On the Angular Distribution of Alpha-Particles Produced in the Li^7-Proton Reaction

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The angular distribution of the alpha-particles ejected in the reaction, $Li^7+H^1\rightarrow 2\ He^4$, is discussed. It is assumed that the Li^7-nucleus is odd, that the incident proton is in a P state and that a broad and a sharp nuclear resonance level participate in the reaction. The angular momentum 0 and 2 $\hbar$ are assigned to the broad and to the sharp levels, respectively. The constants entering into the theory could be determined only by extending the measurements on angular dependence to proton energies higher than 400 kev. The results are in qualitative agreement with the dependence on energy of the reaction yield and of the angular distribution observed so far.

INTRODUCTION

RECENT experiments[1] have revealed a marked departure from spherical symmetry in the distribution of alpha-particles created by the transmutation

$$Li^7+H^1\rightarrow 2He^4. \qquad (1)$$

In these experiments a thin film of lithium was bombarded by protons of energy ranging from 100 to 400 electron kilovolts. The emitted alpha-particles have a total of about 17 Mev kinetic energy. The relative number of alpha-particles was determined at angles of eight different cosines and for eight different energies of the incident protons.

In the center of mass coordinate system the relative number of alpha-particles per unit solid angle is represented very well by the factor $1+A(E)\cos^2\theta$; here θ is the angle between the direction of the emerging particles and the direction of the incident proton beam. The coefficient which gives the departure from spherical symmetry, $A(E)$, is positive and small at 100 kev, rises to a value of $\frac{1}{3}$ at 300 kev and then rises rapidly to about $\frac{3}{4}$ at 350 and 400 kev. A plot of $A(E)$ is reproduced in Fig. 1.

It is greatly to be desired that the experiments be carried out for proton energies greater than 400 kev.

As was pointed out by those who did the experimental work the behavior of $A(E)$ suggests an approach to a resonance level of the compound, Be^8, nucleus. In the present paper the expected theoretical behavior of the angular distribution of alpha-particles in the neighborhood of such a resonance is examined in detail. It is assumed that the alpha-particles which are ejected at energies outside the apparent resonance region are due to a second, but very much broader, resonant level of the compound nucleus.

The energy-dependence of the *yield* curve[2]

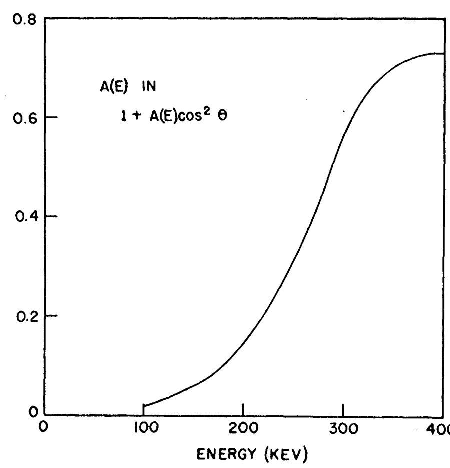

FIG. 1. The variation with bombarding energy of $A(E)$ in the factor, $1+A(E)\cos\theta$, which represents the experimental results on the angular distribution of alpha-particles.

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[1] V. J. Young, A. Ellett and G. J. Plain, Phys. Rev. **58**, 498 (1940).

[2] L. H. Rumbaugh, R. B. Roberts and L. R. Hafstad, Phys. Rev. **54**, 657 (1938).

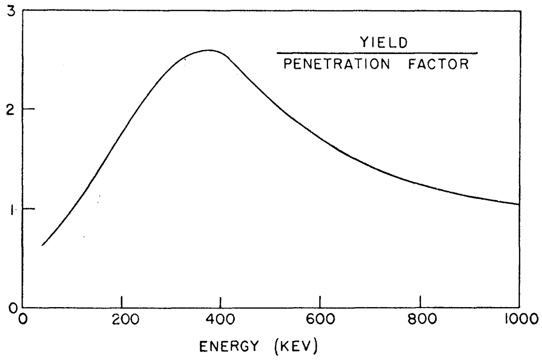

FIG. 2. The ratio of experimentally found yield to the calculated penetration factor as a function of the bombarding energy. Experimental results for energies above 350 kev were taken from reference (2), those between 150 kev and 350 kev from the paper of Herb, Parkinson and Kerst, Phys. Rev. **48**, 118 (1935), and those below 150 kev from the letter of Heydenburg, Zahn and King, Phys. Rev. **49**, 100 (1936). These three sets of results were joined to form a smooth curve on an arbitrary scale. The penetration factor used here is F_1^2/ρ in the notation of reference (3).

measured for alpha-particles ejected at right angles to the incident protons corroborates the presence of two processes giving rise to the emission of the alphas. Figure 2 shows the ratio of the yield and the penetration factor for protons (in arbitrary units). The penetration factor was calculated from the tables of Yost, Wheeler and Breit[3] by assuming a P wave and an effective collision radius of 4×10^{-13} cm (the last two assumptions are of minor importance in determining the shape of the curve). The figure shows the presence of a process that depends on the energy practically only through the penetration factor and a second process which has a resonance around 400 kev. As will appear later we shall associate the former process with a $J=0$ compound state, the latter with a $J=2$ state. The parallelism of the yield curve with the energy dependence of the angular distribution is striking. In view of our conclusions to be given below it would be more satisfactory if the maximum in Fig. 2 occurred at higher energy than the maximum deviation from spherical symmetry. But too much reliance must not be placed upon conclusions based on the ratio in Fig. 2. The resonance in this figure extends over so wide an energy range that the slowly varying influence of the energy on the yield may cause a difference between the position of the apparent maximum and the position of the resonance.

The angular distribution of ejected alpha-particles will thus depend upon the simultaneous influence of two nuclear levels. Under this condition the relative phase of the particles coming from the two resonance levels will be of importance. In most of the considerations involving dispersion theory, intensities alone were important; but in at least one example, the scattering of neutrons by He, phase relations had to be taken into account.[4] In the subsequent discussion the superposition of outgoing alpha-rays with different phases will lead in general to a sigmoid dependence of the quantity $A(E)$ (from the factor $1+A(E)\cos^2\theta$) on the energy of incident protons. This sigmoid dependence is not unlike the dispersion curve in optics. The shape of both curves is due to essentially similar resonance denominators in the dispersion formula and the similarity of the curves helps to emphasize that the dispersion theory of nuclear reactions and the dispersion theory in optics have more in common than a name.

STATISTICS AND PARITY

The disintegration (1) in the energy range involved permits of a fairly simple theoretical discussion. Alpha-particles obey Einstein-Bose statistics so that a pair of them can be produced only in states of even orbital angular momentum and even parity; since the spin is zero the allowed states may be designated ^1S, ^1D, $^1G\cdots$. Furthermore the ground state of Li^7 is well known to have a nuclear spin of $\frac{3}{2}$, whereas the parity of this state has not been experimentally determined. Theoretical predictions both from the individual particle model and the alpha-particle model are that the ground state of the Li^7 nucleus is odd. In this case only P, F, $\cdots$ states of the incident proton will be able to form the even compound nucleus Be^8. Computing the intensities of P and F "regular" solutions to the Schroedinger equation in a Coulomb field[3] it is found that the P wave is over 1000 times stronger than the F wave for particles of 200 kev. Even up to 1 Mev the F wave is inappreciable so if the parity of the

[3] F. L. Yost, J. A. Wheeler and G. Breit, Terr. Mag. 443 (December, 1935).

[4] F. Bloch, Phys. Rev. **58**, 829 (1940).

Li^7 nucleus is odd only the P wave need be taken into account. The relatively small yield of reaction (1) also points to the process being due to an incident proton in a P wave rather than in an S wave.[5]

If it is assumed that the reaction (1) is produced by protons in a P wave the experimental angular distribution, $1+A(E)\cos^2\theta$, can be readily understood. Since the bombarded nuclei are completely unpolarized and since the spins of the incident protons are also unpolarized the emitted beam cannot show more complicated transformation properties under rotation of coordinates than those of the orbital motion in the incident beam. Thus the emitted beam may contain a part which behaves as a vector and gives a $\cos^2\theta$ term but no higher powers of $\cos\theta$ if the incident wave function transforms like a vector, i.e., if it is a P wave. The Bose statistics of the alpha-particles excludes odd powers of $\cos\theta$ so that in fact $1+A(E)\cos^2\theta$ is the most general angular dependence to which incident P waves can give rise.

Although there are many indications that Li^7 nuclei are odd the possibility that they are even is not to be overlooked. In this case the incident proton must have an even wave function and an even angular momentum. In the angular distribution of the alpha-particles the term $A(E)\cos^2\theta$ might be explained as due mainly to an interference term produced by an incident S wave and a less effective incident D wave. The square of the amplitude of the outgoing alpha-particle wave function produced by the incoming D wave contains in general a term proportional to $\cos^4\theta$. But if the incoming D wave is relatively ineffective in producing the reaction, the $\cos^4\theta$ term may have too small a coefficient to be observed experimentally. The $\cos^2\theta$ term depending linearly on the small amplitudes produced by the incident D wave would in fact be much more noticeable than the $\cos^4\theta$ term depending quadratically on the same amplitude. On the basis of present experimental evidence it is not quite impossible that Li^7 nuclei are even and the $1+A(E)\cos^2\theta$ angular dependence is produced by joint action of incident S and D waves. But in principle it is possible to establish the odd parity of the Li^7 nucleus by eliminating experimentally in the reaction products the $\cos^4\theta$ terms to a high accuracy thus demonstrating that the angular dependence is not produced by an incident D wave.

It will be assumed in the following calculations that the Li^7 nucleus is odd and it will be shown that the experimental data obtained so far may be understood with this assumption. The incident proton P wave has a total angular momentum of $\frac{3}{2}\hbar$ at most and this combined with the $\frac{3}{2}\hbar$ of the nucleus can give rise to all integral angular momenta from 0 to $3\hbar$ for the compound nucleus. Outgoing alphas are therefore either in ^1S or ^1D states. From the shape of the experimental curve for $A(E)$ one would judge that a resonance in one or the other of these states is approached as the proton energy increases. Below the apparent resonance, *e.g.*, at 100 kev, the angular distribution approaches spherical symmetry. Barring very special types of interaction which would make the angular distribution due to a compound nucleus of $J=2$ spherically symmetric it appears that the alpha-particles of low bombarding energies are ejected from a compound nucleus of $J=0$.

Dispersion Formula

In view of the probable applicability of the simple conditions discussed above we have calculated the expected angular distribution of alpha-particles in reaction (1). For this purpose the dispersion formula of wave mechanics as presented by Breit and Wigner[6] and Bethe and Placzek[7] has been used. According to the dispersion theory as applied to the calculation of the cross section $\sigma^P{}_Q$ of the nuclear transformation $P\rightarrow Q$

$$\sigma^P{}_Q=4\pi^3\lambda\!\!\!^-{}^2\left|\sum_r \frac{H^P{}_r H^r{}_Q}{E_P-E_r+\frac{1}{2}i\gamma_r}\right|^2. \qquad (2)$$

In Eq. (2) $H^P{}_r$ is the matrix element of the Hamiltonian which couples the initial state P with the compound (resonant) state r; $H^r{}_Q$ is the same thing for the final state Q. Both P and Q may be degenerate, and there are many levels r. E_P is the energy of the initial state, E_r the energy of the intermediate state and γ_r is the level

[5] M. Goldhaber, Proc. Camb. Phil. Soc. **30**, 561 (1934).

[6] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936).

[7] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).

width.[8] In calculating γ_r one must take into account all possible modes of decay of the state r. In the present problem the most probable method is just the emission of two alpha-particles and in this case

$$\gamma_r \cong 2\pi \sum_Q |H^r{}_Q|^2, \qquad (3)$$

where the summation extends over the degenerate states Q.

We are interested in intermediate states, r, of two different angular momenta, 0 and $2\hbar$, which have resonances in the region of the experimental results and which are eigenstates of the angular momentum. The initial state P is a Li^7 nucleus with a total angular momentum $\frac{3}{2}\hbar$ plus an incident proton with an orbital angular momentum $\hbar$ and a spin momentum $\frac{1}{2}\hbar$. Thus the initial state does not appear as an eigenstate of the total angular momentum. By proper linear combinations of the initial states with various orientations of angular momentum (including the three possible orientations of the P wave) one can construct eigenstates of the total angular momentum which we shall call X_r. The subscript r indicates that X_r has the same transformation properties with regard to rotation as the intermediate state r. It will be seen that two functions can be constructed with angular momentum $J=2$. We shall call X_2 that linear combination of them to which the intermediate state r with $J=2$ responds most strongly. The orthogonal linear combination is then not coupled at all with the compound nucleus. The construction of the state X_0 (which is invariant under rotations) is unique.

The matrix element $H^P{}_r$ in the numerator of Eq. (2) may then be written in the form

$$H^P{}_r = \varphi'(E)\alpha'_r(P, X_r). \qquad (4)$$

Here (P, X_r) is the transformation coefficient from the state P to the state X_r. The matrix element connecting X_r and r has been written as a product of two factors: the energy independent complex number α'_r and the energy dependent $\varphi'(E)$. We shall assume $\varphi'(E)$ to be the same function for all intermediate states involved. This assumption is probably approximately valid over the relatively narrow energy region of the resonance. Actually the most important reason for the energy dependence of $\varphi'(E)$ is the variation of the penetration factor. This variation will not depend greatly on the nature of the intermediate state ($J=0$ or 2) over a narrow resonance region as long as the angular momentum of the initial wave (which is a P wave) is the same.

The final state Q is represented by two alpha-particles the wave function of which can be regarded as plane waves which describe the ejection of those alphas in a definite direction characterized by the angle θ. This final state will again not be an eigenstate of the angular momentum. We may introduce an eigenstate Z which can be obtained by superposition of the final states corresponding to various directions θ. The matrix element $H^r{}_Q$ can then be written

$$H^r{}_Q = \alpha''_r \varphi''(E)(Z, Q), \qquad (5)$$

where the symbols have a similar significance as in Eq. (4). The transformation coefficient (Z, Q) is proportional to the surface harmonic (normalized Legendre polynomials) $Y_r(\theta)$. Here the index r refers to the intermediate state and is meant to specify both the angular momentum J and its component, m, around the direction of the incident protons. The surface harmonic $Y_r(\theta)$ can be assumed to be normalized since we still carry the numerical coefficient α''_r. The energy dependence of $\varphi''(E)$ is negligible in the region of resonance due to the high energy of the ejected alpha-particles. It follows from the same reason that according to Eq. (3) we may replace γ_r by an energy-independent quantity Γ_r. The equation for angular distribution then becomes

$$1 + A(E)\cos^2\theta \sim \left| \sum_r \frac{\alpha_r(P, X_r) Y_r(\theta)}{E_P - E_r + \frac{1}{2}i\Gamma_r} \right|^2. \qquad (6)$$

Here $\alpha'_r\alpha''_r$ has been set equal to α_r and the factors $\varphi'(E)\varphi''(E)$ common to all terms on the right-hand side have been dropped.

Spin-Orbit Coupling

The incident proton is assumed to have orbital angular momentum $L=1$ for the reasons given in the introduction. There is certainly no component of this orbital angular momentum about

[8] As has been shown by G. Gamow, *Atomic Nuclei and Nuclear Transformations* (Cambridge, 1937), p. 96, the energy of the unstable state r may be considered to be the complex value $E_r - \frac{1}{2}i\gamma_r$; the $\frac{1}{2}i\gamma_r$ gives the exponential decrease of the amplitude in that state with time.

the direction of bombardment so the states X_r cannot be obtained as a direct superposition of the initial states. From the $j=\frac{3}{2}$ of the Li^7 nucleus and the spin of the incident proton we may compose two states: a threefold degenerate state, P_3, in which the proton spin is opposite to the Li spin and a fivefold degenerate state, P_5, in which the proton spin is parallel to the Li spin. The plane wave describing the proton appears as a factor in both P_3 and P_5. It is evident that only P_3 can form a compound nucleus of $J=0$ and contribute to the resulting S wave. Both P_3 and P_5 are able to form compound states with $J=2$. Thus two intermediate states with $J=2$ can be constructed.

If there is no significant spin-orbit interaction, i.e., if the splitting caused is smaller than the level width it would seem most likely from experimental results on magnetic moment that the ground state of Li^7 nuclei can be described as $^2P^\circ_{1\frac{1}{2}}$. In this case the spin of the odd nuclear particle is parallel to the orbital angular momentum and then P_3 alone can contribute to the ^1D wave of alpha-particles; for the spin in the P_5 is parallel to that in $^2P^\circ_{1\frac{1}{2}}$ and in the absence of spin-orbit interaction the compound nucleus would remain in the triplet state. The two resonant energy levels with $J=2$ coincide when there is no spin-orbit interaction but only P_3 gives rise to alpha-particles.

We shall define strong spin-orbit interaction to be strong enough to separate the two levels with $J=2$ by an amount large compared with the level width. Resonance with one of these levels will be considered as important in the energy range around 400 kev. The more general case in which the levels are separated only by an energy comparable to their widths will not be presented. For the problem at hand the neglected level may be at higher energy than the one revealed so far in the experiments or it may be at negative bombarding energy and thus unattainable in the reaction. Further consideration will be limited to resonance with two compound states, one with $J=0$ which will be designated r_0 and one with $J=2$ which will be called r_2.

As has been stated above, the resonant state r_2 responds most strongly to the linear combination X_2. To any orthogonal linear combination r_2 will not respond at all. In the next section we shall construct two states with $J=2$ which we shall call $X_2^{(3)}$ and $X_2^{(5)}$. The state $X_2^{(3)}$ is constructed to be orthogonal to P_5 and $X_2^{(5)}$ orthogonal to P_3. We may express X_2 as a linear combination of these two states

$$X_2=X_2^{(3)}\cos\xi+X_2^{(5)}\sin\xi,$$

where ξ is a real number.[9] The state

$$X_2^{(3)}\sin\xi-X_2^{(5)}\cos\xi$$

will not be coupled with r_2. It may be noted that the case of no spin-orbit coupling is represented by $\xi=0$. In this case a nuclear state interacting with $X_2^{(5)}$ is present in the resonance region but this state does not disintegrate into two alpha-particles. For strong spin-orbit coupling $\xi=0$ is also possible but in this case a nuclear state interacting with $X_2^{(5)}$, though it can disintegrate into two alphas, falls into a region sufficiently far removed from the one under consideration.

Since P_3 alone is coupled with the nuclear state with $J=0$ one may substitute (P_3, X_0) for (P, X_0) in Eq. (6). Similarly $X_2^{(3)}\cos\xi+X_2^{(5)}\sin\xi$ is alone responsible for the excitation of the nuclear state with $J=2$. Substituting this expression in those terms of Eq. (6) which contain the second surface harmonic we obtain

$$1+A(E)\cos^2\theta \sim \sin^2\xi\sum_{m=-2}^{2}\left|\frac{\alpha_2(P_5, X_2^{(5)m})Y_2^m(\theta)}{E_P-E_2+\frac{1}{2}i\Gamma_2}\right|^2 + \sum_{m=-1}^{1}\left|\frac{\alpha_0(P_3, X_0)Y_0(\theta)}{E_P-E_0+\frac{1}{2}i\Gamma_0} + \cos\xi\frac{\alpha_2(P_3, X_2^{(3)m})Y_2^m(\theta)}{E_P-E_2+\frac{1}{2}i\Gamma_2}\right|^2, \quad (7)$$

where the magnetic quantum number m is made explicit. Since waves of different m values and P_3 and P_5 have no phase relation, mixed products containing different m values and P_3 and P_5 are omitted. Terms $(P_3, X_2^{(5)})$ and $(P_5, X_2^{(3)})$ have been omitted because of the assumed orthogonality relations.

[9] There is no phase relation between the initial states P_3 and P_5 so that interference terms between P_3 and P_5 do not occur. Thus only absolute values of the coefficients of $X_2^{(3)}$ and $X_2^{(5)}$ matter and we can use the sine and cosine of a real number for these coefficients without loss of generality.

TABLE I. *Transformation coefficients.*

m	$(P_3, X_2^{(3)})$	$(P_5, X_2^{(5)})$	ANGULAR DEPENDENCE
2	0	$-\frac{1}{3}\sqrt{6}$	$(15/8)^{\frac{1}{2}}e^{2i\varphi}\sin^2\theta$
1	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{6}\sqrt{6}$	$(15/2)^{\frac{1}{2}}e^{i\varphi}\sin\theta\cos\theta$
0	$\frac{1}{3}\sqrt{6}$	0	$(5/4)^{\frac{1}{2}}(1-3\cos^2\theta)$
-1	$\frac{1}{2}\sqrt{2}$	$\frac{1}{6}\sqrt{6}$	$(15/2)^{\frac{1}{2}}e^{-i\varphi}\sin\theta\cos\theta$
-2	0	$\frac{1}{3}\sqrt{6}$	$(15/8)^{\frac{1}{2}}e^{-2i\varphi}\sin^2\theta$

CALCULATION OF MATRIX ELEMENTS

Let "a" represent the normalized wave function of the nucleus. It is fourfold degenerate having components $a_{1\frac{1}{2}}$, $a_{\frac{1}{2}}$, $a_{-\frac{1}{2}}$, $a_{-1\frac{1}{2}}$. Similarly let b_1, b_0, b_{-1} describe the normalized incident P wave of the proton and $s_{\frac{1}{2}}$, $s_{-\frac{1}{2}}$ the spin state of the incident proton. The subscripts give the z component of the angular momentum of each state. We take the direction of bombardment as the z axis so that the only component of b which appears in P_3 and P_5 is b_0.

Incident waves P_3 and P_5 are then described as follows:

$$\tfrac{1}{2}[a_{\frac{1}{2}}s_{\frac{1}{2}}-\sqrt{3}a_{1\frac{1}{2}}s_{-\frac{1}{2}}]b_0,$$

$$\frac{1}{\sqrt{2}}[a_{-\frac{1}{2}}s_{\frac{1}{2}}-a_{\frac{1}{2}}s_{-\frac{1}{2}}]b_0, \qquad (P_3)$$

$$\tfrac{1}{2}[\sqrt{3}a_{-1\frac{1}{2}}s_{\frac{1}{2}}-a_{-\frac{1}{2}}s_{-\frac{1}{2}}]b_0,$$

$$a_{1\frac{1}{2}}s_{\frac{1}{2}}b_0,$$

$$\tfrac{1}{2}[\sqrt{3}a_{\frac{1}{2}}s_{\frac{1}{2}}+a_{1\frac{1}{2}}s_{-\frac{1}{2}}]b_0,$$

$$\frac{1}{\sqrt{2}}[a_{-\frac{1}{2}}s_{\frac{1}{2}}+a_{\frac{1}{2}}s_{-\frac{1}{2}}]b_0, \qquad (P_5)$$

$$\tfrac{1}{2}[a_{-1\frac{1}{2}}s_{\frac{1}{2}}+\sqrt{3}a_{-\frac{1}{2}}s_{-\frac{1}{2}}]b_0,$$

$$a_{-1\frac{1}{2}}s_{-\frac{1}{2}}b_0.$$

The functions X_0, $X_2^{(3)}$ and $X_2^{(5)}$ can be constructed from the same waves, a, b and s:

$$\frac{1}{2\sqrt{3}}\{-\sqrt{3}a_{-1\frac{1}{2}}b_1s_{\frac{1}{2}}+\sqrt{2}a_{-\frac{1}{2}}b_0s_{\frac{1}{2}}+a_{-\frac{1}{2}}b_1s_{-\frac{1}{2}}-a_{\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-\sqrt{2}a_{\frac{1}{2}}b_0s_{-\frac{1}{2}}+\sqrt{3}a_{1\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}\}, \qquad (X_0)$$

$$\tfrac{1}{2}[a_{\frac{1}{2}}b_1s_{\frac{1}{2}}-\sqrt{3}a_{1\frac{1}{2}}b_1s_{-\frac{1}{2}}],$$

$$\tfrac{1}{4}[2a_{-\frac{1}{2}}b_1s_{\frac{1}{2}}+\sqrt{2}a_{\frac{1}{2}}b_0s_{\frac{1}{2}}-2a_{\frac{1}{2}}b_1s_{-\frac{1}{2}}-(6)^{\frac{1}{2}}a_{1\frac{1}{2}}b_0s_{-\frac{1}{2}}],$$

$$\frac{1}{2(6)^{\frac{1}{2}}}\{\sqrt{3}a_{-1\frac{1}{2}}b_1s_{\frac{1}{2}}+2\sqrt{2}a_{-\frac{1}{2}}b_0s_{\frac{1}{2}}-a_{-\frac{1}{2}}b_1s_{-\frac{1}{2}}+a_{\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-2\sqrt{2}a_{\frac{1}{2}}b_0s_{-\frac{1}{2}}-\sqrt{3}a_{1\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}\}, \qquad (X_2^{(3)})$$

$$\tfrac{1}{4}[(6)^{\frac{1}{2}}a_{-1\frac{1}{2}}b_0s_{\frac{1}{2}}+2a_{-\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-\sqrt{2}a_{-\frac{1}{2}}b_0s_{-\frac{1}{2}}-2a_{\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}],$$

$$\tfrac{1}{2}[\sqrt{3}a_{-1\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-a_{-\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}],$$

$$\frac{1}{2\sqrt{3}}[\sqrt{3}a_{\frac{1}{2}}b_1s_{\frac{1}{2}}-2\sqrt{2}a_{1\frac{1}{2}}b_0s_{\frac{1}{2}}+a_{1\frac{1}{2}}b_1s_{-\frac{1}{2}}],$$

$$\frac{1}{4\sqrt{3}}\{2\sqrt{3}a_{-\frac{1}{2}}b_1s_{\frac{1}{2}}-(6)^{\frac{1}{2}}a_{\frac{1}{2}}b_0s_{\frac{1}{2}}+2\sqrt{3}a_{\frac{1}{2}}b_1s_{-\frac{1}{2}}-4a_{1\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-\sqrt{2}a_{1\frac{1}{2}}b_0s_{-\frac{1}{2}}\},$$

$$\frac{1}{2\sqrt{2}}[a_{-1\frac{1}{2}}b_1s_{\frac{1}{2}}+\sqrt{3}a_{-\frac{1}{2}}b_1s_{-\frac{1}{2}}-\sqrt{3}a_{\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-a_{1\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}], \qquad (X_2^{(5)})$$

$$\frac{1}{4\sqrt{3}}\{\sqrt{2}a_{-1\frac{1}{2}}b_0s_{\frac{1}{2}}+4a_{-1\frac{1}{2}}b_1s_{-\frac{1}{2}}+(6)^{\frac{1}{2}}a_{-\frac{1}{2}}b_0s_{-\frac{1}{2}}-2\sqrt{3}a_{-\frac{1}{2}}b_{-1}s_{\frac{1}{2}}-2\sqrt{3}a_{\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}\},$$

$$\frac{1}{2\sqrt{3}}[-a_{-1\frac{1}{2}}b_{-1}s_{\frac{1}{2}}+2\sqrt{2}a_{-1\frac{1}{2}}b_0s_{-\frac{1}{2}}-\sqrt{3}a_{-\frac{1}{2}}b_{-1}s_{-\frac{1}{2}}].$$

One easily verifies that $(P_3, X_2^{(5)})=(P_5, X_2^{(3)})=(X_2^{(3)}, X_2^{(5)})=0$. The transformation coefficients $(P_3, X_2^{(3)})$ and $(P_5, X_2^{(5)})$ and the normalized angular dependence of the surface harmonics for the different values of magnetic quantum number are given in Table I. $(P_3, X_0)=\frac{1}{3}\sqrt{3}$.

Substituting the results of Table I into Eq. (7) and summing up the coefficients of α_0^2, $\alpha_0\alpha_2$ and α_2^2 one obtains

$$\sigma\sim\frac{1}{3}\left|\frac{\alpha_0^2}{(E_P-E_0+\frac{1}{2}i\Gamma_0)^2}+\frac{2(5/2)^{\frac{1}{2}}\alpha_0\alpha_2(1-3\cos^2\theta)\cos\xi}{(E_P-E_0+\frac{1}{2}i\Gamma_0)(E_P-E_2+\frac{1}{2}i\Gamma_2)}+\frac{(5/2)\alpha_2^2[(1+3\cos^2\theta)\cos^2\xi+3\sin^2\xi(1-\cos^2\theta)]}{(E_P-E_2+\frac{1}{2}i\Gamma_2)^2}\right|. \tag{8}$$

As has been anticipated in the introduction only the zeroth and second power of $\cos\theta$ appear in this formula.

Theoretical Angular Dependence

It is assumed for the reasons given above that the sharp rise of $A(E)$ with increasing energy is due to an approach to resonance with a state of $J=2$. The spherically symmetrical contribution below resonance then comes from a broad resonance to a state of $J=0$. In Eq. (8) we accordingly assume $\Gamma_0 \gg E_P-E_0$ and neglect the dependence of the $J=0$ resonance on energy. The unknown quantities α_2, α_0, Γ_0 and Γ_2 can be lumped into one complex number, $\beta e^{i\delta}$ which is conveniently defined as

$$\beta e^{i\delta}\equiv\tfrac{1}{2}(10)^{\frac{1}{2}}\alpha_2\Gamma_0/\alpha_0\Gamma_2. \tag{9}$$

β and δ are real. Since $\frac{1}{2}\Gamma_2$ is unknown it is expedient to choose it as the energy unit and instead of the energy difference E_P-E_2 we shall use ϵ,

$$\epsilon\equiv(E_P-E_2)/\tfrac{1}{2}\Gamma_2. \tag{10}$$

With the notations defined in Eqs. (9) and (10) we get

$$1+A(\epsilon)\cos^2\theta\sim 1+\epsilon^2+2\beta(1-3\cos^2\theta)(\epsilon\cos\delta+\sin\delta)\cos\xi +\cos^2\xi\beta^2(1+3\cos^2\theta)+3\sin^2\xi\beta^2(1-\cos^2\theta) \tag{11}$$

from which follows at once

$$A(\epsilon)=3\frac{\beta^2\cos 2\xi-2\beta(\epsilon\cos\delta+\sin\delta)\cos\xi}{1+\epsilon^2+2\beta(\epsilon\cos\delta+\sin\delta)\cos\xi+\beta^2(1+2\sin^2\xi)}. \tag{12}$$

The denominator of the right-hand side of Eq. (12) is positive and vanishes only under the particular conditions: $\sin\xi=0$, $\beta\sin\delta\cos\xi=-1$. This means that the emission in the direction of the beam does not vanish at any bombarding energy unless these conditions are fulfilled or unless $A=-1$.[10] In general $A(\epsilon)$ has one real root which occurs for the value $\epsilon=\epsilon_0$, with

$$\epsilon_0=-\tan\delta+\tfrac{1}{2}\beta\sec\delta\cos 2\xi\sec\xi. \tag{13}$$

At the energy corresponding to ϵ_0 the angular distribution is spherically symmetrical. In terms of ϵ_0, $A(\epsilon)$ becomes

$$A(\epsilon)=6\beta\cos\delta\cos\xi\frac{\epsilon_0-\epsilon}{\epsilon^2+1+2(\epsilon-\epsilon_0)\beta\cos\delta\cos\xi+2\beta^2}. \tag{12'}$$

[10] It is interesting to note that if $A(\epsilon)$ falls below -1 in a certain energy range there will be no emission at those angles θ for which $\cos^2\theta=1/|A|$. Thus in this range there will be a cone of zero alpha-particle intensity the opening angle of which rises from zero to a certain maximum value and falls back to zero again as the bombarding energy is increased.

The plot of $A(\epsilon)$ against ϵ is clearly a sigmoid curve asymptotic to the ϵ axis at large absolute values of ϵ and crossing the axis at ϵ_0. There is one maximum and one minimum to this curve and these lie at ϵ_m

$$\epsilon_m=\epsilon_0\pm(\epsilon_0^2+1+2\beta^2)^{\frac{1}{2}}. \quad (14)$$

It is characteristic of this theory that the maximum and the minimum occur at equal energy differences from ϵ_0. Substituting Eq. (14) into Eq. (12′) we find the extreme values of $A(\epsilon)$

$$A(\epsilon_m)=-3\beta\cos\delta\cos\xi/(\epsilon_m+\beta\cos\delta\cos\xi). \quad (15)$$

It will be noted that in general the maximum values of A are not equal in absolute value. If, for instance, the energy corresponding to ϵ_0 does not lie in the resonance region ($|\epsilon_0|\gg 1$), one of the extreme values of A will differ from zero insignificantly. From Eqs. (15) and (10)

$$3/A_{\max}-3/A_{\min} = -(E_{\max}-E_{\min})/\tfrac{1}{2}\Gamma_2\beta\cos\delta\cos\xi, \quad (16)$$

where $E_{\max}$ and $E_{\min}$ are the E values at which the maximum and minimum occur. Thus the measured values of A and E at the maximum and minimum determine $\frac{1}{2}\beta\Gamma_2\cos\delta\cos\xi$ at once. The resonant energy E_2 is then related to the energy at which $A=0$, E^0, by

$$E_2=E^0 +\tfrac{1}{4}\Gamma_2\beta\cos\delta\cos\xi[2+3/A_{\max}+3/A_{\min}]. \quad (17)$$

Actually it is not at all certain that the shape of the $A(E)$ curve has a sigmoid appearance. As has been pointed out above in one-half of the sigmoid curve the value of $A(E)$ may be very small. In fact if the energy corresponding to ϵ_0 should happen to lie at negative energies of the incident protons one-half of the sigmoid curve is missing. Thus in appearance, or even in fact, the dependence of A on the energy may reduce to a simple hump. The observed dependence of the yield of alpha-particles on energy makes it more likely that A as a function of E has a simple maximum rather than a sigmoid behavior. In fact if $A(E)$ would become negative above 400 kev and yet within the resonance region one would expect the maximum in Fig. 2 to lie at higher energies than 400 kev. This is so because a negative value of $A(E)$ within the resonance region would enhance particularly strongly the number of alpha-particles ejected at right angles to the incident beam and the experiments on which Fig. 2 is based were made with just such a geometrical arrangement.

Though in the present calculations explicit assumptions about the nuclear model have been avoided some simplifications had to be introduced. In particular we assumed that only two compound nuclei are involved in the process and we also neglected the energy dependence of the matrix elements involved within the resonance range. It seems however that it is not worth while to make more refined calculations at present. Extension of the measurements on angular dependence to higher energies will probably indicate to what extent the basis of the present calculations should be modified or extended.