On the Passage of Neutrons Through Ferromagnets

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The paper contains a treatment of various phenomena occurring in the passage of neutrons through domains which are regularly or irregularly magnetized. After setting up general equations for the behavior of the neutron spin in different types of magnetic fields which correspond approximately to various stages of magnetization in ferromagnets we obtain very general results for the depolarization of an originally polarized beam. These results can be easily extended to arbitrarily constituted beams. We report next, briefly, on current views concerning the domain structure of ferromagnets and discuss in detail the possibilities of investigating this structure by means of experiments with partially polarized neutron beams. The discussion divides itself naturally into the treatment of single crystals, macroscopically unmagnetized polycrystals and of polycrystals near magnetic saturation. Revising previous unsatisfactory treatments we derive in a very general manner formulae for the change of intensity and polarization of a neutron beam traversing a saturated or quasi-saturated ferromagnetic medium. The formulae obtained constitute the basis for a modified theory of the well-known neutron polarization experiments. A closing paragraph contains a brief comparison between theory and experiment as far as it can be carried out without evaluating certain phenomenological constants, a task reserved for the following paper.

I. INTRODUCTION

THE study of the passage of neutrons through ferromagnets promises to lead to results which will be interesting from two points of view. On the one hand experiments with polarized neutron beams should reveal important information about the magnetic structure of the ferromagnets dependent upon the state of saturation, the mechanical, and the crystallographic condition of the material. On the other hand it will be seen that the magnetic condition of the ferromagnet is of decisive importance for all experiments which aim to produce polarization phenomena with the aid of permanent magnets.

The underlying concept is essentially very simple. Interaction will always be set up between the neutron spin, or more precisely its magnetic moment, and the magnetic field of the ferromagnet. The effects of the interaction will depend obviously on the variation of the magnetic field along the neutron's path in magnitude and direction. One has therefore to examine how the spin of a neutron will behave in magnetic fields the variation of which may follow several different sufficiently simple and general assumptions.

In the same way that magnetic fields influence the state of polarization of a polarized beam they will also affect and in general retard the production of a polarized beam inside of a ferromagnet. The theoretical prediction of the result of a polarization experiment can therefore only be made when these retarding factors are fully taken into account. The revised theory of polarization experiment shows that the effect of depolarization is much greater than previously expected so that, for example, the deviation from magnetic saturation must lie in the neighborhood of 0.1 percent if the experiments are not to be affected appreciably.

II. GENERAL THEORY OF THE SPIN MOTION OF A NEUTRON IN MAGNETIC DOMAINS

The variation of the spin axis of a neutron passing through a magnetic domain is determined by the "classical" equation of motion:

$$d\mathbf{s}/dt=[\boldsymbol{\mu}\times\mathbf{H}(\mathbf{r})]/\hbar=g\mathbf{s}\times\mathbf{H}(\mathbf{r}), \qquad (2.1)$$

where $\mathbf{s}$ is the operator for the spin of the neutron in units of $\hbar$, $\boldsymbol{\mu}$ the magnetic moment of the neutron, $\mathbf{H}$ the magnetic field of the domain, $g=(\boldsymbol{\mu}/\hbar)\mathbf{s}$ the gyromagnetic ratio of the neutron, and $\mathbf{r}$ its location.

From the quantum-mechanical point of view $\mathbf{s}$, $\boldsymbol{\mu}$ and $\mathbf{r}$ are operators. To see that the passage to the classical interpretation will give correct answers, one observes the following: (1) The size of the regions over which the magnetic field of the domains suffers appreciable variation ($>10^{-4}$ cm) is much larger than the wave-length of the neutron ($\sim 10^{-8}$ cm). This circumstance allows one to treat the positional motion classically. (2) As will be shown later, the quantity de-

termining the neutron polarization as measured in the transmission experiments is the *expectation* value of the spin, i.e., the classical observable. (3) Since the equation is linear in the operators (all the other quantities being ordinary numbers), no error is incurred by replacing each by its expectation value. (If there occurred terms of the form $s_x s_y$, this replacement would be incorrect, since $s_x s_y \neq s_y s_x$.) The classical interpretation of (1) will, hence, be used exclusively.

In applying Eq. (2.1) to determine the rotation of the spin axis of a neutron in traversing a magnetic medium one assumes the medium to be made up of a series of regions $(1 \cdots i \cdots n)$ in each of which the magnetic field $\mathbf{H}_i$ is constant in direction. From Eq. (2.1) and the above considerations it is readily seen that in passing through the ith region the change of spin is such that (1) the component parallel to $\mathbf{H}_i$ is unchanged ($\mathbf{s}_{\|i} = \mathbf{s}_{\|(i-1)}$) and (2) the component lying in a plane perpendicular to $\mathbf{H}_i$ precesses about H_i with angular velocity $\omega_i = gH_i$, i.e.

$$\mathbf{s}_{\perp i} = \mathbf{s}_{\perp(i-1)} \cos\left(\int_0^{\tau_i} gH_i dt\right) + \mathbf{s}_{\perp(i-1)} \times \mathbf{H}_i/H_i \sin\left(\int_0^{\tau_i} gH_i dt\right), \tag{2.2}$$

where H_i is the *algebraic* magnitude of $\mathbf{H}_i$ with reference to an arbitrarily defined sense. If we write

$$\mathbf{s}_{\|(i-1)} = (\mathbf{s}_{i-1} \cdot \mathbf{H}_i) \cdot \mathbf{H}_i/H_i^2 \quad \text{and} \quad \mathbf{s}_{\perp(i-1)} = \mathbf{s}_{i-1} - (\mathbf{s}_{i-1} \cdot \mathbf{H}_i)\mathbf{H}_i/H_i^2,$$

$\mathbf{s}_i$ is given by $\mathbf{s}_i = A_i \mathbf{s}_{(i-1)}$ where A_i is the dyadic[1]

$$A_i = \frac{\mathbf{H}_i\mathbf{H}_i}{H_i^2} + \left(1 - \frac{\mathbf{H}_i\mathbf{H}_i}{H_i^2}\right) \cos\left(\int_0^{\tau_i} gH_i dt\right) - \sin\left(\int_0^{\tau_i} gH_i dt\right) \frac{\mathbf{H}_i \times}{H_i}. \tag{2.3}$$

By successive application of this procedure the final value of the spin $\mathbf{s}_f$ is given by $\mathbf{s}_f = A_n \cdots A_i \cdots A_1 \mathbf{s}_0$. This gives the rotation of the spin axis of a neutron traversing a *given path*. The magnitude of the spin is, of course, constant. To obtain the spin of the emerging neutron beam, $\mathbf{s}_f$ must be averaged over all neutron paths. If the initial beam is polarized it is clear that the magnitude of the average spin of the emerging beam is, in general, smaller than that of the incident beam and hence the medium has a depolarizing action on the incident beam. As to the averaging process itself, it will be assumed that the fields $\mathbf{H}_i$ and the magnitude of the intervals τ_i have no statistical connection with each other; under these conditions the $\mathbf{H}_i$ can be averaged before the τ_i, a procedure which will be employed consistently in what follows.

When the averaging process is applied to ferromagnetics there arises the question: What is the magnetic field acting on the neutron? From the atomistic standpoint this field is the Lorentz **H**, the strong variation of which over atomic distances gives rise to magnetic scattering. However, to find the effect of the magnetic field on the *transmitted* beam one must average over a region containing many atoms. This process, however, is precisely that which is used to establish the connection between the equations of Lorentz and Maxwell; the result is the well-known identification of the atomic **H** with the Maxwell **B**.

III. Depolarization of Neutrons in Different Types of Domains[2]

Case 1

The magnetic fields of the domains are directed successively parallel and antiparallel to a fixed axis. In this case it is clear that

$$\mathbf{s}_{\|f} = \mathbf{s}_{\|0}; \quad \mathbf{s}_{\perp f} = \mathbf{s}_{\perp 0} \cos\phi + \mathbf{s} \times \mathbf{B}/B \sin\phi; \quad \phi = \int_0^{\tau} gB dt.$$

[1] In what follows, the usual dyadic notation is employed. A is always of the form $c_1\mathbf{ab} + c_2\mathbf{q}\times$, where c_1 and c_2 are constants, and where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{q}$ are vectors. This symbolism acquires a meaning only when the dyadic operates on a vector, *viz.* $A \cdot \mathbf{v} \equiv c_1\mathbf{ab} \cdot \mathbf{v} + c_2\mathbf{q} \times \mathbf{v}$.

[2] We have abstained in this section from reporting on detailed calculations dealing with fluctuations, etc.; though some of the formulae given below do not maintain their simple form in the case of a more general treatment, their content and applicability remain unaffected.

In averaging over the beam it is necessary to know the values of ϕ for the different paths. If it be assumed that this distribution is symmetrical about the value $\phi=0$, which amounts to assuming that the average value of $\mathbf{B}$ is 0, one has

$$\mathsf{s}_{\parallel f}=\mathsf{s}_{\parallel 0};\quad \mathsf{s}_{\perp f}=\mathsf{s}_{\perp 0}\langle\cos\phi\rangle_{\mathrm{Av}}. \tag{3.1}$$

Case 2

The magnetic field B_i of the ith domain (corresponding to the ith time interval τ_i) is random in direction, but constant both in magnitude and direction throughout τ_i. I.e.,

$$\langle B_{xi}\rangle_{\mathrm{Av}}=\langle B_{yi}\rangle_{\mathrm{Av}}=\langle B_{zi}\rangle_{\mathrm{Av}}=0, \tag{3.2a}$$

$$\langle B_{xi}B_{yi}\rangle_{\mathrm{Av}}=0 \quad \text{for all } x,\ y,\ z \text{ and } i\neq j, \tag{3.2b}$$

$$\langle B_{xi}B_{yi}\rangle_{\mathrm{Av}}=0 \quad \text{for } x\neq y, \tag{3.2c}$$

$$\langle B_x{}^2\rangle_{\mathrm{Av}}=\langle B_y{}^2\rangle_{\mathrm{Av}}=\langle B_z{}^2\rangle_{\mathrm{Av}}=\tfrac{1}{3}\langle B^2\rangle_{\mathrm{Av}}, \tag{3.2d}$$

$$\partial\mathbf{B}_i/\partial t=0. \tag{3.2e}$$

From Eq. (3.2a)

$$A_i=\frac{\mathbf{B}_i\mathbf{B}_i}{B_i{}^2}+\left(1-\frac{\mathbf{B}_i\mathbf{B}_i}{B_i{}^2}\right)\cos gB\tau_i-(\sin gB_i\tau_i)\frac{\mathbf{B}_i\times}{B_i}$$

and from Eq. (3.2b) it can be seen that $\langle A_n\cdots A_1\rangle_{\mathrm{Av}}=\langle A_n\rangle_{\mathrm{Av}}\cdots\langle A_1\rangle_{\mathrm{Av}}$. To find $\langle A_i\rangle_{\mathrm{Av}}$ one observes from (3.2c) and (3.2d) that

$$\begin{aligned}\langle\mathbf{B}_i\mathbf{B}_i\rangle_{\mathrm{Av}}&=\langle(\mathbf{i}B_{xi}+\mathbf{j}B_{yi}+\mathbf{k}B_{zi})(\mathbf{i}B_{xi}+\mathbf{j}B_{yi}+\mathbf{k}B_{zi})\rangle_{\mathrm{Av}}\\&=\tfrac{1}{3}\langle B^2\rangle_{\mathrm{Av}}(\mathbf{ii}+\mathbf{jj}+\mathbf{kk})=\tfrac{1}{3}\langle B^2\rangle_{\mathrm{Av}}\end{aligned}$$

and from (3.2a) that $\langle\mathbf{B}_i\times\rangle_{\mathrm{Av}}=0$. Hence

$$\langle A_i\rangle_{\mathrm{Av}}=\tfrac{1}{3}+\tfrac{2}{3}\cos gB_i\tau_i=1-(4/3)\sin^2\tfrac{1}{2}gB_i\tau_i$$

or

$$\mathsf{s}_f=\pi_i[1-(4/3)\sin^2\tfrac{1}{2}gB_i\tau_i]\mathsf{s}_0. \tag{3.3}$$

The further development of (3.3) depends upon the size of the domains, as expressed by the magnitude of the τ_i. We distinguish two cases.

a. $\tfrac{1}{2}gB_i\tau_i\ll 1$: $\pi_i[1-(4/3)\sin^2\tfrac{1}{2}gB_i\tau_i]\leqq\pi_i(1-\tfrac{1}{3}g^2B_i{}^2\tau_i{}^2)$

$$\leqq\exp[-\tfrac{1}{3}g^2\textstyle\sum_i B_i{}^2\tau_i{}^2]=\exp[-\tfrac{1}{3}g^2\langle B_{\mathrm{Av}}{}^2\rangle\textstyle\sum_i\tau_i{}^2].$$

Hence

$$s_f=s_0\exp[-\tfrac{1}{3}g^2\langle B^2\rangle_{\mathrm{Av}}\langle\tau^2\rangle_{\mathrm{Av}}T/\tau_{\mathrm{Av}}].$$

Since in the applications of this formula only the order of magnitude of the exponent is of importance, one may write

$$s_f=s_0\exp[-\tfrac{1}{3}g^2\langle B^2\rangle_{\mathrm{Av}}T\tau],$$

which, in terms of the thickness of the material traversed d, the average linear dimension of the domain δ, and the velocity of the neutrons v, becomes:

$$s_f=s_0\exp[-\tfrac{1}{3}g^2\langle B^2\rangle_{\mathrm{Av}}d\delta/v^2]. \tag{3.4a}$$

b. $\tfrac{1}{2}gB_i\tau_i\geqq 1$: Using, again, a procedure which gives the order of magnitude of the exponential without regard for numerical factors of order unity, one obtains

$$\pi_i[1-(4/3)\sin^2\tfrac{1}{2}gB_i\tau_i]\leqq[1-(4/3)\sin^2\tfrac{1}{2}gB_i\tau_i]^{T/\tau_{\mathrm{Av}}}\leqq(\tfrac{1}{3})^{T/\tau}\leqq e^{-d/\delta}. \tag{3.4b}$$

The physical distinction between the two cases arises from the fact that, when $\frac{1}{2}gB_i\tau_i \ll 1$, the depolarization is produced by a succession of small rotations the magnitude of which is given by the quantity $gB_i\tau_i$, whereas, when $\frac{1}{2}gB_i\tau_i > 1$, the rotation in each domain is so large that the component of the initial spin perpendicular to $\mathbf{B}_i$, $\mathbf{s}_{\perp(i-1)}$, averages out to zero, leaving only the parallel component $\mathbf{s}_{\parallel i}$, the magnitude of which is independent of $gB_i\tau_i$. Since the average of $\mathbf{s}_{\parallel(i-1)}$ over all directions of the field is $\frac{1}{3}\mathbf{s}_{i-1}$ the depolarization factor is $\frac{1}{3}$ per domain or $3^{-n}=3^{-d/\delta}$ for the whole specimen.

Case 3

The total field $\mathbf{B}$ of each domain is the sum of two fields $\mathbf{B}_0$ and $\mathbf{B}_i$ of which the first, $\mathbf{B}_0$, is constant throughout the time interval T; the second, $\mathbf{B}_i$, is constant within the subinterval τ_i, its direction being perpendicular to that of $\mathbf{B}_0$ but otherwise random. As regards the relative magnitudes, $B_i \ll B_0$. Furthermore, it will be assumed that the initial direction of the spin $\mathbf{s}$ is parallel or anti-parallel to that of B_0. Expressed mathematically these conditions read:

$$B_{zi}=0 \quad (z\text{-axis parallel to } \mathbf{B}_0), \tag{3.5a}$$

$$\langle B_{xi}B_{yj}\rangle_{\text{Av}}=0 \quad \text{unless } x=y,\ i=j, \tag{3.5b}$$

$$\langle B_{xi}^2\rangle_{\text{Av}}=\langle B_{yi}^2\rangle_{\text{Av}}=\tfrac{1}{2}\langle B_i^2\rangle_{\text{Av}}, \tag{3.5c}$$

$$B_i \ll B_0, \tag{3.5d}$$

$$s_x=s_y=0, \tag{3.5e}$$

$$\partial B_i/\partial t=0. \tag{3.5f}$$

By arguments similar to those of Case 2 it is readily seen from (2.3) that

$$\langle A_i\rangle_{\text{Av}}=\left\langle\frac{\mathbf{B}_0\mathbf{B}_0+\mathbf{B}_i\mathbf{B}_i}{B_0^2+B_i^2}\right\rangle_{\text{Av}}+\left\langle 1-\frac{\mathbf{B}_0\mathbf{B}_0+\mathbf{B}_i\mathbf{B}_i}{B_0^2+B_i^2}\right\rangle_{\text{Av}}\cos gB_0\tau_i-\frac{\sin gB_0\tau_i}{(B_0^2+B_i^2)^{\frac{1}{2}}}\langle\mathbf{B}_0\times\rangle_{\text{Av}},$$

where the brackets denote averages over the *directions* of the $\mathbf{B}_i$ (not the magnitude) and where use has been made of (3.5d) to replace $|\mathbf{B}_0+\mathbf{B}_i|$ by B_0 in the arguments of the trigonometric functions.

It is clear from the form of the dyadics that if condition (3.5e) is satisfied for $\mathbf{s}_0$ it will be satisfied for $\mathbf{s}$ at any time; i.e., the average of $\mathbf{s}$ is always parallel to $\mathbf{B}_0$ as it must be since this direction is the only preferred direction of the whole process. This circumstance permits the dropping of the dyadics $\langle\mathbf{B}_i\mathbf{B}_i\rangle_{\text{Av}}$ and $\langle B_0\times\rangle_{\text{Av}}$. Furthermore $\mathbf{s}\cdot\mathbf{B}_0\mathbf{B}_0=B_0^2\mathbf{s}$. Hence

$$\mathbf{s}_f=\pi_i\left[\frac{B_0^2}{B_0^2+B_i^2}+\left(1-\frac{B_0^2}{B_0^2+B_i^2}\right)\cos gB_i\tau_i\right]\mathbf{s}_0=\pi_i\left[1-\left(1-\frac{B_0^2}{B_0^2+B_i^2}\right)(1-\cos gB_0\tau_i)\right]\mathbf{s}_0$$

$$\leqq\pi_i[1-2(B_i^2/B_0^2)\sin^2\tfrac{1}{2}gB_0\tau_i]\mathbf{s}_0\leqq\exp[-2\textstyle\sum_i(B_i^2/B_0^2)\sin^2\frac{1}{2}gB_0\tau_i]\mathbf{s}_0$$

with use of condition (3.5d).

As in Section 2, two cases are of interest: *a*. $\frac{1}{2}gB_0\tau_i<1$. Then,

$$\mathbf{s}_f=\mathbf{s}_0\exp[-\tfrac{1}{2}\textstyle\sum_i g^2B_i^2\tau_i^2]=\mathbf{s}_0\exp[\frac{1}{2}g^2\langle B_i^2\rangle_{\text{Av}}\langle\tau^2\rangle_{\text{Av}}T/\tau_{\text{Av}}].$$

As in 2, one replaces τ^2_{Av} by $\langle\tau\rangle^2_{\text{Av}}\equiv\tau^2$, obtaining

$$\mathbf{s}_f=\mathbf{s}_0\exp[-\tfrac{1}{2}g^2\langle B_i^2\rangle_{\text{Av}}\tau T]=\mathbf{s}_0\exp[-\tfrac{1}{2}g^2\langle B_i^2\rangle_{\text{Av}}\delta d/v^2]. \tag{3.6a}$$

b. $\frac{1}{2}gB_0\tau_i>1$.

$$\mathbf{s}_f=\mathbf{s}_0\exp[-\textstyle\sum_i(B_i^2/B_0^2)]=\mathbf{s}_0\exp[-\langle B_i^2/B_0^2\rangle_{\text{Av}}(T/\tau)]=\mathbf{s}_0\exp[-(\langle B_1^2\rangle_{\text{Av}}/B_0^2)(d/\delta)]. \tag{3.6b}$$

The physical distinction between the cases *a* and *b* can be described as follows: In case *a* $gB_0\tau_i/2<1$, each domain gives rise to a small rotation of the spin $\sim g\tau_i(\mathbf{B}_0+\mathbf{B}_i)\times\mathbf{s}$. Since by hypothesis $\mathbf{s}$ is parallel to $\mathbf{B}_0$, only the random field $\mathbf{B}_i$ contributes to this rotation with the amount $g\tau_iB_i s$. These rotations accumulate according to formula (3.6a). If, on the other hand, $gB_0\tau_i>1$ the component of $\mathbf{s}$ which is perpendicular to $\mathbf{B}_0+\mathbf{B}_i$ undergoes a large rotation, and if we average over B_i and τ_i this component reduces to zero. There remains, therefore, only the component parallel to $\mathbf{B}_0+\mathbf{B}_i$ with a magnitude $\mathbf{s}B_0/|\mathbf{B}_0+\mathbf{B}_i|$. Averaging this vector component over all directions of $\mathbf{B}_i$ leaves us with

a vector parallel to $\mathbf{B}_0$ which has the magnitude

$$s(B_0^2/B_0^2+B_i^2)\sim\exp(-B_i^2/B_0^2).$$

This gives semi-quantitatively a depolarization coefficient *per domain* equal to $\exp(-B_i^2/B_0^2)$ and, correspondingly, a depolarization coefficient $\exp(-nB_i^2/B_0^2)=\exp[-(d/\delta)(B_i^2/B_0^2)]$ for the total thickness of the ferromagnet.

Case 4

(a) The field of each domain is again made up of two components. The first, $\mathbf{B}_0$, is the same for all domains; the second, $\mathbf{B}_i$, constant throughout the time interval τ_i, is completely random in direction. The magnitude of $\mathbf{B}_i$ is now taken to be of the same order as that of $\mathbf{B}_0$.

(b) The Larmor period is large compared to the time spent in a domain: $g|\mathbf{B}_0+\mathbf{B}_i|\tau_i\ll 1$.

(c) The initial direction of the spin vector is the same as that of $\mathbf{B}_0$. The constancy of $\mathbf{B}_i$ in time τ_i and the statistical independence of the $\mathbf{B}_i$ again permit the reduction of (3) to the form

$$\langle A_i\rangle_{\text{Av}}=\left\langle\frac{(\mathbf{B}_0+\mathbf{B}_i)(\mathbf{B}_0+\mathbf{B}_i)}{(\mathbf{B}_0+\mathbf{B}_i)^2}\right\rangle_{\text{Av}}+\left\langle 1-\frac{(\mathbf{B}_0+\mathbf{B}_i)(\mathbf{B}_0+\mathbf{B}_i)}{(\mathbf{B}_0+\mathbf{B}_i)^2}\right\rangle_{\text{Av}}\cos g|\mathbf{B}_0+\mathbf{B}_i|\tau_i$$
$$-\frac{\sin g\tau_i|\mathbf{B}_0+\mathbf{B}_i|}{|\mathbf{B}_i+\mathbf{B}_i|}\langle(\mathbf{B}_0+\mathbf{B}_i)\times\rangle_{\text{Av}}.$$

The mathematics of the averaging process is complicated by the dependence of $|\mathbf{B}_0+\mathbf{B}_i|$ on $\cos(\mathbf{B}_0,\mathbf{B}_i)$ which must be taken into account in averaging over the direction of $\mathbf{B}_i$. One can get around this difficulty by employing condition (b). Expanding $\cos(g|\mathbf{B}_0+\mathbf{B}_i|\tau_i)$ and $\sin(g|\mathbf{B}_0+\mathbf{B}_i|\tau_i)$ in terms of their argument, one obtains:

$$A_i=1-\tfrac{1}{2}g^2\tau_i^2[(\mathbf{B}_0+\mathbf{B}_i)^2-(\mathbf{B}_0+\mathbf{B}_i)(\mathbf{B}_0+\mathbf{B}_i)]-g\tau_i(\mathbf{B}_0+\mathbf{B}_i)\times,$$

which, when averaged over the directions of $\mathbf{B}_i$, gives

$$\langle A_i\rangle_{\text{Av}}=1-\tfrac{1}{2}g^2\langle\tau_i^2[B_0^2+B_i^2-\mathbf{B}_0\mathbf{B}_0-\tfrac{1}{3}B_i^2]\rangle_{\text{Av}}.$$

Now, by condition (c), $\mathbf{B}_0\mathbf{B}_0\cdot\mathbf{s}=B_0^2\mathbf{s}$ hence

$$\langle A_i\rangle_{\text{Av}}=1-\tfrac{1}{3}g^2\tau_i^2\langle B_i^2\rangle_{\text{Av}}=\exp[-\tfrac{1}{3}g^2\langle\tau_i^2B_i^2\rangle_{\text{Av}}]$$

or

$$\mathbf{s}_f=\mathbf{s}_0\exp[-\tfrac{1}{3}\textstyle\sum_i\langle g^2\tau_iB_i^2\rangle_{\text{Av}}]=\mathbf{s}_0\exp[-\tfrac{1}{3}g^2\langle B_i^2\rangle_{\text{Av}}\langle\tau^2\rangle_{\text{Av}}T/\tau]$$
$$\backsimeq\mathbf{s}_0\exp[-\tfrac{1}{3}g^2\langle B_i^2\rangle_{\text{Av}}\tau T]=\mathbf{s}_0\exp[-\tfrac{1}{3}g^2\langle B_i^2\rangle_{\text{Av}}d\delta/v^2]. \quad (3.7)$$

IV. Domain Structure of Ferromagnets

According to present ideas, a ferromagnetic material consists of domains of microscopic size in each of which the atomic spins are lined up to give a *spontaneous* magnetization. In the unmagnetized state the directions of these spontaneous magnetizations are such that the macroscopic magnetization is zero. The application of an external field causes these directions to shift in such a way as to reduce the angle between the magnetization and this field. The net effect is then no longer zero; in other words the system is magnetized.

The nature of the domains is determined, as is to be expected, quite sensitively by the physical state of the material. For the purposes of this paper these states are classified according to the following scheme: (1) single crystals, (2) unmagnetized polycrystals, which will now be discussed in order.[3]

(1) Single crystals

The distinguishing feature of crystal ferromagnetism is *anisotropy* of magnetization. It is

[3] The case of magnetized polycrystals, in which the magnetic inhomogeneities are due to the slight deviations of the magnetizations of the component crystals from the direction of the external field is discussed in Section VII.

found that the increase of magnetization in an applied field is strongly dependent on the direction of this field with respect to the crystal axes. For instance, in the case of iron, a field of only a few gauss suffices to produce saturation parallel to a cubic axis, whereas saturation along a [1, 1, 1] plane requires a field of almost 400 gauss. Thus, for iron, the cubic axes are the *preferred* or "easy" axes of magnetization, whereas the [1, ±1, ±1] planes are the "difficult" axes. For a more complete discussion of the experiments the reader is referred to the standard texts,[4,5] it will suffice, here, to state the concepts which have resulted from them. The process of magnetization is, at present, regarded as proceeding in the following manner. In the unmagnetized state the domains are lined up along the preferred axes in such a way that the macroscopic magnetization is zero. When a field is applied, the domains readily shift their directions to that preferred axis which makes the smallest angle with the field, the ease with which this change is accomplished being determined by the coercive force, which is very small for single crystals. Further increase of the field causes the magnetization vector to be rotated away from the preferred axis, macroscopic saturation being attained when this vector is brought into coincidence with the direction of the field.

Up to this point, the properties of domains have been discussed solely from the standpoint of their role in macroscopic phenomena. The most important progress in recent times, however, has been achieved by the method of colloidal magnetic powders.[6] The principle upon which this method is based is identical with that behind the elementary demonstration of magnetic fields by the use of iron filings: The magnetic powder is acted on by a force which drives it to the place of strongest field strength. In practice, a suspension or colloidal solution of magnetic powder is placed on a smooth surface of the specimen. The resultant pattern, which is determined by the fields arising from the presence of surface domains, is examined with a microscope.

Unfortunately, the detailed interpretation of the results is subject to a serious limitation.[7] It has been found that the patterns formed on mechanically polished surfaces, in general, differ from those formed on naturally smooth surfaces. The question arises as to whether the observed domain structure is really characteristic of the whole crystal, or whether it is merely a surface phenomenon connected with some magneto-mechanical effect of the polishing operation.

The only experiments which seem free from this objection are those of Elmore,[8] who investigated the powder patterns of cobalt. The surfaces on which the patterns were observed were prepared by electrolytic polishing. It was found that mechanical polishing, followed by electrolytic treatment, did not change the type of pattern; hence it was reasonably certain that the surface structure is independent of the treatment and that, consequently, this structure could be identified with that existing in the interior. The observations themselves indicated the existence of plate-like or thread-like domains with magnetization alternately parallel and anti-parallel to the hexagonal axis. (Cobalt possesses a hexagonal crystal structure.) As this axis is the only preferred direction, this result coincides quite satisfactorily with those of the anisotropy experiments. It was also found that the shortest dimension of the domain always lay perpendicular to the hexagonal axis, its magnitude lying between the limits of 1 to 70μ.

(2) Unmagnetized polycrystalline media

The domain structure of polycrystalline media is obviously much more complicated than that of single crystals. The origin of these complications is twofold. First, the axes of easy magnetization vary in a more or less random fashion through the specimen with the result that at the boundary of the crystal grains discontinuities in magnetization give rise to free magnetic charges; these charges, in turn, set up a magnetic field whose reaction on the domain magnetization can be quite appreciable, even to the extent of rotating

[4] F. Bitter, *Introduction to Ferromagnetism* (McGraw-Hill Book Company, Inc., New York and London, 1937), Sections 57–62.

[5] E. C. Stoner, *Magnetism and Matter* (Methuen and Company, London, 1934), Chapter XI, Section 6.

[6] For a comprehensive review of this method, see reference 4, Sections 33, 34.

[7] W. C. Elmore, "Properties of the surface magnetization in ferromagnetic crystals," Phys. Rev. **51**, 982 (1937).

[8] W. C. Elmore, Phys. Rev. **53**, 757 (1938).

this magnetization somewhat from the preferred axes.[9] Secondly, the usual polycrystalline samples are in a more or less strained condition. Strains produce two effects: the setting up of new preferred axes[10] and the increase of the coercive force and hysteresis.

Direct information concerning the domain structure of polycrystals has been obtained almost exclusively from experiments on the Barkhausen effect. It is noticed that when a ferromagnetic material is subjected to an increasing external magnetic field, there occur sudden reversals of the magnetization of the domains; these reversals are detected by amplification of the pulse produced in a secondary coil surrounding the specimen. A systematic investigation of the size and direction of single reversals in iron, conducted by Bozorth and Dillinger[11] has revealed the following: (a) On the steep part of the magnetization curve, the direction of the reversals is mainly parallel to the applied field. The average volume of a single reversal is of the order 10^{-9} to 10^{-8} cm^3. (b) On that part of the curve above the knee ($B\sim 14{,}000$ gauss, $H\sim 5$ gauss) the direction is mainly transverse with respect to the field. The volume of the reversals is of the order of 10^{-12} cm^3. (c) With fields high enough to produce a quasi-saturated condition *no reversals* have been noticed; the approach to saturation seems to take place mainly by the process, described in the preceding section, of a rotation of the magnetization vectors of the individual crystal grains away from the preferred axis toward the direction of the external field. The Barkhausen jumps are thus linked up with the irreversible hysteresis processes occurring mainly in low fields ~ 0 to 20 gauss. (d) The above observations, including the numerical values of the volume of the reversals, hold for all types of iron, ranging from well-annealed to cold-worked specimens, as well as for nickel and Permalloy. Since the size of the crystal grains differs considerably for these various cases, the volume of a Barkhausen reversal may be larger than, equal to, or smaller than the volume of a single grain.

These results have been interpreted by some authors[12] as indicating that the size of the domains is independent of the crystalline structure. On the basis of this concept a single domain would include regions of widely different elastic and crystalline condition, a conclusion which one would accept only with extreme reluctance. If, on the other hand, the domains are of the same size, a single Barkhausen jump must involve the simultaneous reversal of the magnetization of several domains.

V. Applications of Depolarization of Neutrons to the Study of the Domain Structure of Single Crystals

The concept of the domain structure of cobalt as an alternating series of parallel and anti-parallel magnetizations, as obtained from Elmore's experiments, will be used as a starting point in the discussion of depolarization effects in single crystals. The results for this case can be written down immediately from Section III (1), provided it can be shown that the magnetic field $\mathbf{B}$ is also either parallel or anti-parallel to a fixed direction. Since

$$\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}$$

and

$$\mathbf{H}=\nabla\int \mathrm{div}\mathbf{M}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|\,dx'dy'dz',$$

the only deviation from the conditions of III (1), can arise from the existence of a "free charge" on the surface of the crystal; in the interior, the direction of magnetization is always parallel to the domain boundaries, i.e., perpendicular to the direction of variation of $\mathbf{M}$, with the result that $\mathrm{div}\mathbf{M}=0$. It can also be shown that, by suitable experimental arrangement, the surface effects can be avoided. Therefore, one has $\mathbf{B}=4\pi\mathbf{M}$ which, when substituted into (3.1) gives

$$\mathbf{s}_{\perp f}=\mathbf{s}_{\perp 0}\langle\cos 4\pi Mg\theta\rangle, \tag{5.1a}$$
$$\mathbf{s}_{\parallel f}=\mathbf{s}_{\parallel 0}, \tag{5.1b}$$

where θ is the excess time spent in the positively oriented domains. The consequences of these formulae will now be treated in the order enumerated:

[9] When there is no free charge a field $\mathbf{B}=4\pi\mathbf{M}$ is still present; the torque per unit volume exerted by this field, $4\pi\mathbf{M}\times\mathbf{M}$ is, however, zero.

[10] Reference 4, Section 54–58, Chapter VII.

[11] R. M. Bozorth, Phys. Rev. **39**, 353 (1932); R. M. Bozorth and J. F. Dillinger, *ibid.* **34**, 772 (1929); **35**, 733 (1930).

[12] R. M. Bozorth, Phys. Rev. **39**, 353, 375 (1932); E. C. Stoner, reference 5, page 413; W. F. Brown, Phys. Rev. **53**, 482 (1938).

(1) Depolarization of the perpendicular component

As stated in III (1), the magnitude of this effect depends upon the distribution of θ, which, in turn, depends upon the domain structure. Now, whether the domains have the form of layers or threads, it seems highly unlikely that the parallel regions would be exactly compensated by the antiparallel; one would, in general, expect that the excess of one over the other would be of the order $n\delta$ where δ is the average linear dimension of the domain and n a numerical factor of the order of magnitude unity. If the beam is directed perpendicular to the hexagonal axes, a tentative estimate of the uncompensated path can be attained by taking $n=1$ and δ of the order of magnitude of the sizes observed by Elmore 1 to 70μ. Taking the corresponding value of $\theta=\delta/v$ to be the width of the distribution function, $f(\theta)$, of the neutron paths over the possible values of θ, one finds that the argument of the cosine of (5.1a) varies over the interval $|\phi|<4\pi Mg\delta/v$ which, with $M=1400$ gauss (for cobalt), $g=2e/M_nc$ (M_n=mass of neutron), is of the order of magnitude $3\times10^3\delta$. It is thus apparent that any amount of depolarization can be secured depending on the value of δ. In the case of one extreme, $\delta=1\mu$, $\langle|\phi|\rangle_{Av}\sim0.3$ radian, and $\langle\cos\phi\rangle_{Av}\sim1$. For the other extreme $\delta=70\mu$, $\langle|\phi|\rangle_{Av}=20$ radians; tentatively assuming the distribution function for ϕ to be constant up to $\phi=20$ and zero for ϕ greater than this value, one obtains $\langle\cos\phi\rangle_{Av}\sim1/20$ or residual polarization 5 percent of the initial value.

In order to obtain a suitable variation of δ and, hence, of ϕ it is not necessary to employ different samples. If the beam is sent through the crystal with an inclination of χ with respect to the hexagonal axis, all the domain lengths are increased by a factor

$$\csc\chi, \quad \text{i.e.} \quad \phi_x=\phi_0\csc\chi,$$

where ϕ_0 represents the value of ϕ for $\chi=90°$. The depolarization effect for any angle χ is, then,

$$\langle\cos\phi_x\rangle_{Av}=\int\cos(\phi_0\csc\chi)g(\phi_0)d\phi_0,$$

where $g(\phi)$ is the distribution function for ϕ corresponding to $f(\theta)$. By studying the depolarization as a function of χ one can obtain the Fourier components of $g(\phi)$ over a large range and hence obtain some idea of the shape of this function, itself. The width of the distribution, in particular, is determined by the angle χ at which the residual polarization drops off appreciably from unity, i.e., when $\langle\phi_0\rangle_{Av}\csc\chi\sim1$, from which, in turn, one can obtain an estimate of the domain size to be compared with the results of Elmore.

(2) Constancy of the parallel component

The preceding discussion has shown that appreciable depolarization effects are to be expected whenever the magnetizations of the domains are not parallel to the spin of the incident neutrons. Depolarization experiments should therefore provide information concerning the directional properties of single crystal domains. Examples of such experiments are as follows:

(*a*) *Unmagnetized cobalt crystals.*—In this case the magnetizations are parallel to the hexagonal axis—hence no effect on the polarization (cf. (5.1b)). However, heat treatment or the application of stress may cause deviations which could then be detected by depolarization.

(*b*) *Magnetized cobalt crystals.*—One can test for nonparallel domains at all stages of the magnetization curve. One would have to take into account the depolarizing effects arising from the external field; this should not be difficult in view of the extremely small value of this field.

(*c*) *Unmagnetized iron crystals.*—In an ordinary single crystal of iron the preferred directions of magnetization are the three cubic axes. A neutron beam should suffer depolarization on passing through such a medium regardless of the direction of its original polarization. However, it may be possible, by the application of stress along one of the preferred directions, to align the magnetization of all domains parallel and antiparallel to this direction. In that case, one could, as before, arrange an experiment in which no depolarization took place.

(*d*) *Demagnetization of iron.*—Although a crystal without previous magnetic treatment may have domains whose magnetizations are oriented along all of the three preferred axes, the demagnetization of a sample, magnetically saturated parallel to any one of them, might occur unidirectionally, i.e. the total magnetic moment would be

cut down by the formation of domains whose magnetizations point in the opposite direction. This hypothesis could be tested by a depolarization experiment. If such an arrangement of the domains actually exists, one could investigate its stability towards heat treatment and the application of small stresses.

VI. Applications of Depolarization of Neutrons to the Study of Domain Structure of Polycrystalline Media

Formulae describing the depolarization of neutrons in polycrystals are contained in III (2); one has only to investigate how far the assumptions underlying that treatment are fulfilled in this case.

(1) Randomness of domains

The first theory, discussed at the end of IV (2), states that the domain size is independent of the crystalline structure; the direction of magnetization of these domains must then be assumed to be random.[13] As far as the second theory is concerned, the domains in different crystals certainly have no correlation with each other, because of the arbitrariness of orientation of crystal axes in a polycrystal. For domains contained in the same crystal, the discussion of V has shown that there exist domain arrangements of magnetizations parallel and anti-parallel to a fixed axis in the case of cobalt, which has only one easy axis, but not in the case of iron, which has three. The general effect of such arrangements is to cut down the depolarization. The correction to the formulae arising from these considerations will be presented at the end of this section; for the present, it will be assumed that the magnetizations are randomly oriented.

(2) Constancy of the field B_i over the ith domain

This assumption is open to two objections:

(a) The magnetization of the domains in polycrystalline media may not be constant throughout the domain, but may vary gradually so as to give a continuous transition between different domains. In single crystals the domains are lined up parallel to the easy axes; the transition region between any two domains, in which the magnetization rotates from one preferred direction to another, has been estimated theoretically[14] to be $\sim 10^{-6}$ cm, which is compatible with the observations of Elmore. The application of these ideas to polycrystalline media is, however, not advisable, due to the presence of elastic inhomogeneities and internal magnetic fields, both of which may appreciably alter the direction of magnetization so as to result in a broader transition region.

(b) Even if the magnetization were constant over a domain, the same could not be said for the induction since this quantity is not proportional to the magnetization; one must add to the $4\pi\mathbf{M}$ term the field arising from the existence of magnetic poles at the boundaries of domains. In general, this field is not parallel to the magnetization; in fact, the well-known boundary conditions for the fields—normal component of $\mathbf{B}$ tangential component of $\mathbf{H}$ continuous—preclude this relationship except under most artificial conditions.

These objections indicate that a treatment more suitable than III(2), would be one which provided for a continuous variation of the field along a neutron path. However, the authors are of the opinion that this refinement is unnecessary, for the following reasons:

(a) The main characteristic of the motions considered in III is that the time in which the magnetic field changes appreciably is not larger than the Larmor period of the neutron spin, i.e., from the quantum-mechanical point of view, the motion is non-adiabatic. Now, in the present case, even though the transitions between the different magnetic fields may be continuous, i.e., they may require times of the order τ_i, the variation is sufficiently rapid for the motion to be classed as non-adiabatic. This feature is immediately apparent when one recalls that the spin of a thermal neutron precesses, on the average, about 1 to 2 radians in a region of 10^{-3} cm. Since the linear dimensions of the largest domains do not exceed this quantity, the transition from one field to the other will *a fortiori* take place in a small fraction of a complete precession. Thus, the non-adiabatic character of the motion is preserved.

(b) In view of these considerations, the treatment of III (2), should give results of the correct order of magnitude, the only effect of the variation of $\mathbf{B}_i$ in the time interval being a change in the numerical factors of the exponents of the depolarization formulae (8.1a) (8.1b) of the order of magnitude. However, one would not want any greater accuracy since the size of the domains may be expected to vary over a correspondingly wide range.[15]

The assumptions underlying III (2), can then be taken as essentially valid; the existence of magnetic charge will be approximated by

[13] One might think that the domains would tend to line themselves up in such a way as to avoid as much as possible the creation of magnetization charge; this can, however, be produced merely by having the normal component vary continuously across a boundary, the tangential component being free to change by any value. Such an arrangement would not be random in the sense of III (2), but its effect on the depolarization would be the same to within a numerical factor of the order unity, of III (3), case (1).

[14] L. Landau and E. Lifshitz, Physik. Zeits. d. Sowjetunion **8**, 153 (1935).

[15] A more accurate treatment would be necessary if one were to investigate, say, the distribution of domain sizes; such problems are not contemplated in the present paper.

the insertion of a demagnetizing factor, i.e., $B_i \cong (4\pi-\gamma)M$ where $\gamma \sim (4/3)\pi$. Inserting this relation into (3.4a) and (3.4b), one obtains:

(*a*) For $(4\pi-\gamma)Mg\tau/2 \ll 1$,

$$s_f/s_0 = \exp[-\tfrac{1}{3}(4\pi-\gamma)^2(gM/v)^2 d\delta]. \quad (6.1a)$$

(*b*) For $(4\pi-\gamma)Mg\tau/2 \gtrsim 1$,

$$s_f/s_0 = e^{-d/\delta}. \quad (6.1b)$$

These formulae will now be applied to distinguish between the two theories discussed in IV (2).

(*1*) *Domain size* $\sim 10^{-9}-10^{-8}$ *cm*^{3} *independent of crystal structure.*—$(4\pi-\gamma)Mg\tau/2$ has an upper limit of 2. The depolarization is then given by (6.1b) for all materials investigated by Bozorth and Dillinger,[10] in particular for all types of iron. With $\delta \sim 10^{-3}$ cm a sample of thickness 5×10^{-3} cm suffices to depolarize the beam completely.

(*2*) *Domain size of the order or smaller than the size of the crystal grains.*—In this case the depolarization should depend quite strongly on the previous treatment of the specimen. If the crystal grains have linear dimensions $<3\times10^{-4}$ cm, $(4\pi-\gamma)Mg\tau/2$ has an upper limit of 0.25 and the depolarization is then given by (6.1a). It can then be seen that, even if the domains are assumed to be identical with the crystal, a sample the linear size of whose grains is $\sim 10^{-4}$ cm would give a depolarization effect of an altogether smaller order of magnitude from that of (1). In fact:

$$\tfrac{1}{3}[g(4\pi-\gamma)M/v]^2 d\delta \lesssim \tfrac{1}{3}(1.7\times10^3)^2\times10^{-4}d = 10^2 d,$$

and, hence, the polarization of a beam passing through a sample 5×10^{-3} cm should be cut down by a factor not smaller than $e^{-\frac{1}{2}} \sim 60$ percent.

The difference in depolarization predicted by the two theories should be especially noticeable for thin films whose thickness is of the order of 6μ. The first theory then predicts that the linear dimensions of the domains should be of the same order of magnitude as the thickness of the film. One can then obtain results of the correct order of magnitude by employing (6.1a) for both theories. The first, predicting domain sizes of the order of magnitude of the film thicknesses, gives a depolarization effect $s_f/s_0 = e^{-\frac{1}{2}} = 70$ percent. According to the second theory, for which the domains are not larger than the crystals (in this case $<10^{-5}$ cm), no effect should be observed.

Before (6.1a) and (6.1b) can be employed to establish the application of depolarization experiments to more general problems of polycrystalline domain structure, the possibility of ordered arrangements of the domains within one crystal grain must be investigated in some detail. This will now be done for the cases of iron, nickel, and cobalt.

(1) Iron and nickel: In crystal grains composed of these substances, the magnetization may be oriented along any one of several directions ((1, 0, 0) directions for iron, (1, 1, 1) for nickel). The only arrangement which can cut down the depolarization effect is one of alternately parallel and antiparallel domains (see next paragraph). From the energetic standpoint, however, there is no reason why this arrangement is to be preferred. Hence magnetizations along all "easy axes" may be present. The passage of a polarized neutron beam in a region where the directions of the domain magnetizations change by 90° (iron) or by 63° (nickel) gives rise to strong depolarization effects which are, if anything, somewhat underestimated by the assumption of randomness of direction, and thus tend to compensate the reduction due to the presence of parallel-antiparallel arrangements. As far as order of magnitude estimates are concerned, therefore, the existence of ordered domain structures within crystal grains should not affect the validity of (6.1a), (6.1b) for iron and nickel.

(2) Cobalt: Here, the existence of only one preferred axis should cause appreciable cancellation of the rotation of the neutron spin in separate domains. The problem has been treated for two types of arrangements: plate-like and cylindrical domains, (case of $g(4\pi-\gamma)M\delta/(2v) \ll 1$). The results are:

(a) For domains in the form of plates

$$s_f/s_0 = \exp(-\tfrac{3}{4}(4\pi Mg)^2 d\delta/v^2), \quad (6.2a)$$

where δ now represents the average thickness of the plates.

(b) For domains in the form of cylinders:

$$s_f/s_0 = \exp(-\tfrac{1}{2}(4\pi Mg/v)^2\delta d[(\delta/\delta_0)\log 2\delta_0/\delta]), \quad (6.2b)$$

where δ is the average thickness of the cylinders and δ_0 the average linear dimension of the crystal grains.

The resemblance of (6.2a) to (6.1a) is quite satisfactory, the only difference apart from numerical factors being that the δ of (6.1b) is the domain length averaged over its shape, whereas for (6.2a) it is its shortest length. The form of (6.2b) does differ somewhat from these two except in the case $\delta \sim \delta_0$; it may be mentioned in passing, however, that the theoretical treatment of single crystal domains by Landau and Lifshitz[12] indicates that the cylindrical arrangements require more energy than the plate-like and hence should not occur.

Equations (6.1a, b) (6.2a, b) establish the connection between depolarization experiments and domain size. Such experiments, then, should yield important information on the general problem of domain structure in polycrystals, e.g., its dependence on grain size, internal strain, chemical composition (presence of impurities) and previous magnetic history.

VII. Depolarization of Neutrons in Polycrystalline Media Near Saturation

The magnetic domains whose nature has been discussed in the three preceding sections are characteristic of ferromagnets in small external fields $\lesssim 10$ gauss. When this field is increased to a value $\gtrsim 200$ gauss, these domains are wiped out. As was stated at the beginning of VI (1), the anisotropy experiments show that, under such conditions, the main, if not the sole, process governing the variation of magnetization in single crystals is the rotation of the magnetic vector (towards the direction of the external field). The angle between the two is determined by equating the torque exerted by the crystalline field to that arising from the action of the magnetic field; this equation has the form: $\mathbf{T}(\theta, \phi)=\mathbf{M}\times\mathbf{H}$, where $\mathbf{T}(\theta, \phi)$ is a vector function of the polar coordinates, θ and ϕ of the magnetic vector with respect to the crystal axes.[16]

The application of these ideas to the behavior of polycrystalline samples in high fields leads to the following picture. Each crystal grain possesses a magnetization whose magnitude is equal to the saturation value; the direction of this magnetization deviates somewhat from that of the external field. If the specimen has not been subjected to special mechanical treatment, e.g., plastic deformation, the axes of the microcrystals will be oriented at random; it then follows that the deviations of the magnetizations of the separate crystals are also random. The macroscopic magnetization is given by the average of the component of the magnetization parallel to the external field.

This model is readily seen to correspond to that of III (3), as in VI, the assumption that the random field $\mathbf{B}_i$ is constant in time τ_i is not true in the actual physical case; however, again, the variation of the size of crystals in a single specimen and the consequent uncertainty in the depolarization formulae renders any attempts to obtain better accuracy quite purposeless. The field will hence be considered constant, its relationship to the magnetization will be described, as in VI, by a demagnetizing factor. Thus

$$\mathbf{B}_i=(4\pi-\gamma)\delta\mathbf{M}_i\cong(8/3)\pi\delta\mathbf{M}_i,$$

where $\delta\mathbf{M}_i$ is the vector deviation of the magnetic vector of the ith crystal from the macroscopic magnetization. The dependence of the depolarization formulae (3.6a) and (3.6b) on $\mathbf{B}_i$ is given in terms of its average square,

$$\langle B_i^2\rangle_{\text{Av}}\cong(8\pi/3)^2\langle(\delta\mathbf{M}_i)^2\rangle_{\text{Av}}.$$

Now $(\delta\mathbf{M}_i)^2=M_s^2-M_{\parallel i}^2$ where M_s is the saturation value of the magnetization and $M_{\parallel i}$ the component parallel to the macroscopic magnetization, $\mathbf{M}_0$. Then

$$\langle(\delta\mathbf{M}_i)^2\rangle_{\text{Av}}=M_s^2-M_0^2\cong 2M_0\Delta M,$$

where $\Delta M\equiv M_s-M_0$ is the *deviation* of the macroscopic magnetization from its saturation value. $\mathbf{B}_0$ of course, is given by the familiar relation $\mathbf{B}_0=4\pi\mathbf{M}_0+\mathbf{H}$. Thus, from (3.6a, b)

$$s_f/s_0=\exp[-(8\pi/3)^2M_0\Delta M\delta d/v^2],$$
$$\text{if } 2\pi gM_0\delta/v<1; \quad (7.1a)$$
$$s_f/s_0=\exp[-(4/3)(\Delta M/M_0)d/\delta],$$
$$\text{if } 2\pi gM_0\delta/v>1, \quad (7.1b)$$

where δ is the average linear dimension of the crystal grains. The value of δ at which (7.1a) goes into (7.1b) is given by

$$\delta=v/(2\pi gM_0)\cong 1.5\times10^5/2\times10^{-8}$$
$$\sim 7\times10^{-4}\text{ cm}. \quad (7.2)$$

The applicability of these formulae rests on the assumption that the magnetic inhomogeneities arise solely from the angular deviation of the crystal magnetizations from the macroscopic magnetization. Now it is conceivable, although highly unlikely, that even at high fields there persist regions whose magnetizations are oriented at random, i.e., Barkhausen domains which have not yet flipped over. The contribution of such regions to the observed difference between the macroscopic and saturation magnetization has been shown to be negligible. The presence of these domains, therefore, cannot change the depolarization unless they exert an effect disproportionately large compared to their influence on the magnetization.

[16] When strains are present, the energy is dependent upon the direction of the magnetization with respect to axes defined by the strain. The net effect is merely to set up new crystal axes with new anisotropy constants.

The disproof of the existence of such an effect follows readily from Eq. (3.7) of III (4). This equation represents the depolarization due to a medium consisting of randomly oriented domains in the presence of an external field. In the present case these random domains occupy only a fraction of the total volume of the specimen; this fraction is given by $\eta=\Delta M_1/M_s$ where ΔM_1 is the deviation from saturation due to the presence of these domains. Thus the quantity d in the exponent of (3.7) must be replaced by

$$\eta d=(\Delta M_1/M_s)d\leqq(\Delta M_1/M_0)d_0.$$

Furthermore,

$$\mathbf{B}_i\leqq(4\pi-\gamma)\mathbf{M}_i\sim(8/3)\pi\mathbf{M}_i;\quad|\mathbf{M}_i|\sim M_s\leqq M_0.$$

Therefore:

$$s_f/s_0\leqq\exp[-\tfrac{1}{3}g^2(8\pi/3)^2M_0\Delta Md\delta/v^2].$$

The exponent of this equation, apart from numerical constants of the order unity, has the same functional form as that of (7.1a). Hence, for a given ΔM the two cases can differ only in the values of δ the domain, or crystal, size. Now, in most of the practical cases, the crystal sizes are not far from the borderline between (7.1a) and (7.1b) as given by (7.2). Therefore, whatever the size of the domains, their effect per unit ΔM is, at most, of the same order as that of the crystal grains. The depolarization due to unoriented domains is, hence, important only if the corresponding ΔM is an appreciable fraction of the total ΔM, a hypothesis which cannot be admitted.

Finally, it should be mentioned that, in materials possessing an average magnetic moment different from zero, there will be a *polarization* effect which will either add to, or subtract from, the depolarization. The polarization will be discussed fully in the next section; at this point it is sufficient to point out that, both from theoretical estimates, and from the experiments, no appreciable modification of (7.1a), (7.1b) need be feared.

The order of magnitude of the depolarization effect in high fields will now be estimated. If the crystals have linear dimensions not less than 10^{-3} cm, $gB_0\tau/2>1$ and (7.1b) is to be employed. For the usual samples of soft iron, δ ranges from 10^{-3} to 10^{-2} cm. In order, then, for a reduction of the polarization to $1/e$ of the initial value in 1 cm of material to take place, the relative deviation from saturation, $\Delta M/M_0$, need only be 0.1 percent to 1 percent.

In conclusion, (7.1a) and (7.1b) can be used to investigate the properties of ferromagnets near saturation. One determines the depolarization as a function of the macroscopic magnetization. The ratio of $\log(s_f/s_0)$ to ΔM should be a constant $\leqq(4/3)(d/\delta)(1/M_0)$ or $\frac{1}{3}g^2(8\pi/3)^2M_0d\delta$ from which one can obtain an estimate of δ to be compared to the observed grain size. If these predictions are not fulfilled, it will be necessary to revise the concepts which have been advanced to explain the magnetic behavior of ferromagnets in saturation fields.

VIII. Transmission of Neutrons in Saturated Polycrystalline Media

The transmission of neutrons has been treated heretofore as a purely atomic problem. One first calculated the atomic scattering cross section for beams in the different spin states. To find the effect of a differential layer of material on the transmitted beam, one multiplied this quantity by the number of atoms in the layer and subtracted the resultant total scattering intensity from the incident intensity. The results of these calculations expressed in differential form are:

$$d/dx(I/v)=-\lambda I/v+w\sigma, \tag{8.1a}$$

$$d\sigma/dx=-\lambda\sigma+wI/v, \tag{8.1b}$$

where I is the current density, σ the spin density, v the velocity of the neutrons, λ the sum of the absorptions due to capture and total scattering, both magnetic and nuclear, and w a constant depending on the details of the atomic scattering process and the direction of the beam with respect to the magnetic field.[17]

Now, it has been shown in the previous section that depolarization effects can be avoided only in the case of extreme saturation; if this situation does not obtain, the polarization resulting from the scattering process will be cut down. From the standpoint of the treatments heretofore given, this feature can be taken into account by the insertion into the second equation of a term proportional to the polarization, the coefficient

[17] J. S. Schwinger, Phys. Rev. **51**, 544 (1937).

of proportionality being determined by the rate of depolarization as given by (7.1). It is the opinion of the present authors, however, that the transmission formulae are inadequate in other respects, and that it is necessary to reinvestigate the whole problem in detail.

When one is dealing with a case of coherent scattering the fundamental unit is not the atom but the microcrystal. The procedure of multiplying the atomic cross section by the density of atoms to obtain the absorption of the transmitted beam is correct for gases, liquids, and amorphous solids; for crystalline substances the method is applicable only in the case of wave-length small compared to the separation of the atoms. Now, in order to obtain appreciable magnetic scattering, it is necessary that the wave-length of the neutron be *large* compared to atomic radii; otherwise, the interference of the scattered waves arising from different parts of the atom cuts down the amplitude of the magnetic scattering. Since the separation of the atoms is of the same order as their linear dimensions, it is clear that crystal structure plays an important role in the polarization process. *The method of adding over atomic cross sections must therefore be renounced.*[18] The remainder of this section is devoted to a derivation of the transmission formulae from the standpoint of scattering in crystalline media. An important result of the calculations will be the confirmation of (21); the numerical values of λ and w, however, will obviously be different from those given in previous treatments.

Before any calculations are performed, the meaning of the term *microcrystal* must be clarified. From x-ray studies it is known that crystals which are macroscopically perfect are usually composites of much smaller units, whose linear dimensions are of the order 10^{-5} cm. These sub-units have significance only with respect to wave-mechanical properties; in all other respects the crystal acts as a single entity. Indeed, such a system can conceivably be formed of units whose crystal axes are all oriented in the same direction but whose relative positions are variable over a distance of the order of the lattice spacing. The medium is then homogeneous from the standpoint of elastic and magnetic properties, but waves (x-rays, neutrons, electrons, etc.) emanating from the different sub-units do not superpose coherently. In order to obtain the intensity of scattering due to the whole crystal, one must square the amplitude of each of these waves, and sum the resultant intensities. In the calculations which follow, the term *single crystal* denotes a unit of polycrystalline material which is elastically homogeneous, whereas its sub-units, each of which scatter as a coherent whole, will be designated as *microcrystals*.

The Schrödinger equation for the system, microcrystal plus neutron, is

$$-(\hbar^2/2M_n)\nabla^2\psi+V\psi=E\psi,$$

where M_n denotes the mass of the neutron and V the interaction between the neutron and macrocrystal; this interaction is the sum of the potentials of the constituent atoms. Now, the unit cell of an iron crystal, which has a body-centered structure, can be regarded as a superposition of two simple cubic cells. The potential of one of these is

$$V_I=\sum_{\mathbf{g}} V_{\mathbf{g}I}\exp(i\mathbf{g}\cdot\mathbf{r});\quad \mathbf{g}=\mathbf{g}_1n_1+\mathbf{g}_2n_2+\mathbf{g}_3n_3;\quad n_i \text{ integral},$$

where the $\mathbf{g}$'s are reciprocal lattice wave vectors. The potential of the two cells is then

$$V=\sum_{\mathbf{g}}\left(V_{\mathbf{g}I}\exp(i\mathbf{g}\cdot\mathbf{r})+V_{\mathbf{g}I}\exp(i\mathbf{g}\cdot\{\mathbf{r}+\tfrac{1}{2}\mathbf{a}_1+\tfrac{1}{2}\mathbf{a}_2+\tfrac{1}{2}\mathbf{a}_3\})\right),$$

$\frac{1}{2}\mathbf{a}_1+\frac{1}{2}\mathbf{a}_2+\frac{1}{2}\mathbf{a}_3$ being the coordinate of the origin of one of the cubic systems with respect to the origin of the other, expressed in terms of the lattice vectors $\mathbf{a}_i$. It is convenient to define a new quantity $V_{\mathbf{g}}$ such that $V=\sum_{\mathbf{g}} V_{\mathbf{g}}\exp(i\mathbf{g}\cdot\mathbf{r})$; $V_{\mathbf{g}}=V_{\mathbf{g}I}(1+\exp[\frac{1}{2}i\mathbf{g}\cdot(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)])$. Now $\mathbf{g}_1\cdot\mathbf{a}_1=2\pi n_1$. Therefore $V_{\mathbf{g}}=2V_{\mathbf{g}I}$ if $n_1+n_2+n_3$ is even, zero if this sum is odd. $V_{\mathbf{g}I}$ of course, is calculated from $V(\mathbf{r})$ by means of the inverse Fourier integral

[18] Halpern, Hamermesh and Johnson, Phys. Rev. **55**, 1125A (1939).

$$V_{gl}=(1/a^3)\int_{\tau_0} V(\mathbf{r})\exp(-i\mathbf{g}\cdot\mathbf{r})d\tau_0$$

where τ_0 is the volume of the unit cell.

$V(\mathbf{r})$, itself, is the sum of the nuclear and magnetic potentials arising from the atoms of a constituent simple cubic cell:

$$V=V_{\text{nuc}}+V_{\text{mag}}.$$

For V_{nuc} one takes a fictitious form which gives the same scattering for an individual nucleus and which is suitable for the application of the Born approximation. V_{mag} represents the interaction of the neutron magnetic moment with the field set up by the electronic currents, i.e., $V_{\text{mag}}=\mu_n\boldsymbol{\sigma}\cdot\mathbf{B}$, where μ_n is the magnetic moment of the neutron, $\boldsymbol{\sigma}$ the Pauli spin operator[19] and $\mathbf{B}$ the magnetic field of the electrons.

One can now proceed to calculate the scattering of a microcrystal by the well-known Born method. Let

$$\psi=\exp(i\mathbf{K}_0\cdot\mathbf{r})\chi+\psi_1,$$

where χ is a two-component column matrix on which the neutron spin operator acts.[20] Then

$$(\nabla^2+K^2)\psi_1=(2M_n/\hbar^2)V\psi_0;\quad K^2=2M_nE/\hbar^2,$$

the solution of which is:

$$\psi_1(\mathbf{R})=(M_n/2\pi\hbar^2)\int_\tau[\exp(iK|\mathbf{R}-\mathbf{r}|)/|\mathbf{R}-\mathbf{r}|]V(\mathbf{r})\psi_0(\mathbf{r})d\tau,$$

which, for R large compared to the linear dimensions of the microcrystal, becomes:

$$\psi_1(\mathbf{R})=(M_n/2\pi\hbar^2)\sum_{\mathbf{g}} V_{\mathbf{g}}\chi(e^{iKR}/R)\int\exp[i(\mathbf{K}_0+\mathbf{g}-\mathbf{K})\cdot\mathbf{r}]d\tau,$$

where $\mathbf{K}_0$ and $\mathbf{K}$ are the propagation vectors of the incident and scattered waves, respectively. The integration gives:

$$\psi_1(\mathbf{R})=\frac{M_n}{2\pi\hbar^2}\sum_{\mathbf{g}} V_{\mathbf{g}}\chi\frac{e^{iKR}}{R}\left(\frac{1-\exp[i(K_{0x}+g_x-K_x)l]}{i(K_{0x}+g_x-K_x)}\right)\left(\frac{1-\exp[i(K_{0y}+g_y-K_y)l]}{i(K_{0y}+g_y-K_y)}\right)\times\left(\frac{1-\exp[i(K_{0z}+g_z-K_z)l]}{i(K_{0z}+g_z-K_z)}\right),$$

where l^3 is the volume of a microcrystal, assumed, for the sake of simplicity, to be a cube.

The corresponding scattered current per unit area of a sphere of radius R is:[21]

$$s=v\psi_1^*\psi_1=\frac{2M_n^2}{\pi^2\hbar^4}\sum_{\mathbf{g}}\frac{\chi^*|V_{\mathbf{g}}|\chi}{R^2}\left(\frac{1-\cos l(K_{0x}+g_x-K_x)}{(K_{0x}+g_x-K_x)^2}\right)\left(\frac{1-\cos l(K_{0y}+g_y-K_y)}{(K_{0x}+g_x-K_x)^2}\right)\left(\frac{1-\cos l(K_{0z}+g_z-K_z)}{(K_{0z}+g_z-K_z)^2}\right).$$

[19] It is assumed that the spin of the neutron is $\frac{1}{2}$.

[20] In the calculation of the scattered intensity the zeroth Fourier coefficient of the potential is neglected. Ordinarily this feature is trivial since the resultant change in zeroth-order energy is negligible compared to the kinetic energy of the neutron. In this case, however, a certain complication arises due to the fact that the coefficient has opposite signs for the two spin states, i.e., the "indices of refraction" of waves in these states are different. A numerical calculation indicates that the phase difference between the waves becomes appreciable when a distance of the order 10^{-3} cm has been traversed. If the single crystal were composed of one microcrystal, this difference would actually have to be taken into account; since it is assumed that the scattering units have linear dimensions of the order 10^{-5} cm, the zero Fourier coefficient can forthwith be neglected.

[21] The different Laue spots are assumed not to overlap, in which case the cross terms ($\mathbf{g}\neq\mathbf{g}'$) drop out of the expression for the scattered current.

For variations of K_{0x} and K_x of the order of g_x the quantity $(1-\cos Kl)/K^2$ resembles a δ function in that it has appreciable values only for $K=0$. Since

$$\int_{-\infty}^{+\infty} dK(1-\cos Kl)/K^2=\pi l,$$

the scattered current becomes

$$s=(2\pi M_n{}^2l^3/\hbar^4)\sum_{\mathbf{g}}[(\chi^*|V_{\mathbf{g}}|^2\chi)/R^2]\delta(\mathbf{K}_0+\mathbf{g}-\mathbf{K}),$$

where the δ function is three dimensional.

To find the total scattered current, one integrates over a sphere of radius R:

$$S=v\int\psi_1{}^*\psi_1 d\sum=(2\pi M_n{}^2l^3v/\hbar^4)\sum_{\mathbf{g}}(\chi^*|V_{\mathbf{g}}|^2\chi)\int\int[\delta(\mathbf{K}_0+\mathbf{g}-\mathbf{K})/R^2](R/z)dxdy,$$

where $(R/z)dxdy$ gives the expression for a surface element in Cartesian coordinates. Taking the z axis along the direction of $\mathbf{K}_0+\mathbf{g}$ one obtains

$$S=(2\pi M_n{}^2l^3v/\hbar^4)\sum_{\mathbf{g}}(\chi^*|V_{\mathbf{g}}|\chi)\int\int(1/R^2)\delta(Kx/R)\delta(Ky/R)\delta(Kz/R-|\mathbf{K}_0+\mathbf{g}|)(R/z)dxdy,$$

where $K_x=Kx/R$, etc.; since $\delta(ax)=\delta(x)/a$ the result of the integration is:

$$S=(2\pi M_n{}^2l^3v/\hbar^4K^2)\sum_{\mathbf{g}}(\chi^*|V_{\mathbf{g}}|^2\chi)\delta(|\mathbf{K}_0+\mathbf{g}|-K).$$

One now takes cognizance of the mosaic structure of the crystal by adding the integrated intensities originating from the component microcrystals. Since the crystal axes of these smaller units are assumed to cluster around the value indicated by the macroscopic properties of the system, all of the contributions have the same form. Thus

$$s=\sum S=SL^3/l^3=(2\pi M_n{}^2L^3v/\hbar^4K^2)\sum_{\mathbf{g}}(\chi^*|V_{\mathbf{g}}|^2\chi)\delta(|\mathbf{K}_0+\mathbf{g}|-K),$$

L^3 being the volume of a single crystal. Apparently the same formula is obtained if the single crystal is treated as perfect instead of mosaic; the difference, however, is contained in the δ function, whose value and width are still proportional to l and $1/l$, respectively.

One has now to average over all orientations of the single crystal, i.e., over the direction of the vector $\mathbf{g}$. For this purpose the explicit dependence of V_g on $\mathbf{g}$ must be determined. The potential $V(\mathbf{r})$ of one of the two component simple cubic structures of the system is given by the relation

$$\sum_i \alpha a^3\delta(\mathbf{r}-\mathbf{r}_i)-\mu_n\boldsymbol{\sigma}\cdot\mathbf{B}, \tag{8.2}$$

where a is the lattice spacing, α a constant given in terms of the nuclear cross section, and where the summation runs over all atoms contained in a simple cubic lattice. Here, as pointed out at the beginning of this treatment, use has been made of the trick of replacing the actual nuclear potential by a fictitious potential suitable for the application of the Born perturbation method. $\mathbf{B}$ is the magnetic field due to the atomic magnets; it is determined by the equations

$$\mathrm{div}\mathbf{B}=0;\quad \mathrm{curl}\mathbf{B}=4\pi\,\mathrm{curl}\mathbf{M}.$$

Expressed in terms of Fourier components, these equations read[22]

$$\mathbf{g}\cdot\mathbf{B}_{\mathbf{g}}=0;\quad \mathbf{g}\times\mathbf{B}_{\mathbf{g}}=4\pi\mathbf{g}\times\mathbf{M}_{\mathbf{g}}.$$

Taking the cross product of the second equation with $\mathbf{g}/\mathbf{g}^2$ one obtains:

$$\mathbf{B}_{\mathbf{g}}=-(4\pi/g^2)\mathbf{g}\times(\mathbf{g}\times\mathbf{M}_{\mathbf{g}}).$$

[22] The Fourier coefficients of V and $\mathbf{B}$ due to a simple cubic structure are written as $V_{\mathbf{g}}$ and $\mathbf{B}_{\mathbf{g}}$ instead of $V_{\mathbf{g}I}$ and $\mathbf{B}_{\mathbf{g}I}$ as at the beginning of the section.

A superposition of the Fourier components of two cubic systems as described at the beginning of this section gives:

$$V_{g\ \text{total}}=2(\alpha-\mu_n\sigma\cdot B_g);\quad \text{or}\quad V_{g\ \text{total}}=0$$

depending on whether the sum of the components of $\mathbf{g}$ in the reciprocal lattice system, $n_1+n_2+n_3$, is even or odd. One then has:

$$s=(8\pi M_n{}^2L^3v/h^4K^2)\sum_g\delta(|\mathbf{K}_0+\mathbf{g}|-K)\chi^*[|\alpha|^2+\mu_n{}^2\boldsymbol{\sigma}\cdot\mathbf{B}_g{}^*\boldsymbol{\sigma}\cdot\mathbf{B}_g-\mu_n\boldsymbol{\sigma}\cdot\mathbf{B}_g{}^*\alpha-\mu_n\alpha^*\boldsymbol{\sigma}\cdot\mathbf{B}_g]\chi.$$

If each atom is taken to be the center of a unit cubic cell, the quantity

$$\mathbf{M}_g=(1/a^3)\int_{-a/2}^{+a/2}\int_{-a/2}^{+a/2}\int_{-a/2}^{+a/2}\mathbf{M}(\mathbf{r})\exp(-i\mathbf{K}\cdot\mathbf{r})dx_0dy_0dz_0\ [\text{with } dx_0dy_0dz_0=d\tau_0]$$

is real by asymmetry; hence $\mathbf{B}_g$ is real. Thus

$$s=(8\pi M_n{}^2L^3v/h^4K^2)\sum_g\delta(|\mathbf{K}_0+\mathbf{g}|-K)\chi^*[|\alpha|^2+\mu_n{}^2\boldsymbol{\sigma}\cdot\mathbf{B}_g\boldsymbol{\sigma}\cdot\mathbf{B}_g-2\mu_n\boldsymbol{\sigma}\cdot\mathbf{B}_g\alpha_R]\chi,$$

where α_R is the real part of α. The second term can be simplified further by use of the property of the Pauli matrices:

$$(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{c})=\mathbf{A}\cdot\mathbf{c}+i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{c},$$

where $\mathbf{A}$ and $\mathbf{c}$ are any two vectors.[23] Substituting, also, for $\mathbf{B}_g$ in terms of $\mathbf{M}_g$ one has:

$$s=(8\pi M_n{}^2L^3v/h^4K^2)\sum_g\delta(|\mathbf{K}_0+\mathbf{g}|-K)\chi^*[|\alpha|^2+(8\pi/g^2)\mu_n\boldsymbol{\sigma}\cdot\{\mathbf{g}\times(\mathbf{g}\times\mathbf{M}_g)\}\alpha_R$$
$$-(16\pi^2/g^2)\mu_n{}^2\mathbf{M}_g\cdot\{\mathbf{g}\times(\mathbf{g}\times\mathbf{M}_g)\}]\chi. \quad (8.3)$$

From this equation it is apparent that the average of s over the direction of $\mathbf{g}$ is given in terms of averages of the quantities

$$\delta(|\mathbf{K}_0+\mathbf{g}|-K)\quad\text{and}\quad(1/g^2)[\mathbf{g}\times(\mathbf{g}\times)]\delta(|\mathbf{K}_0+\mathbf{g}|-K)\equiv-(1-\mathbf{gg}/g^2)\delta(|\mathbf{K}_0+\mathbf{g}|-K),$$

now.

(a) $$\langle\delta(|\mathbf{K}_0+\mathbf{g}|-\mathbf{K})\rangle_{Av}=2K\langle\delta(K^2-(\mathbf{K}_0+\mathbf{g})^2)\rangle_{Av}=2K\langle\delta(g^2+2\mathbf{K}\cdot\mathbf{g})\rangle_{Av}$$
$$=K\int_0^\pi\delta(g^2+2Kg\cos\theta)\sin\theta d\theta=1/2g\quad\text{if } g<2K=K_0;\ 0 \text{ if } g>2K,$$

(b) $$\langle(\mathbf{gg}/g^2)\delta(|\mathbf{K}_0+\mathbf{g}|-K)\rangle_{Av}=(2K/4\pi)\int_0^\pi\int_0^{2\pi}[(\mathbf{i}\cos\phi\sin\theta+\mathbf{j}\sin\phi\sin\theta+\mathbf{k}\cos\theta)(\mathbf{i}\cos\phi\sin\theta$$
$$+\mathbf{j}\sin\phi\sin\theta+\mathbf{k}\cos\theta)\delta(g^2+2K_0g\cos\theta)\sin\theta d\theta d\varphi,$$

where θ and ϕ are the polar coordinates of $\mathbf{g}$ with respect to a Cartesian system whose z axis is parallel to $\mathbf{K}_0$ and where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are orthogonal unit vectors, such that $\mathbf{k}=\mathbf{K}_0/K$;

$$\langle(\mathbf{gg}/g^2)\delta(|\mathbf{K}_0+\mathbf{g}|-K)\rangle_{Av}=(1/2g)\int_0^\pi[\tfrac{1}{2}(\mathbf{ii}+\mathbf{jj})\sin^2\theta+\mathbf{kk}\cos^2\theta]\delta(g/2K_0+\cos\theta)\sin\theta d\theta$$
$$=(1/2g)\{\tfrac{1}{2}(\mathbf{ii}+\mathbf{jj})(1-g^2/4K^2)+\mathbf{kk}g^2/4K^2\}=(1/4g)[(1-g^2/4K^2)+\mathbf{kk}((3g^2/4K^2)-1)],$$

(c) $$\langle[g\times(g\times)]\delta(|\mathbf{K}_0+\mathbf{g}|-K)/g^2\rangle_{Av}=-\langle(1-\mathbf{gg}/g^2)\delta(|\mathbf{K}_0+\mathbf{g}|-K)\rangle_{Av}$$
$$=-(1/4g)[(1+g^2/4K^2)+\mathbf{kk}(1-3g^2/4K^2)=-(1/4g)[(1+g^2/4K^2)+(\mathbf{K}_0\mathbf{K}_0/K^2)(1-3g^2/4K^2)].$$

[23] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, 1930) first edition, page 248, Eq. (25).

The average of s is, hence,

$$s=(4\pi M_n^2L^3v/\hbar^4K^2)\sum_{g<2K}(1/g)\chi^*[|\alpha|^2-2\pi\mu_n\boldsymbol{\sigma}\cdot\{(1+g^2/4K^2)+(\mathbf{K}_0\mathbf{K}_0/K^2)(1-3g^2/4K^2)\}\cdot\mathbf{M}_g\alpha_R+4\pi^2\mu_n^2\mathbf{M}_g\cdot\{(1+g^2/4K^2)+(\mathbf{K}_0\mathbf{K}_0/K^2)(1-3g^2/4K^2)\}\cdot\mathbf{M}_g]\chi. \quad (8.4)$$

Here the first term in the square brackets represents the nuclear scattering, the third the magnetic scattering, and the second the interference resulting from the superposition of nuclear and magnetic scattered *amplitudes*. It is this interference which leads to a polarization of neutrons; one can see that the sign of the second term and, consequently, the magnitude of the total scattering is different for the different spin states.

The average weakening of the beam is given by the ratio of the scattered current s, given by (8.4), to the incident current $I=vL^2$. Thus

$$A=s/I=L\chi^*[\kappa_n+\kappa_m+\kappa_i\sigma_z]\chi,$$

where

$$\kappa_n=(4\pi M_n^2/\hbar^4K^2)\sum_{g<2K}|\alpha|^2/g, \quad (8.5a)$$

$$\kappa_m=(4\pi M_n^2/\hbar^4K^2)\sum_{g<2K}(4\pi\mu_n^2/g)\mathbf{M}_g^2\{1+g^2/4K^2+(K_{0z}K_{0z}/K^2)(1-3g^2/4K^2)\}, \quad (8.5b)$$

$$\kappa_i=-(4\pi M_n^2/\hbar^4K^2)\sum_{g<2K}(2\pi\mu_n/g)|\mathbf{M}_g|\alpha_R\{1+g^2/4K^2+(K_{0z}K_{0z}/K^2)(1-3g^2/4K^2)\}, \quad (8.5c)$$

the magnetic vector being directed along the z axis. It will be noticed that terms in A proportional to σ_x, σ_y have been dropped; the lack of influence of such terms is due to the precession of the spin about the direction of the magnetic field; the rapidity of this precession obliterates any possible effect of polarization in directions other than that defined by the field.

For the state $\chi_{\frac{1}{2}}$

$$A=L[\kappa_n+\kappa_m+\kappa_i];$$

for the state $\chi_{-\frac{1}{2}}$

$$A=L[\kappa_n+\kappa_m-\kappa_i].$$

Let it now be supposed that, at a depth x, the intensities of $\chi_{\frac{1}{2}}$ and $\chi_{-\frac{1}{2}}$ are, respectively, $I_{\frac{1}{2}}$ and $I_{-\frac{1}{2}}$. Then

$$\begin{aligned}\Delta I&=\Delta(I_{\frac{1}{2}}+I_{-\frac{1}{2}})=[I_{\frac{1}{2}}(X+L)-I_{\frac{1}{2}}(X)]+[I_{-\frac{1}{2}}(X+L)-I_{-\frac{1}{2}}(X)]\\&=-L[(\kappa_n+\kappa_m)(I_{\frac{1}{2}}+I_{-\frac{1}{2}})-\kappa_i(I_{\frac{1}{2}}-I_{-\frac{1}{2}})]\\&=LI(\kappa_n+\kappa_m)-L\kappa_i\zeta,\end{aligned}$$

where $\zeta=I\sigma_z$. For ζ itself, one obtains the difference equation

$$\begin{aligned}\Delta\zeta&=[I_{\frac{1}{2}}(X+L)-I_{\frac{1}{2}}(X)]-[I_{-\frac{1}{2}}(X+L)-I_{-\frac{1}{2}}(X)]\\&=L\zeta(\kappa_n+\kappa_m)-LI\kappa_i.\end{aligned}$$

At this point the effects due to true absorption (capture) and incoherent scattering processes should be included. This is readily done by adding to the quantity $(\kappa_m+\kappa_n)$ the absorption coefficients for these processes, κ_c and κ_s, respectively. Denoting the sum $\kappa_m+\kappa_n+\kappa_c+\kappa_s$ by the single letter κ, and replacing κ_i by the notation $-w$ one has, finally

$$\Delta I/L=I\kappa+w\zeta, \quad \Delta\zeta/L=-\zeta\kappa+wI,$$

which, in differential form, read:

$$dI/dx=-\kappa I+w\zeta, \quad (8.6a)$$

$$d\zeta/dx=-\kappa\zeta+wI \quad (8.6b)$$

to be compared with (8.1a), (8.1b).

In conclusion, the relations (8.5a, b) for the absorption and polarization coefficients will be expressed in terms of the nuclear cross section and the macroscopic magnetization. (a) The quantity α which symbolized the amplitude of nuclear scattering, can immediately be related to the nuclear cross section, σ; one need only calculate this cross section with the potential $V=\alpha a^3\delta(\mathbf{r})$.[24] Using the Born perturbation method, one obtains:

$$\alpha^2=\pi\sigma\hbar^4/M_n{}^2a^6;\quad \alpha_R=[(\pi\sigma)^{\frac{1}{2}}\hbar^2/M_na^3]\cos\delta,$$

where δ is the phase difference between the incident and scattered nuclear wave. Due to the non-existence of a resonance level of iron for thermal neutrons, $|\cos\delta|$ is expected to be quite close to unity. (b) The quantity, which represents the magnetic scattering, is defined by the integral $(1/a^3)\int\mathbf{M}\exp(-i\mathbf{g}\cdot\mathbf{r})d\tau_0$ where $\mathbf{M}$ is the density of atomic magnetization. Remembering that this integral goes over only one atom, located in the center of the volume of integration, one obtains

$$\mathbf{M}_g=(1/a^3)\mathfrak{y}_aF(\mathbf{g}).$$

Here, $\mathfrak{y}_a$ is the atomic magnetic moment and $F(\mathbf{g})=\int\exp(i\mathbf{g}\cdot\mathbf{r})\rho_md\tau_0$ where ρ_m is the distribution, normalized to unity, of the atomic magnetization. $\mathfrak{y}_a$ itself can be expressed in terms of the macroscopic magnetization $\mathbf{M}_0=N\mathfrak{y}_a$ where $N=2/a^3$ is the number of atoms per cubic centimeter.[25] Hence $\mathbf{M}_g=\mathbf{M}_0F(\mathbf{g})/2$. (c) Inserting these values for α and $\mathbf{M}_g$ into (8.5) one obtains

$$\kappa_n=(N\sigma/4\pi)(\lambda^2/a^2)\sum_{l<2a/\lambda}1/l, \tag{8.7a}$$

$$\kappa_m=(M_n{}^2\lambda^2a\mu_n{}^2M_0{}^2/2\hbar^4)\sum_{l<2a/\lambda}(1/l)[F(1/a)]^2\{1+\lambda^2l^2/4a^2+(\cos^2\theta)(1-3\lambda^2/4a^2)\}, \tag{8.7b}$$

$$w=(M_n\lambda^2M_0\cos\delta(\pi\sigma)^{\frac{1}{2}}/(2\pi\hbar^2a^2))\sum_{l<2a/\lambda}(1/l)F(1/a)\{1+\lambda^2l^2/4a^2+(\cos^2\theta)(1-3\lambda^2l^2/4a^2)\}, \tag{8.7c}$$

where θ is the angle between the magnetization and the direction of the beam, $\lambda=2\pi/K$ is the wave-length of the neutrons, and the summation over $\mathbf{l}=\mathbf{g}a/2\pi$, whose components are integers, goes over values of $\mathbf{l}$ for which $l_1+l_2+l_3$ is even.

IX. Transmission of Neutrons in Incompletely Saturated Polycrystalline Ferromagnetic Media

Equations (8.6a) (8.6b) describe the transmission and polarization of neutrons in a medium in which the magnetic induction $\mathbf{B}$ of Maxwell's equations, i.e., the average over atomic distances of the Lorentzian magnetic field $\mathbf{H}$ always points in one direction. For polycrystals, as has been seen in Section VII, this condition obtains only when the medium is completely saturated; in all other cases, the deviations of the fields of the constituent crystals give rise to depolarization effects as given by (7.1a) and (7.1b). These equations have the general form:

$$s_{fz}=s_{0z}e^{-px}$$

from which one obtains, for the differential change of the polarization:

$$ds_z/dx=-ps_z$$

or, in terms of the variable $\zeta\equiv\sigma_zI=2s_zI$

$$d\zeta/dx=-p\zeta.$$

To obtain the effect of this depolarization on the transmission of neutrons, one merely inserts the

[24] Cf. (8.2).
[25] The factor 2 comes in because of the body-centered structure.

term $-p\zeta$ into (8.6b). The complete transmission equations now read:[26]

$$dI/dx=-\kappa I+w\zeta, \tag{9.1a}$$

$$d\zeta/dx=-\kappa\zeta-p\zeta+wI. \tag{9.1b}$$

These equations can be simplified by the substitution

$$I=je^{-\kappa x};\quad \zeta=\eta e^{-\kappa x}$$

to give:

$$dj/dx=w\eta, \tag{9.2a}$$

$$d\eta/dx=-p\eta+wj. \tag{9.2b}$$

Since the quantity of interest in the experiments is the relative change in transmitted intensity $\Delta I/I=\Delta j/j$, one need not refer back to this substitution, but merely treat j and η as the particle and spin current, respectively.

Equations (9.2) will first be solved for the important case $wx\ll 1$, i.e., transmission effect small. In this case, one can replace j in (9.2b) by j_0, the initial current. Thus

$$d\eta/dx=-p\eta+wj_0,$$

which has the solution

$$\eta=\eta_0e^{-px}+(wj_0/p)(1-e^{-px}), \tag{9.3a}$$

where η_0 is the initial polarization. Substituting in (9.2a), one obtains:

$$j=j_0(1+(w^2/p^2)(px-1+e^{-px}))+(\eta_0w/p)(1-e^{-px}). \tag{9.3b}$$

Equations (9.2) can also be solved exactly; the result is

$$j=j_0e^{-px}\left\{\frac{\sinh[x((p/2)^2+w^2)^{\frac{1}{2}}]}{((2w/p)^2+1)^{\frac{1}{2}}}+\cosh[x((p/2)^2+w^2)^{\frac{1}{2}}]\right\}+\frac{\eta_0e^{-px}\sinh[x((p/2)^2+w^2)^{\frac{1}{2}}]}{((p/2w)^2+1)^{\frac{1}{2}}}, \tag{9.4a}$$

$$\eta=\eta_0e^{-px}\left\{\cosh[x((p/2)^2+w^2)^{\frac{1}{2}}]-\frac{\sinh[x((p/2)^2+w^2)^{\frac{1}{2}}]}{((2w/p)^2+1)^{\frac{1}{2}}}\right\}+j_0e^{-px}\frac{\sinh[x((p/2)^2+w^2)^{\frac{1}{2}}]}{((p/2w)^2+1)^{\frac{1}{2}}}. \tag{9.4b}$$

Two cases are of practical importance:

(1) $$wx\ll 1,\quad px\sim 1,\quad \text{i.e.}\quad p\gg w.$$

Here (9.4) reduces directly to (9.3).

(2) $$wx\ll 1,\quad p\sim w,\quad \text{i.e.}\quad px\ll 1.$$

In this case (9.4) reduces to

$$j=j_0\{1+w^2x^2/2\}+\eta_0wx,$$
$$\eta=\eta_0+j_0wx,$$

which can also be obtained from (9.3) for $px\ll 1$ by expansion of the exponentials in these equations. Thus (9.4) reduces to (9.3) for $wx\ll 1$ and *all* values of p.

The applications of (9.3) to the different types of transmission experiments will now be discussed.

(1) Single transmission

This experiment consists of the measurement of neutron transmission through a ferromagnetic magnetized and unmagnetized; the relative difference is called the *single transmission effect*. The

[26] In what follows, the constant κ and w will be treated phenomenologically; their numerical evaluation will appear in the following paper by O. Halpern, M. Hamermesh, and M. H. Johnson.

theoretical estimate of this effect is obtained directly from (31) with the initial condition $\eta=0$. Thus

$$\Delta j/j_0 \equiv (j-j_0)/j_0 = (w^2/p^2)\{px-1+e^{-px}\} = (w^2x^2/2)f(px), \tag{9.5}$$

where

$$f(\xi) = 2(\xi-1+e^{-\xi})/\xi^2. \tag{9.6}$$

When there is no depolarization $f(px)=1$ and $(\Delta j/j_0)=w^2x^2/2$ in conformance with previous results. In all other cases the maximum transmission effect is reduced by the factor $f(px)$. For small px, $f(px)\cong 1-px/3$; for large px, $f(px)\cong 2/px$.

To illustrate the sensitivity of the single transmission effect to the degree of magnetic saturation, let it be supposed that the average crystal size is 10^{-8} cm^3,[27] i.e., $\delta=2\times10^{-3}$ cm. The corresponding value of p is, by (7.1b), $(\frac{2}{3})\times10^3\Delta M/M_0$. If $\Delta M/M_0=1$ percent and the thickness of the sample $d=1$ cm, $pd=20/3$ and the transmission effect drops to 30 percent of its maximum value.

The presence of depolarization is most easily detected experimentally by observations on the dependence of the transmission effect upon the thickness of the ferromagnet. In the absence of depolarization, this dependence is quadratic $(\Delta j=j_0w^2x^2/2)$; the depolarization has the effect of cutting down the increase with thickness; in the limit $px\gg1$ the dependence is linear.

(2) Double transmission

In this experiment one compares the transmission through two ferromagnetic plates magnetized in the same direction to that obtained with the magnetizations oppositely directed. Let d_1 and d_2 be the thickness of the two plates, w_1 and w_2 their polarization coefficients, which are equal in magnitude but may differ in sign depending on whether the magnetizations are parallel or antiparallel, and p_1, p_2 their depolarization coefficients. From (9.2) the current and polarization of the beam emerging from the first plate are

$$j_1 = j_0\{1+(w_1^2d_1^2/2)f(p_1d_1)\}, \tag{9.7a}$$

$$\eta_1 = (w_1j_0/p_1)\{1-\exp[-p_1d_1]\}. \tag{9.7b}$$

Using these values as initial conditions for the second transmission, one obtains:

$$j_2 = j_1\{1+(w_2^2d_2^2/2)f(p_2d_2)\}+(\eta_1w_2/p_2)\{1-\exp[-p_1d_2]\}$$

or

$$j_2 = j_0\{1+(w_2^2d_2^2/2)f(p_2d_2)\}\left\{1+\frac{w_1^2d_1^2}{2}f(p_1d_1)\right\}+j_0(w_1w_2/p_1p_2)\{1-\exp[-p_1d_1]\}\{1-\exp[-p_2d_2]\}. \tag{9.8}$$

The double transmission effect is the relative difference in j_2 for the two cases $w_1=w_2$ and $w_1=-w_2$. Thus:

$$\Delta j/j_0 = (2w^2/p_1p_2)\{1-\exp[-p_1d_1\}\{1-\exp[-p_2d_2]\}$$

$$= 2w^2d_1d_2g(p_1d_1)g(p_2d_2), \tag{9.9}$$

where $g(\xi)=(1-e^{-\xi})/\xi$. It should be noticed that only when $p=0$ does one obtain the maximum effect $\Delta j/j_0=2w^2d_1d_2$, in conformance with previous estimates; in all other cases the effect is cut down by a factor $g(p_1d_1)g(p_2d_2)$, which, for large pd becomes $1/(p_1d_1p_2d_2)$.

It can be seen that the depolarization is much more destructive in double transmission, as compared to single transmission. Using the previous illustration of $\Delta M/M_0=1$ percent, $\delta=2\times10^{-3}$ cm, one obtains, with $d_1=d_2=1$ cm, a reduction by a factor $(3/20)^2=0.025$. One thus sees that very small deviations from magnetic saturation suffice to wipe out any double transmission effects which are due to the magnetic scattering in ferromagnets.

[27] This estimate is appropriate to the case of Armco iron, as communicated to the authors by Dr. T. D. Yensen of Westinghouse Research Laboratories.

X. Comparison with Experiments

The application of the formulae developed in Section IX cannot be carried out immediately, essentially for two reasons. The quantity w as mentioned before, is here treated phenomenologically, while it will be evaluated on a strictly atomistic basis in the following paper.[26] Secondarily, it so happens that experimental papers in the field do not contain precise information on the deviation from saturation, which determines according to, e.g., (7.1), the depolarization constant p. Still, a few general statements of a semi-quantitative nature can be made.

The earliest experiments[28] were mostly carried out with a magnetization which probably did not exceed 70 percent. It is clear without any detailed calculation that the value of p in this case becomes so large that no sensible value of w could lead to an observable polarization. This is the more true since it will be shown in the following investigation[26] that even for ideal saturation the theoretical value of w is rather small.

A similar objection must be raised against later experiments[29] in which also poor saturation $\Delta M/M_0 > 10$ percent was obtained. One is forced to the conclusion that the observations are theoretically inexplicable for any reasonable value of w.

The most detailed observations so far reported are those by Powers[30] who studied single as well as double transmission effects with various samples of iron and under various conditions of magnetization. His single transmission experiments made with Armco iron were carried out with a deviation from saturation which, according to the author, amounted to approximately 1.5 percent. According to formulae (9.5), (9.6) and (7.1a), and with the experimentally well-established value[27] for the crystal size the reduction factor due to depolarization should have been $\frac{1}{5}$ or smaller.

Powers' double transmission experiments were carried out with Swedish iron and a value of $\Delta M/M_0 \sim 8\times 10^{-2}$. The great sensitivity of double transmission experiments to deviations from saturation (cf. (9.9) and (9.10)) makes it hard to understand how under such conditions a double transmission should have been observable. (9.9), (9.10) and (7.1b) indicate that for any reasonable value of the most accurately known crystal size in Swedish iron, the observed effect should have been a fraction of a percent of the value obtainable with complete saturation.

If the picture of the constitution of ferromagnets and of the neutron's magnetic moment as used in this paper should be maintained, we are forced to conclude that the experimental data need revision. It may be that the saturation achieved in some of the experiments is far better than reported, though it is not easy to see how the theoretically required value of $\Delta M/M_0 \sim 10^{-3}$ can be reconciled with the reported value of $\Delta M/M_0 \sim 10^{-1}$.

The authors are indebted to Professor E. Fermi for the discussion of an interesting possibility, which, though not actually helpful, deserves analysis in this difficult situation. Obviously (cf. (7.1b)) the depolarization would be diminished if the number of incompletely magnetized domains could be diminished or, in other words, the size of the domains increased. In the present paper the size of each domain was identified with the size of the crystallite which, in turn, is in certain cases[27] experimentally determined. This seemed almost inevitable since every crystallite has its own direction of easy magnetization which lies nearest to the direction of the magnetizing field.

The new assumption would introduce blocks comprising many crystallites with the direction of easiest magnetization being determined *for the whole block* by internal stresses. Obviously, the existence of such sufficiently large blocks would diminish depolarization strongly.

There exist, on the other hand, difficulties which make the assumption of such blocks rather impossible. It is not easy to see how the existence of such large stresses which overcome the crystalline energies, will affect the magnetization only, without influencing appreciably the elastic and crystallographic structure. Furthermore, the saturation curve of polycrystalline iron can be semi-quantitatively understood by using the magnetization curve for single crystals of iron and averaging over all directions of the

[28] Reported in Frisch, von Halban and Koch, Phys. Rev. **53**, 719 (1938).
[29] Hoffman, Livingston and Bethe, Phys. Rev. **51**, 214 (1937).
[30] P. N. Powers, Phys. Rev. **54**, 827 (1938).

crystalline axis. This fact seems to indicate that "the turning of the magnetization into the direction of the field" has to overcome mainly crystalline forces. Finally it has been pointed out by Becker and Doering[31] that stresses sufficiently large to overcome the crystal orientation energy cannot be realized in iron since they would have to exceed the breaking strength of the material.

[31] R. Becker and W. Doering, *Ferromagnetismus* (Julius Springer, Berlin, 1939), p. 104.

It should be emphasized as a guide for future experiments that it is not at all necessary to use completely polarized neutron beams for the investigation of the properties of ferromagnets. Polarizations as claimed to be present in the various experiments discussed would be amply sufficient to allow investigations of ferromagnetic structures.

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The Passage of Neutrons Through Crystals and Polycrystals

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We investigate in this paper phenomena, occurring in the passage of neutrons through matter, which originate from the crystalline or polycrystalline constitution. While the procedure is in many respects closely similar to that followed in the theory of x-rays, important and quantitatively decisive differences arise from the complicated coherence properties of the *atomic* scattering. The theoretical formulae developed in the first two paragraphs permit us to interpret in a quantitative manner a series of experiments which show deviations from the so far almost always assumed additivity of nuclear cross sections. We also obtain information concerning the relative phases of the scattering amplitudes of nuclear isotopes. We next show for illustrative purposes how the Larmor precession of the spin of the neutron passing through the magnetized medium, and the well-known differentiation between the action of the vectors "B" and "H" can be explained as a simple dispersion phenomenon. The preceding formulae permit us to determine quantitatively the transmission and polarization of neutron beams passing through ferromagnetic bodies. The main uncertainties which enter into attempts to evaluate experiments, arise from our incomplete knowledge of the velocity distribution of the incident beam and of the form factor which enters into the formulae for magnetic scattering. We present a detailed discussion of these uncertainties. Even neglecting depolarization effects which are due to incomplete saturation and which would still further diminish the theoretical value for transmission effects, we find that the observed values are considerably higher than those theoretically predicted and are not in good agreement with each other.

INTRODUCTION*

CONTINUING earlier investigations[1] on the transmission of neutrons through macroscopic bodies, and in particular on macroscopic and microscopic magnetic effects, we present in this paper a discussion of the influence of crystal structure on scattering and polarization of neutrons. It has been the accustomed procedure to consider the nuclear cross section for the scattering of slow neutrons as a strictly additive property independent of the physical state or the chemical constitution of the sample investigated. On this basis the total cross section of a chemical compound could, for example, be determined from a knowledge of the cross sections of the constituent elements.

Such a procedure is valid only if we are dealing with a substance of strictly amorphous structure (a gas). In all other cases the crystalline structure of the material becomes significant; it will turn out to be of importance even for substances

* A preliminary report of this work appeared in the abstract: Halpern, Hamermesh and Johnson, Phys. Rev. **55**, 1125A (1939).

[1] I. O. Halpern and M. H. Johnson, Phys. Rev. **51**, 992 (1937); II. *ibid.* **52**, 52 (1937); III. *ibid.* **55**, 898 (1939); IV. Halpern and Holstein, *ibid.* **55**, 601 (1939); V. *ibid.* **59**, 960 (1941) (this issue). These are referred to throughout the paper by the corresponding Roman numerals.