They did obtain, however, the correct value for the $4d^9\,{}^2D_{3,2}$ splitting of the ground state, namely 5810 cm^{-1}.

The term values of Cd IV, In V and Sn VI are listed relative to the $4d^9\,{}^2D_3$ level of the ground state of each ion and are to be found in Tables IV, VI and VIII. The estimated intensities of all classified lines are given in Tables V, VII and IX along with the wave-lengths and corresponding frequencies reduced to vacuum.

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Finite Self-Energies in Radiation Theory. Part I

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According to Dirac, electric particles display a finite radius $r_0=2e^2/3mc^2$ as the result of the damping term $(2e^2/3mc^3)d^3x/dt^3$ in the equation of motion. If the finite radius is due to radiative damping, the same must necessarily be true for the finite self-energy that is inversely proportional to the radius. An infinitely large self-energy and an infinitely small radius (Coulomb's law e^2/r) results from *Fermi's* Fourier representation of classical electrodynamics. A certain change is necessary, but the change is to produce at once a finite self-energy and a finite radius r_0. Now, an electric particle vibrating in a field of frequency ν suffers a reduction R_ν of its vibrational energy due to radiative damping, the energy reduction factor being $R_\nu=1/[1+(\nu/\nu_0)^2]$ where $\nu_0=3mc^3/4\pi e^2$. In view of the uncertainty of position due to damping we propose that the Fourier terms in the expression for the *energy* in Fermi's classical radiation theory be reduced by the same factor R_ν with Doppler effect for particles in motion. The result of this reduction is that Dirac's finite radius r_0 now occurs in a modified Coulomb energy $(e^2/r)[1-\exp(-r/r_0)]$, and the finite self-energy of a single particle becomes $e^2/2r_0=(3/4)mc^2$. Whereas the force between charged particles of finite mass remains finite for $r=0$, the force on an ideal test charge of infinite mass becomes infinite for $r=0$. This is analogous to the difference between the field E and the displacement D in Born's unitary field theory. Of interest for nuclear reactions are the electrostatic forces between particles of different masses m and M. The results are related to Sommerfeld's fine-structure constant and to the theory of mesons.

1. Introduction

Electric particles can be treated from the unitary or dualistic point of view. In the unitary theory a particle is but a spherically symmetric solution of certain modified field equations, without singularity at $r=0$. Born-Infeld's new field equations yield a finite maximum field e/r_0^2 at $r=0$. The electronic radius r_0 can be adjusted so that the total field energy is $\kappa\cdot mc^2$; the fraction κ can be chosen at will. This adjustable parameter is a disadvantage since we cannot know beforehand what fraction of the total mass is of electromagnetic origin. We prefer the dualistic point of view in which particles of various masses m are taken for granted, and the field produced by them, the "radius" and the self-energy, are to be expressed in terms of e and m.

One general point is common to all theories of electric particles. The smaller the radius, the larger the mass, the product r_0mc^2 being proportional to the square of the universal charge. However, the accepted (dualistic) radiation theory leads to an infinitely large self-energy and to an infinitely small radius, as expressed in Coulomb's energy $e^2/r=\infty$ for $r=0$. *If there are any reasons for having a finite radius then the same reasons must also be responsible for the finite self-energy.*

A radius dependent on the charge and mass occurs in Thomson's formula for the scattering cross section of an electric particle as the result of *radiative damping*. A similar radius occurs in Dirac's re-examination of the classical Lorentz theory.[1] Due to the damping term

[1] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).

$(2e^2/3mc^3)d^3x/dt^3$ in the equation of motion, the universal length

$$r_0 = 2e^2/3mc^2$$

plays the rôle of an apparent radius in Dirac's discussion. The same radius measures the uncertainty of position due to the natural line breadth. *If the finite electronic radius is the result of radiative damping*, the same cause then must also be responsible for the finite electromagnetic energy. Our task is to find the quantitative relation between radius and self-energy in terms of e and m.

In particular we have tried to answer the following questions. (1) If m is the total inertia of a particle, what part of m is of electromagnetic origin? (in contrast to the infinite field energy of the present theory). (2) What is the mutual electrostatic energy of two electric particles of equal or different masses? (in contrast to the Coulomb energy e^2/r that is ∞ for $r=0$).

An answer different from the impossible result of radiation theory can only be obtained by means of a *formal* change of this theory, introduced *ad hoc* for the purpose of getting rid of the infinities.

We are proposing a cutting-off method based on the analogy to the classical Lorentz theory. If an electric particle of charge e and mass m is put into a periodic electric field of frequency ν then the vibrational energy of the particle depends on whether we do or do not account for radiative damping. With damping the energy is reduced (§ 2) by a factor

$$R_\nu = 1/[1+(\nu/\nu_0)^2], \quad \text{where} \quad \nu_0 = 3mc^3/4\pi e^2.$$

On the other hand, in Fermi's classical field theory the electromagnetic energy produced by the particles consists of Fourier terms of wavelength c/ν. Our hypothesis is that each of these Fourier terms of the energy is to be reduced by the same factor R_ν that reduces the energy of a vibrating electron because of radiative damping. For a particle in motion ν is replaced by the Doppler frequency ν' for reasons of invariance.

The result of this formal modification is so simple that it might be considered correct even if the method of deduction is not yet satisfactory. The mutual energy of two equal particles at distance r is found (§ 3) to be

$$E_{jk} = (e^2/r)[1-\exp(-r/r_0)]$$

where

$$r_0 = 2e^2/3mc^2.$$

The same $\exp(-r/r_0)$ appears in Dirac's investigation. For $r=0$ we have the finite value

$$E_{jk}(r=0) = e^2/r_0.$$

For the self-energy of one particle at rest we obtain

$$E_{jj} = \tfrac{1}{2}e^2/r_0,$$

intimately connected with the modification of Coulomb's mutual energy. With the former value of r_0 we also can write $E^0 = \frac{3}{4}mc^2$. That is, if the modified Coulomb law is correct (in view of its great simplicity it probably is) then only $\frac{3}{4}$ of the total energy mc^2 can be of electromagnetic origin. A result like this is not unexpected in view of neutral particles that have mass without field.

Our discussion throws an interesting sidelight on Sommerfeld's *constant* $\alpha = 1/137$. The characteristic frequency ν_0 occurring in R_ν may be considered as the *central frequency* between $\nu=0$ and $\nu=\infty$ for the particle. Indeed, according to Dirac[1] Eq. (35) the spectral intensity emitted by a self-accelerated particle is proportional to R_ν and the total recorded intensity $\int R_\nu d\nu$ consists of two equal parts, integrals from $\nu=0$ to ν_0, and from ν_0 to $\nu=\infty$. The characteristic time period $t_0 = 1/\nu_0 = 4\pi e^2/3mc^3$ leads to an approximative value ($\alpha \sim 1/140$) of the fine-structure constant when used in the proper value theory[2] of the electronic charge.

There is also a relation between our results and the *theory of the meson*. The modified Coulomb potential resulting from the damping effect consists of two terms

$$V - U = (e/r) - (e/r)\exp(-r/r_0).$$

V yields the ordinary Coulomb repulsion between like charges whereas U leads to an attraction between like charges at nuclear distances. V

[2] A. Landé, Phys. Rev. **59**, 434 (1941); J. Frank. Inst. **229**, 767 (1940); **231**, 63 (1940). Part III is to appear soon. The proper value theory of M. Born starts from a different background and arrives at different results (Proc. Roy. Soc. Edinburgh **59**, 219 (1939) **60**, 100 and 141 (1940).

and U are solutions of the differential equations

$$\nabla^2 V - \partial^2 V/\partial ct^2 = -4\pi\rho,$$

$$\nabla^2 U - \partial^2 U/\partial ct^2 = -4\pi\rho + k^2 U,$$

with

$$k = 1/r_0 = 3mc^2/2e^2.$$

The latter differential equation of Yukawa is the Schrödinger-Klein-Gordon equation for a free particle of mass

$$M = kh/2\pi c = (3mc^2/2e^2)(h/2\pi c) = (3m/2)(hc/2\pi e^2) \doteq m\cdot(3/2)\cdot 137 = m\cdot 205,$$

perhaps identical with the meson mass.

2. Energy of Scattering Particles

The equation of motion for a particle of charge e and mass m is

$$m\ddot{x} - (2e^2/3c^3)d^3x/dt^3 = eE \qquad (1)$$

for small accelerations. For a periodic field

$$E = E_\omega \cos(\omega t)$$

the solution reads

$$x = (eE_\omega/m\omega^2)\cos\zeta_\omega \cos(\omega t - \zeta_\omega), \qquad (1')$$

where

$$\zeta_\omega = \mathrm{tg}^{-1}(\omega/\omega_0), \quad \cos\zeta_\omega = [1 + (\omega/\omega_0)^2]^{-\frac{1}{2}},$$
$$2\pi\nu_0 = \omega_0 = 3mc^3/2e^2. \qquad (2)$$

The average vibrational energy thus is

$$\tfrac{1}{2}m(\dot{x})^2_{\mathrm{Av}} = (e^2E^2_\omega/4m\omega^2)\cos^2\zeta_\omega. \qquad (3)$$

$\cos^2\zeta_\omega$ is identical with the reduction factor R_ν mentioned before. Periodic terms also occur in the classical Fermi theory[3] under the heading "mutual energy." There, however, the terms appear without the reducing factor $\cos\zeta_\omega$ and without the phase lag ζ_ω. We are trying to rectify this situation in an *invariant* way. For this purpose we need the proper value ζ'_{sj} of the phase shift for a particle e_j moving with velocity β_j in the direction of ϑ_{sj} through the wave s of frequency ω_s. Since the proper frequency felt by the particle according to Doppler is

$$\omega'_{sj} = \omega_s(1 - \beta_j\cos\vartheta_{sj})(1 - \beta_j^2)^{-\frac{1}{2}}, \qquad (4)$$

we have a phase lag and amplitude reduction

$$\zeta'_{sj} = \mathrm{tg}^{-1}(\omega'/\omega_0); \quad \cos\zeta'_{sj} = [1 + (\omega'/\omega_0)^2]^{-\frac{1}{2}}. \qquad (4')$$

The terms describing the mutual energy between field and particles in Fermi's classical theory of radiation can be transformed (§7) into

$$H' = 4\pi c^2\Omega^{-1}\textstyle\sum_s \omega_s^{-2}\{(\sum_i e_i\beta_i \sin\vartheta_{si}\sin\Gamma_{si})^2 + (\sum_i e_i\cos\Gamma_{si})^2\}. \qquad (5)$$

Here Ω is the total volume in which the proper vibrations of frequency ω_s take place; ϑ_{si} is the angle between the wave s and the velocity β_i, and

$$\Gamma_{si} = \omega_s r_i\cos\theta_{si}/c + \text{phase}$$

contains the angle θ_{si} between the wave direction s and the radius vector r_i from the zero point to the particle e_i. Replacing the summation over all waves s by an integration over $(\Omega/2\pi^2c^3)\omega_s^2 d\omega_s$ one obtains

$$H' = (e_1e_2/r_{12}) + \cdots + \text{infinite self-energies}.$$

This is Fermi's explanation of the Coulomb law in wave fashion, accompanied by infinite self-energies of every single particle.

We propose to modify Fermi's mutual energy (5) into the form:

$$H' = 4\pi c^2\Omega^{-1}\textstyle\sum_s \omega_s^{-2} \times\{(\sum_i e_i\beta_i\sin\vartheta_{si}\cos\zeta'_{si}\sin(\Gamma_{si}+\zeta'_{si}))^2 + (\sum_i e_i\cos\zeta'_{si}\cos(\Gamma_{si}+\zeta'_{si}))^2\} \qquad (6)$$

making use of the invariant phase shift and amplitude reduction (4) (4′). We remark that all *electrostatic* results obtained from (6) could as well be obtained from an energy expression half-way between (5) and (6) in which the reduction ζ'_{si} is applied only to *one* factor of the squares $(\sum\cdots)^2$. We even could omit the phase lag altogether.

Another justification for the factor $\cos^2\zeta' = R_\nu$ is this. Due to the classical uncertainty $\delta\lambda = r_0$ wave functions at the place X are to be replaced by their averages with density $\exp(-r/r_0)$, *viz.*

$$\sin(\omega x/c)_{\mathrm{Av}} = \frac{1}{2}\int_{-\infty}^{\infty}\exp(-|\xi|r_0^{-1})\cdot\sin[\omega(x+\xi)/c]d\xi = \sin(\omega x/c)\cdot[1+(\omega/\omega_0)^2]^{-1}.$$

[3] E. Fermi, Rev. Mod. Phys. 4, 87 (1932).

3. Self-Energy and Mutual Energy of Particles

For a single particle at rest, $\beta_i=0$ and $\zeta'_{si}=\zeta_s$ we obtain from (6)

$$E_i^0=(4\pi c^2/\Omega)e_i^2\sum_s\omega_s^{-2}\cos^2\zeta_s\cos^2(\Gamma_{si}+\zeta_s).$$

Replacing the sum by an integral we obtain

$$E_i^0=\frac{e_i^2}{2c}\cdot\frac{2}{\pi}\int_0^\infty\left(1+\frac{\omega^2}{\omega_0^2}\right)^{-1}d\omega=\frac{e_i^2\omega_0}{2c}=\tfrac{3}{4}m_ic^2.\quad(7)$$

Next we consider the mutual energy of two particles at rest in the distance r_{ij}. Here

$$\cos(\Gamma_{si}+\zeta_s)\cos(\Gamma_{sj}+\zeta_s)=\tfrac{1}{2}\cos(\omega_sr/c\cos\theta_s)+\tfrac{1}{2}\cos(\text{phase})$$

where θ_s is the angle between the wave s and the direction of r_{ij}. Replacing the summation by the integration $\frac{1}{2}d(\cos\theta_s)(\Omega/2\pi^2c^3)\omega_s^2d\omega_s$, and using the abbreviations

$$\begin{aligned}&\omega/\omega_0=u,\quad \omega r/c=ux_0,\quad x_0=r/r_0,\\ &r_0=2e^2/3mc^2,\quad \cos\zeta=(1+u^2)^{-\frac{1}{2}},\end{aligned}\quad(8)$$

we obtain from (6) the mutual energy

$$\begin{aligned}\overset{0}{E}_{ij}&=(e^2/r)(2/\pi)\int_0^\infty du\,\sin(ux_0)u^{-1}(1+u^2)^{-1}\\&=(e^2/r)[1-\exp(-r/r_0)].\end{aligned}\quad(9)$$

For large r this is the Coulomb energy. For small r the mutual energy tends toward the constant value $e^2/r_0=\frac{3}{2}mc^2$. The mutual plus the self-energies of a pair of electrons at the distance zero is equal to the self-energy of a charge $2e$. If an electron and a positron approach to $r=0$ the energy $2\cdot(\frac{3}{4}mc^2)$ is released, but the system would still retain its original non-electromagnetic mass. This process has nothing to do with annihilation. In general, a point charge acts on another point charge like an exponentially shading-off charge cloud of radius $r_0=2e^2/3mc^2$.

For the mutual energy of two electric particles of masses m and M we obtain from (6) with $X_0=r/R_0$ and $R_0=2e^2/3Mc^2$

$$\overset{0}{E}_{mM}=(e^2/r)(2/\pi)\int_0^\infty dw\,\sin w\,w^{-1}\times(1+w^2/x_0^2)^{-\frac{1}{2}}(1+w^2/X_0^2)^{-\frac{1}{2}}.\quad(10)$$

We discuss this integral in the limit $M=\infty$ where $X_0=\infty$. Here (10) reduces to

$$\begin{aligned}\overset{0}{E}_{m\infty}&=(e^2/r)(2/\pi)\int_0^\infty dw\,\sin w\,w^{-1}(1+w^2/x_0^2)^{-\frac{1}{2}}\\&=(e^2/r)(2/\pi)\int_0^{x_0}dw\int_0^\infty dt\,\cos(wt)(1+t^2)^{-\frac{1}{2}}\\&=(e^2/r)(2/\pi)\int_0^{x_0}dwK_0(w),\end{aligned}$$

where K_0 is the Bessel function. For $x_0=r/r_0=\infty$ this reduces to e^2/r. In general one can say that a test charge of mass ∞ in the distance r from the charge e of mass m feels a Coulomb potential of the smaller charge

$$e'=e(2/\pi)\int_0^{r/r_0}dwK_0(w)<e.\quad(11)$$

Since $K_0(w)$ for small w is $-\lg w+\lg 2-\gamma=-\lg w+0.11593$, the potential energy at small distance becomes

$$\begin{aligned}\overset{0}{E}_{m\infty}&=(e^2/r_0)(2/\pi)[1.11593+\lg(r_0/r)]\\&=(3mc^2/\pi)\lg(3.052\,r_0/r).\end{aligned}$$

For $M\gg m$ one obtains approximately

$$\overset{0}{E}_{M\gg m}=(3mc^2/\pi)\lg(3.052\,M/m).\quad(12)$$

Whether the formulae (10, 11, 12) for particles of different masses have any physical significance depends on whether the protonic mass is or is not of electromagnetic character. It seems more reasonable to assume that the mass of the proton is mainly that of a neutron, plus a small electron mass. In this case the *electric* forces between electrons and protons, and between protons and protons, would be of the same type (9) as between electronic particles, with r_0 being the *electronic* radius.

The potential energy between two electrons remains finite and differentiable even for $r=0$, whereas the potential energy $\overset{0}{E}_{m\infty}$ between the electron and a test charge of infinite mass ($R_0=0$) becomes logarithmically infinite for $r=0$. This is a distinction similar to that between the potential of the vector E (which remains finite), and the potential of the vector D (that becomes infinite for $r=0$) in the unitary theory of *Born* and *Infeld*[4]).

[4] M. Born, Proc. Roy. Soc. **A143**, 410 (1934).

4. Self-Energy of Particles in Motion

In order to find the electromagnetic self-energy of a particle in motion we need the average of $(\cos\zeta'_{sj})^2$ over all angles ϑ_{sj}. Expanding (4′) into powers of β^2 we obtain the average

$$(\cos\zeta'_{sj})_{\mathrm{Av}}=\cos^2\zeta_s+\frac{4}{3}\beta^2\left(\frac{\omega_s^2}{\omega_0^2}\cos^4\zeta_s-\frac{\omega_s^4}{\omega_0^4}\cos^2\zeta_s\right)+\cdots,$$

if we neglect terms in β^4. Remembering that the average of $\cos^2\vartheta_{sj}$ is $\frac{1}{3}$ and that of $\sin^2\vartheta_{sj}$ is $\frac{2}{3}$ we obtain from (6) the self-energy of a moving particle

$$E=(2e^2\omega_0/c)\left[\beta^2\frac{2}{3}\frac{1}{2}\frac{\pi}{2}+\frac{1}{2}\left(\frac{\pi}{2}+\frac{4}{3}\beta^2\frac{3\pi}{16}-\frac{4}{3}\beta^2\frac{\pi}{4}\right)\right]$$

$$=\tfrac{3}{4}mc^2(1+\tfrac{1}{2}\beta^2) \qquad (13)$$

in agreement with relativity if β^4 is neglected. The result is mainly due to the *invariance* of the phase and amplitude reduction.

Our considerations yield definite values for the self- and mutual-energies. We cannot expect that quantum theory will change these values materially. In particular, there is no reason why the self-energy should be multiplied by a factor of order 1000 on account of the magnetic dipole energy of a spinning charge cloud. This expectation would be just as wrong as the expectation of an infinite self-energy from the picture of a point charge.

In conclusion: The electromagnetic self-energy of a charged particle is finite. Hence there must also be a deviation from Coulomb's law so as to eliminate Coulomb's singularity for $r=0$. We have tried to find the modified interaction energy and the corresponding finite self-energy by a modification of Fermi's radiation theory, taking account of the classical uncertainty of position due to the natural line breadth.

Our next task is that of trying to deduce the modified energy expression (6) from a corresponding modification of the general set-up of radiation theory. Fermi's theory is based on standing waves rather than on incoming and outgoing waves. Since standing waves account for retarded and advanced potentials in a symmetrical fashion, they cannot adequately describe the very radiation damping that was the starting point of our approach. Indeed, radiation damping is due to transforming an incoming plane wave into an outgoing spherical wave. (The corresponding difficulties in the unreduced theory are much greater due to the *infinite* self-field of the electron.)

5. Modified Fermi Theory

Fermi[3] Eq. (140) introduces a Fourier expansion of the scalar and vector potential into standing waves in a large volume Ω:

$$V(r,t)=c(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_s Q_s(t)\cos\Gamma_{sr},$$

$$U(r,t)=c(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_s\left[\alpha_s\chi(t)+A_s q_s(t)\right]\sin\Gamma_{sr}, \qquad (14)$$

$$\Gamma_{sr}=\omega_s(\alpha_s\cdot r)/c+\text{phase}.$$

α_s and A_s are unit vectors longitudinal and transversal to the wave s. Substitution of (14) into the Maxwell equations with $\operatorname{div}U-\partial V/\partial ct=0$ leads to the following differential equations for Q_s, χ_s and q_s:

$$d^2Q_s/dt^2+\omega_s^2Q_s=c(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_i e_i\cos\Gamma_{si},$$

$$d^2\chi_s/dt^2+\omega_s^2\chi_s=(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_i e_i(\alpha_s\dot{r}_i)\sin\Gamma_{si}, \qquad (15)$$

$$d^2q_s/dt^2+\omega_s^2q_s=(8\pi/\Omega)^{\frac{1}{2}}\textstyle\sum_i e_i(A_s\dot{r}_i)\sin\Gamma_{si}.$$

Equation (15) has homogeneous solutions Q^0, χ_s^0, q_s^0 superposed by solutions of the inhomogeneous equation which are

$$Q_s^1(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\omega_s^{-2}(1-\beta_i^2\cos^2\vartheta_{si})^{-1}\cos\Gamma_{si},$$

$$\begin{matrix}\chi^1\\ q_s\end{matrix}(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\beta_i \begin{matrix}\sin\\ \cos\end{matrix}\vartheta_{si}\omega_s^{-2}(1-\beta_i^2\cos^2\vartheta_{si})^{-1}\sin\Gamma_{si}, \quad (16)$$

when we suppose that $\dot{r}_i/c=\beta_i$ and $\ddot{r}_i=0$. ϑ_{si} is the angle between s and β_i. By virtue of (16) and (14) the field at every point r is composed of contributions of every single particle in a *symmetric* fashion, aside from the pure field derived from Q^0, χ^0, q^0.

Referring to the amplitude reduction and the phase lag (4′) we now define reduced quantities $\mathbf{Q}_s$, $\boldsymbol{\chi}_s$, $\mathbf{q}_s$ as the solutions of the following differential equation

$$d^2\mathbf{Q}_s/dt^2+\omega_s^2\mathbf{Q}_s=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\cos\zeta'_{si}\cos(\Gamma_{si}+\zeta'_{si}), \quad (17)$$

etc. Compare these with (15). The solutions are $\mathbf{Q}_s=\mathbf{Q}_s^0+\mathbf{Q}_s^1$, etc. Here $\mathbf{Q}_s^0$ is identical with the former Q_s^0 whereas $\mathbf{Q}_s^1$ is reduced in amplitude and lags in phase:

$$\mathbf{Q}^1(t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i\omega_s^{-2}(1-\beta_i^2\cos^2\vartheta_{si})^{-1}\cos\zeta'_{si}\cos(\Gamma_{si}+\zeta'_{si}), \quad (17')$$

etc., to be compared with (16). We also define reduced potentials at the place of the particle e_i.

$$\mathbf{V}(i,t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_s \mathbf{Q}_s\cos\zeta'_{si}\cos(\Gamma_{si}+\zeta'_{si}),$$
$$\mathbf{U}(i,t)=c(8\pi/\Omega)^{\frac{1}{2}}\sum_s(\alpha_s\boldsymbol{\chi}_s+A_s\mathbf{q}_s)\cos\zeta'_{si}\sin(\Gamma_{si}+\zeta'_{si}). \quad (17'')$$

Given positions and given velocities of the particles produce Fourier components (17′) satisfying (17). Equations (15) are the Fourier representation of Maxwell's equations including $\operatorname{div}U-\partial V/\partial ct=0$. Since (15) for $\mathbf{Q}_s$ is equivalent to (17) for Q_s, Eqs. (17) are only another form of Maxwell's equations as long as $\ddot{r}=0$.

6. Energy of Particles in a Field

In Fermi's (154) we write $\dot{r}_i$ for $-c\gamma_i$, 1 for δ_i, which is immaterial for our main task, which is to *deduce* (6) *from a Hamiltonian*. We propose the following modification of Fermi's Hamiltonian (154):

$$\begin{aligned}H=\sum_i m_ic^2+\sum_s\sum_i(\dot{r}_ip_i)+c(8\pi/\Omega)^{\frac{1}{2}}\sum_s\mathbf{Q}_s\sum_i e_i\cos\zeta'_{si}\cos(\Gamma_{si}+\zeta'_{si})\\ -(8\pi/\Omega)\sum_s\sum_i(\dot{r}_i,\alpha_s\chi_s+A_s\mathbf{q}_s)\cos\zeta'_{si}\sin(\Gamma_{si}+\zeta'_{si})\\ +\tfrac{1}{2}\sum_s[(\mathbf{p}_s^2+\xi_s^2-\mathbf{P}_s^2)+\omega_s^2(\mathbf{q}_s^2+\chi_s^2-\mathbf{Q}_s^2)]. \quad (18)\end{aligned}$$

H differs in the terms containing ζ'_{si} from the unreduced Fermi Hamiltonian. Nevertheless, as we shall see in (20), the energy of the pure field is the same as in Maxwell's theory, and only the mutual energy is changed. As "coordinates" we consider the reduced quantities $\mathbf{Q}_s$, χ, $\mathbf{q}_s$; the conjugate "momenta" are $\mathbf{P}_s$, ξ_s, $\mathbf{p}_s$. As equations of motion $\partial H/\partial\mathbf{Q}_s=-d\mathbf{P}_s/dt$, $\partial H/\partial\mathbf{P}_s=d\mathbf{Q}_s/dt$ we obtain the former Eqs. (17) which, as we saw before, are equivalent to Maxwell's equations for $\ddot{r}_i=0$. Furthermore, the first four terms of (18) can be written

$$\text{Energy}=\text{rest}+\text{potential}+\text{kinetic energy}$$
$$E=\sum_i[m_ic^2+e_i\mathbf{V}(i)+(\beta_i,p_i-e_i\mathbf{U}(i)/c)], \quad (18')$$

representing the Hamiltonian of a system of electric particles under the reduced potentials $\mathbf{V}$ and $\mathbf{U}$, defined in (17″) with the Fourier amplitudes $\mathbf{Q}_s=\mathbf{Q}_s^0+\mathbf{Q}_s^1$, etc. and $\mathbf{Q}_s^0=Q_s^0$. According to (18′) a monochromatic external potential Q_s acts on the particle with a potential $\mathbf{V}$ that is reduced by the factor $\cos\zeta'_{si}$ and has a phase lag ζ'_{si} behind the phase of Q_s. At the same time there is no damping term in (18′). The motion of the electron under this reduced external potential then is the same as its motion according to the usual theory where the effect of the unreduced potential is accompanied by that of a damping term. The terms $\mathbf{Q}_s^1$ in $\mathbf{V}$ calculated for vibrating electrons have no phase lag and will not produce an additional damping effect. This result seems very inconsistent indeed. But we must remember that the method of standing waves accounts for advanced and retarded potentials simultaneously, not only for an external potential Q_s^0 but also for the potential $\mathbf{Q}_s^1$ produced by a particle on itself. The whole problem needs further clarification.

7. Transformation of the Energy

We now turn to transforming the energy (18) into a more convenient form in which the Hamiltonian form is abandoned, however. p_i is the total momentum, $p_i - e_i\mathbf{U}(i)/c$ is the kinetic momentum $(p_i)^{\text{kin}}$. It does not matter in (17) that $(p_i)^{\text{kin}}$ and m_ic^2 split up into an electromagnetic and a mechanical part in the ratio of 3 to 1 (see below). Only the mechanical part is to be carried in m_ic^2 and $(\dot{r}_ip_i)^{\text{kin}}$, the electromagnetic part being contained in the other terms of (18). The potential momentum is a part of the second sum in (18) and cancels the negative potential momentum represented by the fourth sum of (18). This leaves for the total energy:

$$E=\sum_i m_ic^2+\sum_i (\dot{r}_ip_i)^{\text{kin}}+c(8\pi/\Omega)^{\frac{1}{2}}\sum_i e_i \sum_s \mathbf{Q}_s \cos(\Gamma_{si}+\zeta'_{si})\cos\zeta'_{si}$$
$$\times\tfrac{1}{2}\sum_s [(\mathbf{p}_s^2+\boldsymbol{\xi}_s^2-\mathbf{P}_s^2)+\omega_s^2(\mathbf{q}_s^2+\boldsymbol{\chi}_s^2-\mathbf{Q}_s^2)]. \quad (19)$$

A further simplification is obtained by virtue of [Fermi's (160)(161)]

$$\omega_s\boldsymbol{\chi}_s-\mathbf{P}_s=0,\quad \boldsymbol{\xi}_s=\omega_s\mathbf{Q}_s-(c/\omega_s)(8\pi/\Omega)^{\frac{1}{2}}\sum_j e_j\cos\zeta'_{sj}\cos(\Gamma_{sj}+\zeta'_{sj}),$$

which results in

$$E=\sum_i m_ic^2+\sum_i (\dot{r}_ip_i)^{\text{kin}}+(4\pi c^2/\Omega)\sum_s \omega_s^{-2}(\sum_i e_i\cos\zeta'_{si}\cos(\Gamma_{si}+\zeta'_{si})+\tfrac{1}{2}\sum_s(\mathbf{p}_s^2+\omega_s^2\mathbf{q}_s^2).$$

At last we can use the relations $\dot{\mathbf{q}}_s=\mathbf{p}_s$ so that

$$\mathbf{q}_s^2=(\mathbf{q}_s^0)^2+(\mathbf{q}_s^1)^2+2\mathbf{q}_s^0\mathbf{q}_s^1,$$
$$\mathbf{p}_s^2=(\dot{\mathbf{q}}_s^0)^2+(\dot{\mathbf{q}}_s^1)^2+2\dot{\mathbf{q}}_s^0\dot{\mathbf{q}}_s^1.$$

When summing over s the double products vanish because of independent phases. q_s' can be taken from (16) whereas $(\dot{q}_s')^2$ is small and of order β^4. Thus the energy becomes (if we neglect β^4)

$$E=\sum_i m_ic^2+\sum_i (\dot{r}_ip_i)^{\text{kin}}+\tfrac{1}{2}\sum_s[(p_s^0)^2+\omega_s^2(q_s^0)^2]$$
$$+(4\pi c^2/\Omega)\sum_s \omega_s^{-2}\{(\sum_i e_i\beta_i\sin\vartheta_{si}\sin(\Gamma_{si}+\zeta'_{si})\cos\zeta'_{si})^2+(e_i\cos(\Gamma_{si}+\zeta'_{si})\cos\zeta'_{si})^2\}. \quad (20)$$

E consists of the mechanical rest and kinetic energy of the particles, the pure field energy, and the mutual energy between field and particles. For the latter (20) yields the expression used before in (6) *q.e.d.*

The attempt to incorporate the phase lag and the amplitude reduction into the general set-up of the radiation theory cannot be considered as satisfactory because of inherent difficulties of the standing wave method. Nevertheless the results obtained in the case of particles at rest or in uniform motion may be considered as a supplement to Dirac's investigation[1] of the classical electron. Dirac has shown that the "own radiation field" of a particle leads to a self-accelerated motion in which the term $\exp(-r/r_0)$ plays a major rôle. We have tried to show here that radiation damping when accounted for by the modified energy (6), leads to a mutual energy between two particles of the extremely simple form

$$E_{12}=(e^2/r)[1-\exp(-r/r_0)].$$

The corresponding electromagnetic self-energy ($E_1=\frac{1}{2}E_{12}$ for $r=0$) is $\frac{3}{4}mc^2$ contradicting those who think that the mass ought to be completely of electromagnetic origin, and confirming the present idea[1,5] that only a part of the mass can be electromagnetic.

[5] W. Heitler, *Quantum Theory of Radiation* (Oxford, 1936), p. 33. See also reference 1.