A METHOD OF MEASURING HEAT CONDUCTIVITIES.[1]

By R. W. King.

UNDER the boundary conditions

$$\theta(0, t) = \theta_1 + \theta_2 \cos \omega t,$$
$$\theta(\infty, t) = 0,$$

the solution of the differential equation

$$\frac{\partial \theta}{\partial t} + \mu\theta = k\frac{\partial^2 \theta}{\partial x^2}, \tag{1}$$

expressing the linear flow of heat is

$$\theta(x, t) = \theta_1 e^{-p_1 x} + \theta_2 e^{-p_2 x} \cos(\omega t - qx + e), \tag{2}$$

in which

$$p_1 = \sqrt{\frac{\mu}{k}},$$

$$p_2 = \frac{1}{k}\sqrt{\frac{\mu + \sqrt{\mu^2 + \omega^2}}{2}},$$

$$q = \frac{1}{k}\sqrt{\frac{-\mu + \sqrt{\mu^2 + \omega^2}}{2}}.$$

In these equations k is the ratio of the thermal conductivity to the product of density and specific heat, and μ is a term giving the rate of radiation from the surface of the specimen considered. From the value of q it follows that the velocity of propagation of the heat wave represented by equation (2) is a function of the thermal conductivity, density, specific heat, the period $2\pi/\omega$ of the impressed temperature variation, and the rate of radiation from the surface of the rod. Of these five factors, the only uncertain one is the last; and this can be readily eliminated and

[1] The method of measuring heat conductivities which is described in the following pages is, it is thought, original. It however bears a certain resemblance to the methods of Ångström and Neumann, and it also appears from a recent article by T. Barratt (Proc. Phys. Soc. London, p. 347, 1914) that in some of the experimental features, the present method is similar to his. This however is a coincidence.

For an excellent summary of practically all the various methods of measuring thermal conductivities and for a very complete bibliography the reader is referred to Chwolson's Traité de Physique.

the conductivity found if the velocities corresponding to two separate periods are known. We have, calling v the velocity of propagation of the heat wave, λ its wave length and T its period,

$$v = \frac{\lambda}{T} = \frac{2\pi}{Tq},$$

whence

$$v^2 = \frac{8\pi^2 k}{T^2(-\mu + \sqrt{\mu^2 + \omega^2})}$$

or

$$\mu = \frac{\omega^2 v^2 T^2}{16\pi^2 k} - \frac{4\pi^2 k}{T^2 v^2}.$$

Hence, if, for a constant value of μ, the velocities v_1, v_2 correspond to the periods T_1, T_2 respectively, then

$$\frac{\omega_1^2 v_1^2 T_1^2 - \omega_2^2 v_2^2 T_2^2}{16\pi^2} = 4\pi^2 k^2 \left(\frac{1}{T_1^2 v_1^2} - \frac{1}{T_2^2 v_2^2}\right)$$

and

$$k = \frac{1}{4\pi}\left(\frac{T_1^2 T_2^2 v_1^2 v_2^2 (v_1^2 - v_2^2)}{T_2 v_2^2 - T_1^2 v_1^2}\right)^{1/2}.$$

The following is an account of some experimental work undertaken with the idea of reproducing the conditions of the problem just outlined.

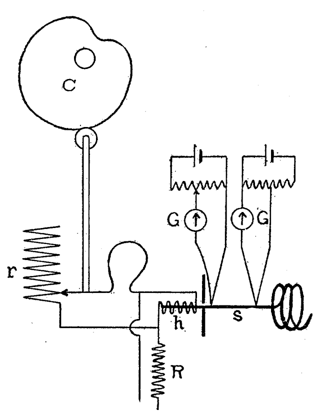

Fig. 1.

The essential parts of the apparatus are shown in Fig. 1. The specimen S is in the form of a wire, usually about 2.5 mm. in diameter and from 25 to 50 cm. in length. The greater portion of this wire is coiled up so that, all together, it occupies a space about 7 cm. long. One end of the specimen is inserted in a small resistance coil h which serves to supply heat to it. The coil h is connected in multiple with a variable resistance r, the two being attached through a larger resistance R to a steady source of current (in the present case, 110-volt D.-C. mains). The value of r is varied by the cam C in such a way that the current I through h is given by the expression

$$I = I_0 \left| \sin \frac{\omega}{2} t \right|.$$

(The design of the cam will be discussed in detail later.) To measure the velocities at which the impressed waves of heat travel along the specimen, two thermo-elements are attached a few centimeters apart to the end of the specimen at which heat is supplied. Each element is in series with a short-period galvanometer and a counter electromotive force equal to the mean value of the electromotive force of the element. By means of this arrangement, the motions of the galvanometers correspond to the variations of the temperatures of the points a and b from their respective mean temperatures; and to determine the velocity of wave propagation it is only necessary, knowing the distance between a and b, to determine by observation the time-lag between the motions of the galvanometers.

II. Details of the Apparatus.

The Specimen.—Exact dimensions of the specimens used will be found among the data. In the case of the specimen of tin, the thermo-junctions were between the tin itself and No. 40 constantan wire fused into its surface; and in the case of the copper the junctions were between the copper itself and No. 30 constantan wire inserted in very small holes in he copper and fastened with a solder of pure silver. For the purpose of checking the behavior of the apparatus, some specimens were equipped with three junctions about equally spaced, and with each of these it was found that the wave velocity between the first and second junctions was the same as that between the second and third.

The device found most convenient for holding the specimen and heating coil h is shown in Fig. 2. It consists of three circular disks of asbestos

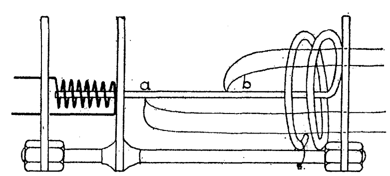

Fig. 2.

board fastened to a $\frac{1}{4}$-inch iron rod. The two end disks, in order to be readily removable, are clamped by nuts, and the middle one is cemented in place. The specimen is inserted in a hole just large enough to receive it in the central disk, and the opposite end is steadied by means of a bit of wire which ties it to the iron rod. The junction wires pass out through small holes in the right-hand disk. During use the whole arrangement is placed in a small wooden box, or wrapped with a sheet of paper, or inserted in an electric furnace in the cavity of which it just fits.

The Heating Coil h.—This coil, which is wound from No. 20 nichrome wire, has a diameter of 1 cm. and length of 2.5 cm. Its resistance at room temperature is 1.5 ohms. In the writer's work the maximum current in h has been in the neighborhood of 2.5 amperes.

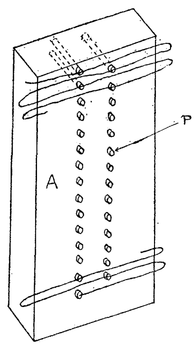

Fig. 3.

The Variable Resistance r.—This resistance is built up in fifty-two equal sections from No. 18 manganin wire, the manner of construction being shown in Fig. 3. A is a fiber block through which two columns of copper pins P are driven, and the wire, being arranged in a zig-zag form, is soldered to the ends of these pins which project from one face of A. The sliding contact which the cam actuates presses against the other ends of the pins. As a result of using a variable resistance of this kind, the current through the coil h, instead of varying continuously in agreement with the equation

$$I = I_0 \left| \sin \frac{\omega}{2} t \right| .$$

actually varies by a series of little jumps which approximate closely to the continuous value. However, the result is satisfactory, for with a period as great as eight minutes no lack of smoothness can be detected in the resultant heat-wave in the specimen. The form of this resultant wave has been tested at different times and been found to be strictly sinusoidal.

The Cam.—The cam must be of such a shape as to cause the temperature of h to vary in the manner specified by the first boundary condition. Call c the heat capacity of h, μ its radiation constant, and let $H = H(t)$ represent the rate of development of heat due to a current in h. Then the change of temperature of h is given by the equation

$$H(t)dt - \mu\theta dt = cd\theta,$$

and since θ must vary according to the relation

$$\theta = \theta_0(\alpha - \cos \omega t), \qquad \alpha \geqq 1,$$

we have

$$\begin{aligned} H(t) &= c\theta_0\omega \sin \omega t + \mu\theta_0(\alpha - \cos \omega t) \\ &= \theta_0[\mu\alpha - A \sin (\omega t + \varphi)], \end{aligned}$$

where

$$A = \sqrt{\mu^2 + c^2\omega^2}$$

and

$$\varphi = \tan^{-1}\left(- \frac{\mu}{c\omega} \right) .$$

Dropping the phase angle φ and remembering that when $I = 0$, $H(t) = 0$, we may write

$$H(t) = A(1 - \cos \omega t)$$

$$= 2A \sin^2 \frac{\omega}{2} t,$$

and since $H(t)$ varies as the square of I,

$$I = \sqrt{2A} \sin \frac{\omega}{2} t,$$

or what, so far as the requirements of the present work are concerned, is equivalent,

$$I = \left| \sqrt{2A} \sin \frac{\omega}{2} t \right|.$$

To get such a current, consider two multiply-connected resistances, h and r, in series with a resistance R and a steady electromotive force. Assume the resistance R to be so large in comparison with h and r that, when h is held fixed and r is varied, the current I_0 supplied by the electromotive force is practically constant. Then, calling I the current through h,

$$I = \frac{I_0 r}{r + h} = a \left| \sin \frac{\omega}{2} t \right|$$

whence

$$r = \frac{ha \left| \sin \frac{\omega}{2} t \right|}{I_0 - a \left| \sin \frac{\omega}{2} t \right|}.$$

Or taking $a = \frac{1}{2} I_0$,

$$r = h \frac{\left| \sin \frac{\omega}{2} t \right|}{2 - \left| \sin \frac{\omega}{2} t \right|}.[1]$$

Since the value of r is proportional to the displacement of the sliding contact, the radii of the cam (except for an arbitrary additive constant) are given at once by this last expression. The writer has used for the values of ten radii equally spaced in an interval of 180°,

3.75 + 0.000 cm.	3.75 + 5.725 cm.
3.75 + 0.878 "	3.75 + 7.050 "
3.75 + 1.907 "	3.75 + 8.180 "
3.75 + 3.075 "	3.75 + 8.960 "
3.75 + 4.370 "	3.75 + 9.230 "

The cam was run by an electromotor operating through a reduction

[1] All of this work assumes that the changes of resistance of h and r due to changes of temperature are negligible, and apparently this assumption may be safely made.

gearing. The motor was run at as constant a voltage as possible and the change of period of the cam was secured by changing the ratio of the gearing.

The Galvanometers.—These instruments had sensitivities of 118 and 194 megohms, the more sensitive one being connected with the rear junction. They were critically damped, and having equal periods of 3.5 seconds, it was unnecessary to correct for their lag behind the electromotive forces of their respective junctions. They were arranged to cast images of a Nernst filament upon a scale five meters distant.

The time-lag of the rear galvanometer behind the leading one was determined by recording on a chronograph the times of transits of the images of the filament through three reference positions chosen near the center of the swings. The reason for using three positions rather than one will be clear upon referring to the data given upon page 443, for it will be noted that the three values of the time-lag determined during a single complete vibration often vary almost as much from one another as those taken in vibrations several minutes apart. With three reference positions it thus becomes possible to arrive at a given degree of accuracy in about one third the time that would be required when using but one. On the other hand, it was not found desirable to use more than three as the uncertainty incurred by having so many positions to watch counteracted the gain secured by the greater number of readings. The chronograph records were extended to include six to eight vibrations with a period of 130 seconds and from four to six vibrations with a period of 295 seconds. Longer records were not made because slow and irregular changes in the value of μ then had a noticeable tendency to affect the results.

The one condition, which above all others seems to be necessary to an accurate measurement of the time-lag, is that the mean temperature of the junctions remain constant, and until this condition has been attained it is of very little use to take readings.

More sensitive galvanometers than the two the writer has used would doubtless be desirable as it has been found necessary, with the present arrangement, to use a temperature variation at the leading junction of from 20 to 30 degrees Centigrade. Furthermore, with the rear galvanometer considerably more sensitive than the one used, it would be possible, not only to cut down the temperature amplitude, but also to place the junctions farther apart. This would of course increase the time lag and thereby increase the accuracy of the results. With the wires thus far used, damping of the heat wave has been considerable but not so great as was expected. The extent to which this damping occurs does not of course affect the time lag between the galvanometers. Their motions are of the type known as forced vibrations and it can readily be shown that the time lag is therefore determined entirely by their free periods,

TABLE I.

Tin.

Diam. .25 cm. Length 25 cm. Distance between junctions 2.535 cm. Conductivity .1567.

Period.	Time-lag.			
	7.97	9.04	10.14	
	13.07	11.84	10.82	
	21.04	20.88	20.96	
	9.86	10.90	11.98	
	10.94	9.75	8.68	
133.0 secs	20.80	20.65	20.66	
	10.13	11.23	12.39	
	10.63	9.61	8.42	
	20.73	20.84	20.81	
	9.10	10.30	11.31	
	11.76	10.60	9.56	
	20.86	20.90	20.87	
	9.52	10.54	11.62	
	11.48	10.35	9.26	
	21.00	20.89	20.88	
	9.60	10.79	11.92	
	11.58	10.43	9.33	
	21.18	21.22	21.25	Average = 20.90
				= 22.64 secs.
	12.43	13.50	14.37	
	14.08	13.00	11.97	
	26.51	26.50	26.34	
	13.28	14.23	15.19	
	13.34	12.20	11.19	
	26.62	26.43	26.38	
297.6 secs.				
	12.02	13.05	13.94	
	14.53	13.58	12.51	
	26.55	26.63	26.45	
	12.60	13.51	14.48	
	13.89	12.83	11.79	Average = 26.46
	26.49	26.34	26.27	= 28.66 secs.

the degree of their mechanical and electrical damping, and the period of the impressed electromotive force.

III. Data and Results.

Table I. gives a sample set of readings as taken directly from a chronograph record. The three columns give respectively the time-lag (in an arbitrary scale) as determined from the transits occurring at the three reference positions. Motion from the left hand position (corresponding to the left hand column) to the right represents a cooling of the junctions.

Table II. gives in summarized form the results for copper and tin, the two metals upon which thus far accurate measurements have been made.

Table II.

Copper.

Diam. .25 cm. Length 30 cm. Distance between junctions 4.24 cm. Density 8.93.

Date.	Mean Temperature.	Specific Heat.	Period in Seconds.	Time-Lag in Seconds.	Conductivity.[1]	
Mar. 26	35°	.0928	130.7 293.0	11.77 15.52	...910	.9085 av.
Apr. 14	35	.0928	129.3 291.3	12.12 16.60	...907	
Apr. 16	60	.0938	129.8 293.0	12.26 16.97	...906	.9035 av.
Apr. 30	60	.0938	128.8 290.3	12.43 17.55	...901	

[1] Unit of conductivity is the calorie per centimeter per second per degree Centigrade.

Tin.

Diam. .25 cm. Length 25 cm. Distance between junctions 2.535 cm. Density 7.28. Mean temperature 35°. Specific Heat .0547.

Date.	Period in Seconds.	Time-Lag in Seconds.	Conductivity.
Apr. 5	129.7 290.0	11.30 14.36	1551
Apr. 12	130.1 294.5	11.41 14.66	1543
Apr. 12	128.3 287.7	11.22 14.31	1553
Apr. 12	133.0 297.6	11.32 14.33	....1567....1554 av.

The value of the temperature coefficient for copper as determined (only roughly of course) between the temperatures of 35° and 60° is —.00022, a value which agrees quite satisfactorily with a value found by Jäger and Diesselhorst[1] for one of their specimens of copper. The values

[1] Wiss. Abh. d. Phys. Techn. Reichsanstalt, 3, 269, 1900.

of the specific heat of copper which I have used are those of Naccari,[1] and of the specific heat of tin are those of Schimpff.[1] The values of the densities were taken from the *Tabellen* of Landolt and Börnstein.

Preliminary experiments indicate that the method here described may be advantageously applied to the measurement of the thermal conductivity at high and low temperatures and in a magnetic field, and more extensive measurements are now in progress. Work done with tellurium seems to show that the method will work satisfactorily for conductivities as low as .01.

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[1] Landolt u. Börnstein, Tabellen.