last two terms of (27) is of the same order as the difference between the emission current and the calorimetric values, and larger than the uncertainty in either of these. For an accurate comparison of the two work functions both the surface structure of the emitter and its Thomson coefficient will have to be known. And because of the uncertainty in the last three terms of (28), any attempt to calculate A^* from the cooling effect for a polycrystalline metal is at present even more hopeless.[18]

For the sake of simplicity, all the equations of this section have been derived neglecting any possible variation of the mean reflection coefficient $\bar{r}$ with temperature. It is easily shown that if there is a reflection coefficient for an electron with normal momentum p_x, given by

$$r=\exp[-p_x^2/2m\omega],$$

as has been proposed by Nottingham,[19] a term $\omega^2kT/(\omega+kT)^2$ should be added to the right of (27), and a term $-k\omega^2/(\omega+kT)^2$ should be added to the right of (28). If $\omega=0.2$ ev, as proposed by Nottingham,[19] the former correction is 0.05 ev at 2320°K and the latter is -2.2×10^{-5} ev/deg. These values are undoubtedly much too large to represent the reflection effect, since Nottingham's "reflection coefficient" includes the effects of patch fields in addition to true reflection.

[18] Using an incorrect formula, Krüger and Stabenow obtained from their data the low value 0.66 amp./cm^2 deg.^{2} for A*.

[19] W. B. Nottingham, Phys. Rev. **49**, 78 (1936.)

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On the Measurement of Observables in Relativistic Quantum Mechanics

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The paper discusses the question how far observables in quantum theory permit accurate measurement if *no assumptions* are made about the values of conjugate variables. The physical connection between the uncertainty principle and the commutation relations is briefly discussed. A treatment of the γ-ray microscope makes it appear that even with small-angle diffraction no great improvement beyond the Compton wave-length can be obtained. An analysis of an ideal arrangement to measure electric field strengths leads to the result that, admitting the existence of arbitrarily constituted test bodies, the accuracy of the measurement still cannot exceed certain limits which are mainly defined by the wave-length of the field and the spatial and temporal domain of measurement. These restrictions are due to the properties of the "vacuum," which are changed as a consequence of the possibility of pair production.

1. Introduction

It may perhaps be said to be a distinctive feature of the mathematical theory of quantum mechanics that quantities of physical significance are assumed to be measurable individually with arbitrary accuracy, at least in principle. This feature is expressed mathematically by the assumption that it is possible to transform any single arbitrary function of the dynamical variables to its principle axes. Restrictions on the simultaneous measurement of two or more observables occur if their mathematical representations do not commute.

The discussions of the many ideal experiments which have been carried through, mainly for the verification of the uncertainty relations in special cases, are thus important in testing general postulates in particular cases. They derive their power of conviction that other methods might not lead to different results primarily from the fact that on the one hand they conform to the general principles of the theory, and on the other that the experimental arrangements employed are usually of so general a nature that imagination fails to provide us with suggestions leading to experiments based on different principles.

In *relativistic quantum mechanics* important conceptual difficulties arise from the fact that into every description of a measurement there enters a statement concerning the time at which the measurement is carried out. Interpreting the relativistic commutation relations rigidly we might feel inclined to say that the three components of momentum of a free electron should be measurable independently of the time interval allowed, while there should exist restrictions on the accurate measurement of the energy within an arbitrarily short time interval. But the knowledge of the three momentum components of a free particle would immediately yield that of its energy. On the other hand also, we know from the discussion of ideal experiments that a momentum component is not measurable with unlimited accuracy in an indefinitely short time.

These well-known facts seem sufficient to shatter confidence in the direct and unambiguous connection between commutation relations and experimentally verifiable uncertainty relations. The solution of this apparent paradox, as it is probably accepted by many physicists at present, can be seen in the fact that the proper use of the Dirac equation prevents the treatment of the electron as a one-body problem. The large number of complications in the properties of the vacuum due to the phenomenon of pair production as postulated by every relativistic theory, prevents the simple application of the rules and postulates found valid in nonrelativistic quantum mechanics.

We intend to show in the next paragraph how the change in electrodynamics necessitated by relativistic quantum mechanics tends to destroy the possibility of an arbitrarily accurate determination of the position of an electron with the gamma-ray microscope. Although this line of reasoning, if valid, can only make futile the special arrangement used, we feel inclined to generalize the result on the basis of the experience gained from the previous discussions of ideal experiments and the uncertainty relations.

Our result will probably be accepted as confirmation of views or expectations now held by many physicists with reference to the properties of the Dirac electron. On the other hand, the investigation of paragraph three will refer to a case in which no large velocities need occur. Nevertheless, the properties of the vacuum seem to invalidate the concept of observable in a field in which it has generally been supposed to apply without restriction.[1] We refer to the measurability of an electric field strength in an electromagnetic field variable in space and time. This problem has been given its most comprehensive treatment by Bohr and Rosenfeld,[2] who have arrived at the result that the average value of a field strength over an indefinitely large space-time domain can be measured with unlimited accuracy provided no atomistic restrictions are imposed on the test body used in the ideal experiment. But they did not take into account the later discovered properties of the vacuum due to pair production.[3] In our opinion the measurement of an individual component of a field strength is not possible if the newly discovered properties of the vacuum are taken into account even in the most qualitative manner. It goes without saying that these new facts gain importance only when the space-time domain over which we average is sufficiently small; no objection can be raised against the asymptotic approach of the quantum theoretical concepts towards the classical concepts of field strength in large world domains.

2. A Gamma-Ray Microscope for Measurements of Arbitrarily High Accuracy

It has been pointed out in previous discussions that in the measurement of the position of an electron by means of the gamma-ray microscope, a natural limitation seems to occur due to the Compton shift of the scattered radiation. The wave-length of a ray scattered through an angle ϑ is given by the relation

$$\lambda=\lambda_0+(2h/mc)\sin^2\tfrac{1}{2}\theta \qquad (1)$$

[1] Through the discussion of an ideal experiment Cox and Myers (Nature **142**, 394 (1938)) arrive at the result that the electron's spin (magnetic moment) cannot be measured *even if the electron is bound*. This thesis cannot be maintained but is due to an error in the physical discussion leading to it.

[2] N. Bohr and Rosenfeld, Proc. Dan. Acad. Sci., Copenhagen, 1933.

[3] The phenomenon of pair production is mentioned in this paper as the most characteristic consequence of any relativistic quantum theory and also because it plays the most important part in the experiments analyzed by us. The possibility should be kept in mind that in the case of other ideal experiments, other relativistic features become more important while pair production may remain latent.

with the usual symbols. On the other hand, since the inaccuracy Δx of a measurement of position is limited by the relation

$$\Delta x \sim \lambda/\epsilon \tag{2}$$

in which the numerical aperture $\epsilon \ll 1$ is connected with the *maximum* angle of scattering ϑ_0 by $\epsilon = 2\vartheta_0$, many authors have assumed that an accuracy in the neighborhood of the Compton wave-length would be the limit for the gamma-ray microscope.

Now a closer discussion of frequency conditions and resolving power for scattering under small angles leads to a different result. It seems that the gamma-ray microscope should allow a position measurement with *arbitrarily high accuracy* in spite of the fact that a frequency shift due to the Compton recoil must be taken into account. This statement becomes incorrect, and therefore the position not accurately measurable, when we consider the changes in the theory due to the phenomena associated with pair production.[3]

To prove this statement we first observe that the shift in wave-length for *small angle* scattering is *quadratic* in the scattering angle

$$\lambda = \lambda_0 + \tfrac{1}{2}\Lambda\vartheta^2 \qquad (\Lambda = h/mc). \tag{3}$$

Suppose now that we use as a primary wave-length

$$\lambda_0 = \Lambda\epsilon\delta, \tag{4}$$

where δ is a pure number to be specified later. The spread in wave-lengths over the opening of the microscope according to Eq. (3) is smaller than $\Lambda\epsilon^2/8$. By Eq. (2) the accuracy obtainable is given by the relation

$$\Delta x \sim \lambda/\epsilon \sim (\lambda_0 + \tfrac{1}{2}\Lambda\vartheta_0^2)/\epsilon \sim \Lambda(\delta + \tfrac{1}{8}\epsilon). \tag{5}$$

Since for small angles the scattered intensity varies as $(1+\Lambda\epsilon^2/2\lambda_0)^{-3}$ we must take $\delta \gg \epsilon$ in order to make full use of the aperture. For example we can choose $\delta = \epsilon^{\frac{1}{2}}$. We then note from Eq. (5) that the accuracy of measurement can be made an arbitrarily small fraction of the Compton wave-length provided that the aperture, and with it the primary wave-length, be made sufficiently small.

By this reasoning the possibility of an arbitrarily accurate measurement of a position coordinate seems established provided the phenomena are described correctly by the theory of optics as it was accepted before pair production was discovered. If it is required, however, that a second or "control" measurement be possible, then a restriction must be imposed on the time needed for carrying out the experiment, and it will be shown that this in turn requires a modification of the whole reasoning since it brings in the possibility of pair formation.

Since for the *high frequencies* used, the lateral recoil velocity of the electron is of order $c\delta$, the accuracy Δx can be achieved only if the process takes a time τ smaller than $\Delta x/c\delta$. The frequency of the wave train will be defined only within a latitude $\Delta\nu \sim c\delta/\Delta x$. Since $\Delta\nu$ is larger than the maximum Compton shift for the aperture used, it is obvious that this shift will be of no significance in the present arrangement. But in addition to this, we shall have to use an intensity in the primary radiation which makes it certain that *at least* one photon will be scattered into the spherical angle $\Omega \sim \epsilon^2$ during the time τ. Neglecting factors of the order unity, and remembering that the differential cross section for sufficiently small angles takes on the Thomson value, we obtain the following relation

$$cE^2\epsilon^2(e^4/m^2c^4)\cdot(\Delta x/c\delta) \sim hc/\lambda_0 \tag{6}$$

in which E is the field strength in the incident wave. Making use of Eqs. (2), (4) and (5) we obtain

$$E^2 \sim (m^4c^8/e^6)\alpha(1/\epsilon^3\delta), \tag{7}$$

α being the fine-structure constant.

We learn from Eq. (7) that for measurements of this type with an accuracy exceeding the Compton wave-length it is necessary to use field strengths exceeding the "critical value" which enters into the new tentative formulation of quantum electrodynamics. The square root of the first factor on the right side of (7) can be interpreted as "the field strength of the electronic field at the electron's surface." It is known that for values of the field strength larger than this critical value, the phenomena cannot be described, even approximately, by older electrodynamic theories. At present we do not have a satisfactory substitute and it does not seem worth while, even if feasible, to attempt anything like a quantitative discussion. We shall limit ourselves to the statement that the (extremely strong) production of (virtual) pairs will extend over a distance from the electron which will certainly equal a Compton wave-length, and that these virtual pairs will in turn take part in the scattering of the primary radiation. Even from a purely qualitative standpoint it seems hardly possible that the electron should act like a point source of scattered radiation, a condition which is necessary for the efficacy of the microscope arrangement.

3. Measurement of Field Strength

Field strengths are measured by determination of the ponderomotive action on a charge, q. We therefore have to discuss the measurement of the changes of momentum imparted to a test body. In the discussion presented in this paragraph we shall follow in all general assumptions the discussion of Bohr and Rosenfeld.[2] We shall not be concerned with a renewed discussion of certain fine points presented in their investigation, but shall attempt to show that accepting the results there obtained we shall have to modify them on account of the new electrodynamics. We shall furthermore limit our discussion to the simple case of the measurement of a component of the electric field strength.

The experimental arrangement shall be as follows: A test body of mass M occupying a volume V shall carry a uniformly distributed charge q. *No atomistic restrictions shall be made concerning any property of the test body.* We measure with the aid of the Doppler effect the velocity of the test body at the time t_1; it is assumed that the time of measurement does not exceed τ.

During the following time interval T we let the body be exposed to the action of the electric field E_x which we want to measure. At the time $t_1+\tau+T$ we carry out a second determination of the velocity as we did previously with the aid of the Doppler effect, during the time interval τ. From these measurements we deduce the average field strength $\langle E_x\rangle_{\mathrm{Av}}$ which is defined by the relation

$$\langle E_x\rangle_{\mathrm{Av}}=(1/TV)\int E_x dt dV$$

with the aid of the formula

$$q\langle E_x\rangle_{\mathrm{Av}}=\frac{M(v_2-v_1)}{T}. \tag{8}$$

Following the discussion of Bohr and Rosenfeld we also shall only be concerned with statements about $\langle E_x\rangle_{\mathrm{Av}}$ and not about any instantaneous point values of E_x.

We now have to discuss the possible accuracy which can be obtained from (8) for $\langle E_x\rangle_{\mathrm{Av}}$. Bohr and Rosenfeld have discussed in detail why τ must always be taken very small compared to T. The question of the relative magnitude of cT and $V^{\frac{1}{3}}$ which, for other reasons, was of great importance in the paper of Bohr and Rosenfeld does not concern us here; we shall assume that we are dealing with the physically most important case of

$$cT\sim V^{\frac{1}{3}}.$$

The inaccuracy in the determination of $\langle E_x\rangle_{\mathrm{Av}}$ is now mainly determined by the inaccuracy of the measurements of v_1 and v_2. Since we want the test body to be localized in a region of the approximate size of $V\sim L^3$ the uncertainty principle gives us for the inaccuracy in the measurement of Mv_1 and Mv_2 the approximate values

$$M\Delta v_1\sim M\Delta v_2\sim h/L.$$

We can therefore say that $\langle E_x\rangle_{\mathrm{Av}}$ could be measured with an inaccuracy determined by

$$\Delta\langle E_x\rangle_{\mathrm{Av}}\sim h/qTL. \tag{9}$$

If we measure the field strength of a light wave of the wave-length λ it will be reasonable to assume the spatial domain L and the time interval T which is connected with it by $L\sim ct$ in such a way that L is only a fraction of λ. Only then can we be certain that a value obtained for our average field strength has something to do with the primitive concept of a field strength as referring to a point, i.e., an arbitrarily small domain in space; otherwise the actions of a field will be averaged to zero when taken over a domain comprising many wave-lengths. For the same reason we have chosen $L\sim ct$. Inserting these relations into (9) we obtain for the inaccuracy in the measurement of an average field strength

$$\langle\Delta E_x\rangle_{\mathrm{Av}}\sim hc/q\lambda^2. \tag{10}$$

Assuming that we express the wave-length in units of the Compton wave-length of the electron

$$\lambda = \delta(h/mc) = \delta\Lambda$$

and the charge carried by the test body in units of the electron charge $q = ze$ we can write (10)

$$\langle \Delta E_x \rangle_{\text{Av}} \sim \frac{1}{\delta^2 z} \frac{e}{\Lambda^2} \cdot \left(\frac{hc}{e^2} \right). \tag{11}$$

From (11) Bohr and Rosenfeld have concluded that $\langle E_x \rangle_{\text{Av}}$ can be measured with an arbitrary accuracy as long as we do not impose any atomistic restrictions upon the test body. By assuming z suitably large, the expression for $\langle \Delta E_x \rangle_{\text{Av}}$ can obviously be made arbitrarily small.

It is to be admitted that it is, logically, fully consistent to use such an idealized test body. Quantum mechanics in its present stage takes into account only the finiteness of h and not the atomistic structure of the charge e. Whatever the future theory may be which will *combine* the finiteness of h with the atomistic structure of e it seems at present undoubtedly of great importance to test the internal consistency of a theory which admittedly gives a correct picture of only part of our knowledge of nature.

But quantum mechanics on the other hand has in a certain way already exceeded its own limits through the discovery of the phenomena of pair production and the changes in electrodynamics which though only incompletely formulated, have thereby become necessary. An atomistic concept has crept in concerning the mass and charge of the particles created out of the vacuum and we shall see immediately that this phenomenon of pair creation destroys the possibility of an arbitrarily accurate measurement of the average field strength.

Let us inquire into the order of magnitude of z which is necessary to make the right side of (11) a small number. Assuming mainly for the sake of illustration that we are dealing with an incident wave-length equal to the Compton wave-length of the electron we then find that $\langle \Delta E_x \rangle_{\text{Av}}$ will be small compared to 1 if $z \gg 10^{12}$. Even admitting in the sense of the hypothesis underlying the whole argument, that a test body carrying a charge of 10^{12} in a volume smaller than Λ^3 can be constructed, we must not forget that such a test body will not retain its charge permanently even in a vacuum. There will be a leakage due to pair production and though we do not feel that the present state of theory allows us to make any reliable quantitative estimate we can be sure that this leakage will be a rapid process. It can be easily visualized as being due to a production of pairs of which one member will be drawn towards the test body and diminish its total charge while the other will go off to infinity. The energy necessary for the creation of a pair will be supplied by the gain in potential energy, which is due to the partial neutralization of the test body. The process mentioned is an illustration of the well-known mathematical fact that in the presence of an external field, states of positive and negative energy of the electron get mixed up with each other. The process quite obviously will continue until the gain in potential energy due to adding a charge to the test body will have become smaller than $2mc^2$. We so arrive at the concept of a limiting charge which cannot permanently be exceeded on a body of given radius.

A very crude estimate of the order of magnitude of this limiting charge can be obtained as follows: The gain in potential energy obviously equals ze^2/L while the energy which is necessary for the creation of a pair exceeds $2mc^2$. Equating these two expressions we obtain for the limiting charge

$$z_0 = 2mc^2 L/e^2 \tag{12}$$

or with $a = e^2/mc^2$ (a = electron radius)

$$z_0 = 2L/a. \tag{13}$$

The relation (13) seems to lead to absurd results if applied to the case of nuclei which carry a charge larger than the limiting z_0. This contradiction is only apparent since in the deliberations leading to (13) we had failed to include the kinetic energy of the particles into the energy necessary for the creation of a pair. This was approximately justified as long as the wave-length of a particle exceeded the Compton wave-length and the radius L of the test body was sufficiently large. If the test body (nucleus) becomes too small the kinetic energy must be taken into account and the gain in potential energy diminishes. The whole argument obviously is closely related to the fact that there do not exist stationary states for a Dirac electron under the influence of a charge $z>118$.

This phenomenon of leakage makes it impossible for us to use a test body which carries an arbitrarily large charge. We see from (11) and (13) that the limit is reached long before z has taken on a sufficiently large value. Inserting for a moment the limiting value z_0 into (11) we obtain for the maximum accuracy (minimum error) of $\langle\Delta E_x\rangle_{\text{Av}}$ the relation

$$\langle\Delta E_x\rangle_{\text{Av}} \sim eh/L^3mc. \qquad (14)$$

Field measurements therefore can be carried out with great accuracy for long wave-lengths and large time intervals; they become increasingly and finally absurdly inaccurate if we pass to small wave-lengths and short time limits. This production of momentum during pair production and its reaction on the momentum of the test body accounts for the impossibility of measuring accurately the momentum change produced in the test body by the field.

If one wants to visualize other disturbances which arise through pair production it can, perhaps, be done as follows: The high frequency radiation which is incident upon the body at the time t_1 during a time interval of τ will create pairs under the influence of the large electrostatic field of the test body. These pairs together with the pairs created by the test body itself will exert forces upon the test body. If the particles are created in a distance of the order of magnitude L the momentum due to these forces created during the time L/c will be of the order of magnitude of

$$e^2z/Lc.$$

We will therefore obtain, per pair, an inaccuracy in the field strength of an order of magnitude of

$$\langle\Delta E_x\rangle_{\text{Av}} \sim \frac{e^2z}{Lc}\frac{c}{ezL} \sim \frac{e}{L^2}.$$

Not too much weight should be given to these deliberations because they probably are too "classical" and because we have no exact way of predicting the completely different phenomena which will occur in such excessively large fields.