c\{R\sin\beta\cdot(r^2/2r_1^2)+L_1r^3\cos\theta+L_1'r^3\sin\theta+\sum_{n=2}^{\infty}[(K_nr^n+L_nr^{n+2})\cos n\theta+(K_n'r^n+L_n'r^{n+2})\sin n\theta]\},\quad(7)$$

where

$$\begin{aligned}&L_1=C_1r_1^{-4},\quad L_1'=C_1'r_1^{-4}-\tfrac{1}{2}r_1^{-2};\\&K_n=-(n+1)C_nr_1^{-2n}-nD_nr_1^{-2n+2}\quad(n>1),\\&L_n=nC_nr_1^{-2n-2}+(n-1)D_nr_1^{-2n}\quad(n>1),\end{aligned}\quad(8)$$

with similar equations in the primed constants.

The solution for an arbitrary distribution of dislocations may be found by superposition of the individual solutions with due regard to the sign of each dislocation.[12] The coefficients in the resultant boundary stress function χ_2 contain such sums as $\sum_i s_iR_i^n\cos n\beta_i$, where s_i is $+1$ or -1, according as the ith dislocation is positive or negative. For a macroscopically uniform distribution of dislocations, with an excess N of positive ones, approximate values of the coefficients may be obtained by replacing the sums by integrals, or in other words by treating the dislocation distribution as *continuous* in calculating the effect of the boundary. If this is done, most of the coefficients vanish, and there remains only

$$\chi_b=(Nc/4r_1^2)r^3\sin\theta,\quad(9)$$

whence

$$(X_y)_b=-(Nc/2r_1^2)r\cos\theta.\quad(10)$$

The function $\chi=-c\sum_i s_iy_i'\ln r_i'+\chi_b$ is a good approximation to the actual stress function, except in a region near the boundary, whose width δ is a few times larger than the distance between dislocations. This may be seen as follows. Let A be the region $r<r_1-\delta$ and B the region

[9] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1939), p. 64.

[10] J. H. Michell, Proc. Lond. Math. Soc. **31**, 100 (1899).

[11] This is not equivalent to the stress produced by an image dislocation at the point $(r_1^2/R, \beta)$. The failure of the method of images is easily demonstrated in the case of a dislocation at $(a, 0)$ in a semi-infinite medium bounded by a free surface at $x=0$. If r_1 is distance from the image at $(-a, 0)$ and r_2 is distance from the original dislocation, the stress function is $cy[\ln(r_1/r_2)+\frac{1}{2}(r_2/r_1)^2]$; the image method gives only the first term, which satisfies the condition $X_x=0$, but not the condition $X_y=0$ at the boundary.

[12] The problem of a single dislocation in a cylinder can be solved more simply in bipolar coordinates: J. S. Koehler, Phys. Rev. **59**, 477A (1941); G. B. Jeffery, Phil. Trans. Roy. Soc. **A221**, 265 (1920). A different coordinate system is required for each dislocation, and there is, therefore, no advantage in introducing such coordinates in the present calculation.

$r_1-\delta<r<r_1$. The dislocations in A are indistinguishable, on the boundary, from a continuous distribution, and may therefore be treated as such in calculating the contribution of the boundary under their influence alone; the result is a good approximation in B as well as in A. This is not true of the dislocations in B. But the effect produced *in* A by a dislocation in B, in the presence of the boundary, is not appreciably affected by a slight change in the position of the dislocation. Therefore the dislocations in B may be replaced by a continuous distribution without greatly altering the stresses in A, though the stresses in the boundary region B are not given correctly by this method.

In the magnetic calculation it is convenient to treat the specimen as if it extended to infinity, but had at $r=r_1$ a boundary incapable of transmitting stress, with no dislocations outside the boundary. If the boundary could transmit stress, χ, in the external region, would be given by Eq. (4) summed over the dislocations. Except within a distance δ of the boundary, this may be replaced by its value for a continuous distribution, $-Nc\sin\theta\cdot[r\ln r+r_1^2/4r]$. The function $\chi=-c\sum_i s_i y_i'\ln r_i'+\chi_b'$, with

$$\chi_b'=+Nc\sin\theta\cdot[r\ln r+r_1^2/4r], \qquad (11)$$

is, therefore, a good approximation to the actual (zero) stress function for $r>r_1+\delta$. Thus the actual stresses in both regions may be approximated closely, everywhere except within a distance δ of either side of the boundary, by treating the dislocations as if they were surrounded by an infinite medium, but adding to the stress function a term χ_b for $r<r_1$ and a term χ_b' for $r>r_1$. The corresponding term in X_y for $r>r_1$ is

$$(X_y)_b'=\tfrac{1}{2}Nc[-r^{-1}(\cos\theta+\cos3\theta)+r_1^2r^{-3}\cos3\theta]. \qquad (12)$$

It is also convenient, in the magnetic calculation, to express X_y as a Fourier integral,

$$X_y=f(r,\theta)=(2\pi)^{-1}\int g(\rho,\phi)\exp[i\boldsymbol{\varrho}\cdot\mathbf{r}]d\boldsymbol{\varrho}, \qquad (13)$$

where

$$\mathbf{r}=r(\mathbf{i}\cos\theta+\mathbf{j}\sin\theta),\quad \boldsymbol{\varrho}=\rho(\mathbf{i}\cos\phi+\mathbf{j}\sin\phi),$$

and $d\boldsymbol{\varrho}=\rho d\rho d\phi$. The Fourier transform g is given by

$$g(\rho,\phi)=(2\pi)^{-1}\int f(r,\theta)\exp[-i\boldsymbol{\varrho}\cdot\mathbf{r}]d\mathbf{r}, \qquad (14)$$

where $d\mathbf{r}=rdrd\theta$. For a positive dislocation at the origin in an infinite medium,[13]

$$g=ic(-\cos\phi+\cos3\phi)/2\rho. \qquad (15)$$

For a dislocation at $\mathbf{R}$, an additional factor $\exp[-i\boldsymbol{\varrho}\cdot\mathbf{R}]$ must be inserted; a distribution of dislocations of signs s_i at points $\mathbf{R}_i$ requires a factor $\sum_i s_i\exp[-i\boldsymbol{\varrho}\cdot\mathbf{R}_i]$. The stress described by $(X_y)_b$ at interior points, and by $(X_y)_b'$ at exterior, has the Fourier transform

$$g=i(Nc/r_1)\rho^{-2}J_1(\rho r_1)(\cos\phi-\cos3\phi). \qquad (16)$$

2. Single Dislocations and Doublets

Suppose now that the specimen, with dislocation lines parallel to Oz, is subjected to a large magnetic field H parallel to Oy. This corresponds roughly to the situation in a small portion of a twisted wire, illustrated in Fig. 1(e); this same situation will now be supposed to prevail throughout the specimen.

The deviation of the magnetization vector from the field direction is measured by its direction cosine α with respect to the x axis. The equations that determine α were derived in an earlier article;[14] they are

$$\nabla^2\alpha-\partial u/\partial x-\eta\alpha=f_1, \qquad (17)$$

$$\nabla^2u=h\partial\alpha/\partial x, \qquad (18)$$

where $\eta=HJ_s/C$, $u=UJ_s/C$, $h=4\pi J_s^2/C$, $f_1=C^{-1}(dw/d\alpha)_{\alpha=0}$; C is a constant proportional to the strength of the interatomic coupling forces, U is the potential of the internal field produced

[13] The integration over θ may be carried out by taking as a new variable of integration the angle between $\boldsymbol{\varrho}$ and $\mathbf{r}$; the same method may be used for the integration over ϕ in §§3 and 4. The results may be expressed in terms of Bessel functions by means of formulas given by Watson, *Theory of Bessel Functions* (Cambridge, 1922), pp. 20–21. In the subsequent integration over r or ρ, only elementary properties of Bessel functions are required for Eqs. (15) and (16); but for such integrals as those of Eqs. (29) and (30), use must be made of special formulas given by Watson, Chapter XIII, §§13.4 ff., 13.51–13.53, 13.6. Partial fractions and the differentiation method of Eq. (47) are also useful.

[14] W. F. Brown, Jr., Phys. Rev. **58**, 736 (1940), Eqs. (50), with the change $(x,y,z)\rightarrow(z,x,y)$. Since w contains no linear term in γ, $f_3=0$ and (to the approximation adopted here) $\mathbf{J}$ remains in the xy plane.

by the non-uniform magnetization, and w is the anisotropy energy density associated with the stress. In Eq. (17), the first term represents the coupling torque exerted by neighboring atoms, the second the torque due to the component of internal field perpendicular to the magnetization, the third the torque due to the component of magnetization perpendicular to the external field; the sum of these must balance the anisotropy torque represented by f_1. Equation (18) is Poisson's equation. For $H \gg 4\pi J_s$ the internal field may be neglected; α is then determined by the equation

$$\nabla^2\alpha - \eta\alpha = f_1. \tag{17a}$$

For arbitrary direction cosines (α, β, γ) in a material with isotropic negative saturation magnetostriction $-\lambda_\infty$, w is given by[15]

$$w = \tfrac{3}{2}\lambda_\infty[X_x\alpha^2 + Y_y\beta^2 + Z_z\gamma^2 + 2(Y_z\beta\gamma + Z_x\gamma\alpha + X_y\alpha\beta)]; \tag{19}$$

when $Y_z = Z_x = 0$ and $\beta \fallingdotseq 1$, this becomes const. $+3\lambda_\infty X_y\alpha$, whence $f_1 = (3\lambda_\infty/C)X_y$. Insertion of the value of X_y from Eq. (3) gives, for a single positive dislocation at $x = y = 0$,

$$f_1 = k(\cos\theta + \cos 3\theta)/2r, \tag{20}$$

where $k = (3\lambda_\infty/C)c = 3\lambda_\infty G'\lambda_0/\pi C$. The solution of Eq. (17a), subject to finiteness conditions at $r = 0$ and $r = \infty$, is by direct integration

$$\alpha = (k/2\lambda')\{[K_1(\lambda' r) - 1/\lambda' r]\cos\theta - [K_3(\lambda' r) + 1/\lambda' r - 8/(\lambda' r)^3]\cos 3\theta\}, \tag{21}$$

where[16] $\lambda' = \eta^{\frac{1}{2}}$; or, by use of the Fourier expansion (15),

$$\alpha = (2\pi)^{-1}\int G \exp[i\boldsymbol{\varrho}\cdot\mathbf{r}]d\boldsymbol{\varrho}, \tag{22}$$

where

$$G = \tfrac{1}{2}ik(\cos\phi - \cos 3\phi)/\rho(\rho^2 + \lambda'^2). \tag{23}$$

The decay distance $1/\lambda'$ is about 500 lattice spacings for nickel at $H = 100$. For $r \gtrsim 1/\lambda'$ the coupling forces, which are responsible for the terms in $1/r^3$, K_1, and K_3, are important; but for $r \gg 1/\lambda'$ the value of α is the same as if the coupling forces were not present ($C = 0$). The magnetization at any point is $J_s(1 - \frac{1}{2}\alpha^2)$. If an attempt is made to average this by integration over the xy plane, it is at once evident that the boundary must be taken into account, for the infinite integral diverges logarithmically at $r = \infty$.

If, however, the dislocations occur in pairs, with the positive one a short distance—$\mathbf{R} = \mathbf{i}X + \mathbf{j}Y$—from the negative, this difficulty does not occur. The value of α for such a *doublet* may be found by applying the operator $-\mathbf{R}\cdot\nabla$ to the value for a singlet. If the latter is expressed in the form (22), an additional factor $-i\boldsymbol{\varrho}\cdot\mathbf{R}$ in (23) gives G for a doublet, and

$$\int \alpha^2 d\mathbf{r} = \int |G|^2 d\boldsymbol{\varrho}. \tag{24}$$

The value of $(\alpha^2)_{Av}$ for n' dislocations per cm^2, distributed at random, is n' times this integral. The integration is straightforward and gives

$$1 - J/J_s = \tfrac{1}{2}(\alpha^2)_{Av} = n'k^2\pi(3X^2 + 5Y^2)/64\eta. \tag{25}$$

Such a distribution of dislocation doublets therefore gives a $1/H$ law of approach to saturation, provided H is high enough to justify the neglect of the internal field. The effect of the latter at lower H will be considered in §4.

3. Paired Dislocations

Consider, next, a more general distribution of dislocations, arbitrary except that equal numbers of positives and negatives are present in any macroscopic area of the xy plane. This restriction eliminates boundary difficulties and is probably not a bad approximation, for there is no physical reason why one type of dislocation should be produced faster than the other in the hardening process.

Superposition of the effects of all the dislocations gives α in the form (22), with G replaced by

$$G_1 = G\sum_i s_i \exp[-i\boldsymbol{\varrho}\cdot\mathbf{R}_i]. \tag{26}$$

Hence

$$\int \alpha^2 d\mathbf{r} = \int |G_1|^2 d\boldsymbol{\varrho} = \int |G|^2 \sum_i \sum_j s_i s_j \times \exp[i\boldsymbol{\varrho}\cdot\mathbf{R}_{ij}]d\boldsymbol{\varrho},$$

[15] Reference 1, p. 146.

[16] The symbols λ and k of reference 14 have been replaced by λ' and k' in the present article. The following must be distinguished: λ_0 = elementary slip distance, λ = elastic constant, λ_∞ = magnitude of saturation magnetostriction, $\lambda' = \eta^{\frac{1}{2}}$ = reciprocal of magnetic decay distance. A few symbols (β, γ, χ) have been used for different purposes in different parts of the discussion.

where $\mathbf{R}_{ij}=\mathbf{R}_j-\mathbf{R}_i$; or since $s_i=-\sum_j' s_j$,

$$\int \alpha^2 d\mathbf{r}=\int |G|^2 \sum_i s_i[s_i+\sum_j' s_j \exp(i\boldsymbol{\varrho}\cdot\mathbf{R}_{ij})]d\boldsymbol{\varrho}$$
$$=-\sum_i \sum_j' s_i s_j \int |G|^2[1-\exp(i\boldsymbol{\varrho}\cdot\mathbf{R}_{ij})]d\boldsymbol{\varrho}. \quad (27)$$

After integration over ϕ and transformation to a new variable of integration $x=R_{ij}\rho$, this becomes[13]

$$\int \alpha^2 d\mathbf{r}=-\tfrac{1}{2}\pi k^2 \sum_i \sum_j' s_i s_j R_{ij}^4\{f_0(\lambda' R_{ij})$$
$$-\tfrac{1}{2}f_2(\lambda' R_{ij})\cos 2\theta_{ij}$$
$$+f_4(\lambda' R_{ij})\cos 4\theta_{ij}+\tfrac{1}{2}f_6(\lambda' R_{ij})\cos 6\theta_{ij}\}$$
$$=-\tfrac{1}{2}\pi k^2 \sum_i \sum_j' s_i s_j F(\mathbf{R}_{ij}), \quad (28)$$

where $\theta_{ij}=\measuredangle(\mathbf{R}_{ij}, \mathbf{i})$, and

$$f_0(\mu)=\int_0^\infty [1-J_0(x)][x(x^2+\mu^2)^2]^{-1}dx, \quad (29)$$

$$f_n(\mu)=\int_0^\infty J_n(x)[x(x^2+\mu^2)^2]^{-1}dx \quad (n>0). \quad (30)$$

Integration gives[13]

$$2\mu^3 f_0(\mu)=K_1(\mu)+2\mu^{-1}K_0(\mu)$$
$$+2\mu^{-1}[(\gamma-\tfrac{1}{2})+\ln(\mu/2)], \quad (31)$$
$$2\mu^3 f_2(\mu)=K_1(\mu)+4\mu^{-1}K_2(\mu)+\mu^{-1}-8\mu^{-3},$$
$$2\mu^3 f_4(\mu)=-K_3(\mu)-6\mu^{-1}K_4(\mu)$$
$$+\tfrac{1}{2}\mu^{-1}-16\mu^{-3}+288\mu^{-5},$$
$$2\mu^3 f_6(\mu)=K_5(\mu)+8\mu^{-1}K_6(\mu)+\tfrac{1}{3}\mu^{-1} \quad (32)$$
$$-24\mu^{-3}+1152\mu^{-5}-30{,}720\mu^{-7};$$

γ is Euler's constant. For $\mu\ll 1$,

$$f_0(\mu)=\tfrac{1}{8}\mu^2, \quad f_2(\mu)=\tfrac{1}{16}\mu^2,$$
$$f_4(\mu) \text{ and } f_6(\mu)\ll 1/\mu^2. \quad (33)$$

For $\mu\gg 1$,

$$f_0(\mu)=\mu^{-4}[(\gamma-\tfrac{1}{2})+\ln(\mu/2)],$$
$$2f_2(\mu)=4f_4(\mu)=6f_6(\mu)=\mu^{-4}. \quad (34)$$

In the case of f_0 and f_2, these approximations are reliable to within about 2 percent except in the range $\frac{1}{5}<\mu<5$. Use of (33) at $\mu=1$ results in an error of 16 percent in f_0 and 19 percent in f_2; (34) is useless below $\mu=3$ for f_0 and below $\mu=20$ for f_6.

Let n be the number of dislocations per cm^2 of the xy plane, and S the area of the specimen in this plane; then $\sum_i \sum_j' s_i s_j F(\mathbf{R}_{ij})$ may be replaced by $nS[s_i \sum_j' s_j F(\mathbf{R}_{ij})]_{Av}$, the average being taken over all dislocations i. Let the dislocations about any given one be divided into three groups: (1) those at a distance small enough to permit the use of the approximation (33) over the range of fields under investigation; (2) those at a distance large enough to permit the use of (34); (3) those at an intermediate distance. For rough calculations, if the chief contribution at large distances is from f_0, group 3 need include only values of R_{ij} between the smallest $1/\lambda'$ and the largest $3/\lambda'$ for the range.

For the first group,

$$F(\mathbf{R}_{ij})=\tfrac{1}{8}\lambda'^{-2}R_{ij}^2(1-\tfrac{1}{4}\cos 2\theta_{ij})$$
$$=(3X_{ij}^2+5Y_{ij}^2)/32\eta; \quad (35)$$

this makes a positive contribution to $(\alpha^2)_{Av}$ if dislocations i and j have opposite signs, a negative contribution if they have the same sign. Let f_1 be the average *net* number of dislocations j *of opposite sign* to dislocation i, in group 1; then the sum over all i's and over j's within group 1 gives

$$[(\alpha^2)_{Av}]_1=(1/64)\pi k^2 n f_1 \eta^{-1}(3X^2+5Y^2), \quad (36)$$

where X^2 and Y^2 are average values of X_{ij}^2 and Y_{ij}^2 for *near* pairs of dislocations.

For the second group,

$$F(\mathbf{R}_{ij})=\lambda'^{-4}\{[(\gamma-\tfrac{1}{2})+\ln(\lambda' R_{ij}/2)]-\tfrac{1}{4}\cos 2\theta_{ij}$$
$$+\tfrac{1}{4}\cos 4\theta_{ij}+\tfrac{1}{12}\cos 6\theta_{ij}\}; \quad (37)$$

and if f_2 is defined for this group as f_1 was for the first, the contribution to $(\alpha^2)_{Av}$ is

$$[(\alpha^2)_{Av}]_2=\tfrac{1}{2}\pi k^2 n f_2 \eta^{-2}[c_0+\ln(\lambda' R/2)], \quad (38)$$

where $\ln R$ is an average of the values of $\ln R_{ij}$ for this group, and

$$c_0=(\gamma-\tfrac{1}{2})-\tfrac{1}{4}(\cos 2\theta_{ij})_{Av}+\tfrac{1}{4}(\cos 4\theta_{ij})_{Av}$$
$$+\tfrac{1}{12}(\cos 6\theta_{ij})_{Av}. \quad (39)$$

If the distribution of dislocations j about a given dislocation i is approximately isotropic at large distances, as seems likely, the angular terms will be small, and $c_0=\gamma-\frac{1}{2}=0.0772$. In this case $\lambda' R$ must be at least 3 by the definition of this group;

but a larger order of magnitude than 10 for $\lambda' R$ would require appreciable noncancelling contributions to $\ln R$ from R_{ij}'s greater than the "block" dimensions or interflaw distances ($\geqq 10^{-4}$ cm). Such correlation of dislocation positions in different blocks seems unlikely, and it may therefore be assumed that $\lambda' R \leqq 10$; then $\ln(\lambda' R/2) \leqq 1$.

Thus, finally,

$$1-J/J_s=\tfrac{1}{2}(\alpha^2)_{Av}$$
$$=(1/128)\pi k^2 n f_1 \eta^{-1}(3X^2+5Y^2)$$
$$+\tfrac{1}{4}\pi k^2 n f_2 \eta^{-2}[c_0+\ln(\lambda' R/2)]+\psi(\eta), \quad (40)$$

where $\psi(\eta)$ is the contribution of dislocation pairs separated by intermediate distances and is a more complicated function than the other two terms; or

$$J=J_s-a/H-b/H^2-\chi(H), \quad (41)$$

where

$$a=nf_1\cdot 9(\lambda_\infty G'\lambda_0)^2(3X^2+5Y^2)/128\pi C, \quad (42)$$

$$b=nf_2\cdot 9(\lambda_\infty G'\lambda_0)^2[c_0+\ln(\lambda' R/2)]/4\pi J_s, \quad (43)$$

$$\lambda'=(HJ_s/C)^{\frac{1}{2}}, \quad (44)$$

and $\chi(H)$ is a more complicated function. If only group 1 contributes, the formula reduces to that for doublets, since $n=2n'$. It is clear that: (1) dislocations separated by distances smaller than the magnetic decay distance contribute a $1/H$ term to the law of approach to saturation; (2) dislocations separated by distances at least several times the magnetic decay distance contribute, not a $1/H^2$ term, but a term $b(H)/H^2$, in which $b(H)$ varies logarithmically with H; (3) dislocations separated by intermediate distances contribute a more complicated approach to saturation.

4. Effect of Magnetic Interactions and of the Boundary

To take account of the internal field, Eqs. (17) and (18) must be used instead of Eq. (17a). The factor $\rho^2+\lambda'^2$ in the denominator of G must then be replaced by $\rho^2+\lambda'^2+h\cos^2\phi=\rho^2+\lambda'^2\sin^2\phi+k'^2\cos^2\phi$, where $k'^2=\lambda'^2+h$. The calculation for the doublets of §2 is simple enough to be carried out without difficulty; the result is[17]

$$1-J/J_s=n'(\pi k^2 C/8J_s)[H^{\frac{1}{2}}+B^{\frac{1}{2}}]^{-4}$$
$$\times[(H+4H^{\frac{1}{2}}B^{\frac{1}{2}}+B)X^2$$
$$+(H+4H^{\frac{1}{2}}B^{\frac{1}{2}}+5B)Y^2], \quad (45)$$

where $B=H+4\pi J_s$ is the induction. This reduces to the previous formula for $H\gg 4\pi J_s$ and gives an approximately constant deviation from saturation for $H\ll 4\pi J_s$. In the range $0.1<H/4\pi J_s<1$, in which data are available, the right member of (45) is only very roughly equivalent to an expression of the form $a'/H+b'/H^2$; if it is fitted to such a formula as well as is possible, the required a' is of the same order of magnitude as a (though somewhat smaller), and the required b' is not large enough to affect the order of magnitude of the observed b.

To estimate the effect of the boundary when positive and negative dislocations are not exactly equal in numbers, consider the finite cylinder for which the stress problem was solved in §1. The direct effect of the boundary on α, for a given stress, is unimportant compared with its effect on the stress itself; the former is limited to a region within a distance $\delta' \leqq 1/\lambda'$ of the boundary,[14] the latter extends throughout the specimen. The specimen may, therefore, be considered infinite provided the extra boundary term of Eqs. (10), (12), (16) is added to the stress. The value of α calculated in this way is the correct value inside the actual specimen and vanishes outside, except within a distance $\delta+\delta'$ of the boundary. The calculation of $\int \alpha^2 d\mathbf{r}$ proceeds as in §3. The correction to be added to the previous value, Eq. (28), includes terms of the form $\sum_i s_i F_n(R_i)\cos n\beta_i$, which may be neglected, and a term

$$\left[\int \alpha^2 d\mathbf{r}\right]_b=\tfrac{1}{2}\pi k^2 N\int_0^\infty \{N+4N[J_1(\rho r_1)/\rho r_1]^2$$
$$-4[J_1(\rho r_1)/\rho r_1]\sum_i s_i J_0(\rho R_i)\}$$
$$\times\rho^{-1}(\rho^2+\lambda'^2)^{-2}d\rho, \quad (46)$$

which may be evaluated approximately by re-

[17] The integration over ρ is elementary. The integration over ϕ may be carried out by expressing the numerator in the integrand as a sum of terms $A_n\cos^n\phi\cos n\phi$. See D. Bierens de Haan, *Nouvelles Tables d' Intégrales Définies* (Leyden, 1867), p. 77, Table 47, No. 20.

placing the sum by an integral.[18] The third term in braces becomes twice the negative of the second; the integral may be evaluated by operating with $-(1/2\mu)d/d\mu$ on the simpler integral[13]

$$\psi(\mu)=\int_0^\infty \{1-4[J_1(x)/x]^2\}x^{-1}(x^2+\mu^2)^{-1}dx$$
$$=\mu^{-2}\{(\gamma-\tfrac{1}{4})+\ln(\mu/2)+4\mu^{-2}[\tfrac{1}{2}-I_1(\mu)K_1(\mu)]\}. \quad (47)$$

Since $\lambda' r_1 \gg 1$, the correction to $1-J/J_s$ reduces to

$$[1-J/J_s]_b=(\pi k^2/4S)(N^2/\lambda'^4)\times(\gamma-\tfrac{3}{4}+\ln\tfrac{1}{2}\lambda' r_1), \quad (48)$$

which is of the same general form as the contribution of pairs of dislocations separated by a large distance. One may consider each unneutralized dislocation to be accompanied by an induced or image dislocation on or outside the boundary, although a quantitative calculation by the method of images is not possible.[11]

It has been assumed throughout §§2–4 that, despite the severe lattice distortion at the dislocation, the interatomic coupling forces are able to maintain approximate saturation there as elsewhere.[19] Any weakening of the interatomic forces at a dislocation would produce an appreciable effect only within a distance $1/\lambda'$ of the dislocation; it would not affect the contribution of distant pairs of dislocations, but might seriously alter the contribution of near pairs.

5. Interpretation of Experiments

The calculations of §§2–4 justify the following statements. (1) Even a drastically simplified model, in which it is assumed that $H \gg 4\pi J_s$ and that positive and negative dislocations are present in exactly equal numbers, leads to a more complicated law than the empirical one; but a/H and b/H^2 terms are definitely present and important. (2) Although b is not strictly a constant, it may be replaced by a constant over a limited range because of its slow variation with H. (3) The assumption of equal numbers of positive and negative dislocations is not a serious restriction, for the boundary terms introduced by a lack of such equality are of the same form as the terms due to dislocation pairs separated by large distances. (4) For $H<4\pi J_s$, the $a/H+b/H^2$ law becomes an even worse approximation because of magnetic interactions; but the empirical a and b in this range are probably of the same order of magnitude as the theoretical a and b calculated for $H \gg 4\pi J_s$. In view of (3) and (4), the numerical calculations will be based on the simplified model of §3.

Kaufmann's data on twisted nickel,[20] reproduced in Fig. 3 of reference 14, show an approximately linear increase of a with plastic twist. From the specimen dimensions (given in the caption of the same figure), twist may be converted to plastic shear γ averaged over the cross section. Thus $da/d\gamma=5.6\times10^4$ e.m.u. with torque still acting, $=3.3\times10^4$ after its release. Similarly, $db/d\gamma=8.2\times10^7$ e.m.u. with torque acting, $=1.70\times10^7$ after its release; the former value is largely due to *elastic* strain and is in fair agreement with the theoretical value.

Consider first the part of b not due to anisotropy or to elastic stress, of amount 1.70×10^7 per unit of plastic shear. If this is given by Eq. (43), then $nf_2/\gamma=4\pi J_s(db/d\gamma)/9(\lambda_\infty G'\lambda_0)^2(c_0+\ln\frac{1}{2}\lambda' R)$. For definiteness take $R=10^{-4}$ cm and $\lambda'^2=1000J_s/C$ (Kaufmann's field range was 600 to 6000 oersteds); $c_0=0.07$; $J_s=500$ e.m.u., $\lambda_\infty=3.9\times10^{-5}$, $\lambda_0=2\times10^{-8}$ cm, $G'=6.5\times10^{11}$ dynes/cm^2,

[18] This step requires defense for two reasons. (1) The terms in the integral do not converge separately at $\rho=0$; therefore a slight change in one term might conceivably introduce a large error. (2) The difference between a continuous and a discontinuous distribution of dislocations might be important at large ρ because of the oscillations in $J_0(\rho R_i)$. Objection (1) may be met by expanding the integrand in powers of ρ: it vanishes for $\rho=0$ by virtue of $\Sigma_i s_i=N$, whatever the distribution of dislocations. Objection (2) does not apply until $\rho \cong 1/l$, where l is the distance between dislocations; but by then $\rho r_1 \gg 1$, so that the contribution to the integral from this range of ρ values is small compared with the contribution from the range where the substitution is permissible. A rigorous calculation shows that the error is of order of magnitude $2\Sigma_i s_i R_i^2/Nr_1^2-1$.

[19] The contrary possibility was suggested at the end of reference 14, because of an apparent failure of magnetostrictive forces alone to give a large enough order of magnitude. But in the tentative calculation given there, the forces were supposed to be concentrated in a cross-sectional area of atomic dimensions at the dislocation; when they are properly localized in the stress field about the dislocation, the difficulty disappears.

[20] A. R. Kaufmann, Phys. Rev. **57**, 1089A (1940) and private communication.

$C=10^{-5}$ erg/cm. Then $nf_2/\gamma=2\times10^{10}$ dislocations per cm^2 per unit of plastic shear.

Consider next the value of a after release of the applied torque. Here positive and negative dislocations seek equilibrium positions in the same vertical plane, so that, approximately, $X=0$. To explain the linearity of a with γ and hence with n on the basis of Eq. (42), it may be supposed that dislocations are produced at certain favored positions on "block" walls (planes perpendicular to Ox) and are propagated along Ox, positives to the right and negatives to the left from opposite walls, until they reach a common value of x and become caught in each other's attraction. The plastic shear is then $\gamma=n'\lambda_0 L$, where L is the block length and $n'=$number of doublets per cm$^2=\frac{1}{2}nf_1$. Thus nf_1 is proportional to γ, but Y is determined by conditions at the block walls, and therefore $a\propto\gamma$. If a value of Y is assumed in Eq. (42), nf_1/γ may be calculated from the experimental $da/d\gamma$.

The largest Y that is consistent with $Y<1/\lambda'$ throughout the range is about 2×10^{-6} cm. If Y has this value, $nf_1/\gamma=3\times10^{12}$ dislocations per cm^2 per unit of plastic shear. Then $L=2\gamma/nf_1\lambda_0=\frac{1}{3}\times10^{-4}$ cm. A smaller assumed value of Y would increase still further the value of nf_1, which is already 100 times as large as nf_2. Thus it appears that $f_1\gg f_2$: that is, most of the dislocations are accompanied by a companion dislocation of opposite sign no more than 2×10^{-6} cm away. Very closely, therefore, $f_1=1$; and with Y as assumed, $n/\gamma=3\times10^{12}$, and $f_2=2\times10^{10}/3\times10^{12}=0.007$.

In these calculations, no use has been made of Taylor's formula for the stress produced by an array of dislocations or of Kaufmann's data on the applied torque required to produce the plastic twist. An independent calculation of n/γ and L may be made by using Taylor's formula and the torque data and by assuming that after plastic flow the stress due to the dislocations balances the applied stress. This calculation gives $n/\gamma=1.4\times10^{12}$, $L=0.4\times10^{-4}$. Taylor's formula, besides being subject to the error[21] pointed out in §1, is based on a somewhat different picture from the "doublet" picture adopted here;[22] consequently an agreement in order of magnitude is all that can be expected. The only unsatisfactory aspect of the theory is that to obtain such agreement, a value of Y must be assumed; and the value required, though reasonable in other respects, necessitates pushing the approximation (33) to the limit of its applicability.[23]

The larger value of a under applied stress may be due to a larger n or to a nonvanishing X. It should be possible to decide this question when more is known about the distribution of dislocations and about their behavior under applied stress.

Magnetic measurements at higher fields are desirable. The analysis of data in the range $H<4\pi J_s$ is complicated by another factor in addition to the necessity for taking account of magnetic interactions. The theory neglects second-order terms in α and therefore becomes a poor approximation if the value of α given by Eq. (21) is an appreciable fraction of unity for any value of r. It follows that k/λ' must be small, or $\sqrt{(H/H_1)}$ must be large, where $H_1=(3\lambda_\infty G'\lambda_0)^2/\pi^2 CJ_s\doteq50$ oersteds for nickel. This condition is not very well satisfied in the lower part of Kaufmann's range.

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[21] This error has been corrected in calculations recently completed and briefly reported by J. S. Koehler, Phys. Rev. **59**, 943A (1941). I am indebted to Dr. Koehler and to Mr. H. B. Huntington for helpful discussions and for the opportunity to compare my stress formulas with theirs in advance of publication.

[22] The doublet picture would probably not lead to Taylor's parabolic relation between stress and plastic strain: see reference 2, p. 174. The experimental mechanical data plotted in reference 14, Fig. 3, show a small but systematic deviation from such a parabolic relation.—Taylor's stress formula, combined with Kersten's theory of the magnetic effect of internal stress, leads to a formula $b=c'n(\lambda_\infty G'\lambda_0)^2/J_s$, with $c'\doteq3/125=0.024$ (W. F. Brown, Jr., Phys. Rev. **59**, 528 (1941)). The factor in Eq. (43) that corresponds to c' is $(9f_2/4\pi)(c_0+\ln\frac{1}{2}\lambda' R)$, and its numerical value according to the calculations of the last paragraph is 0.013.

[23] The order of magnitude $Y\leqq10^{-6}$ cm may be arrived at by the following independent argument, which, however, is highly speculative. Suppose that the "favorable" points on block walls are distributed at random but with an average spacing (along y) of m atomic distances λ_0. Then one out of m of the favorable points on the left wall is in the same atomic plane with a favorable point on the right wall. On these particular atomic planes, positive and negative dislocations can be generated at opposite ends and combine in the middle, and a large slip can result; elsewhere, dislocations can be produced but not annihilated, and the process is that described before. The spacing of these doubly favored planes, $m^2\lambda_0$, may be identified with that of the slip bands, $D\leqq10^{-4}$ cm; then $m=\sqrt{(D/\lambda_0)}$ and $Y=m\lambda_0=\sqrt{(D\lambda_0)}\leqq10^{-6}$ cm.