$$\overset{l}{\mathfrak{F}}_{l,\gamma}(x)=\left(\frac{2a}{\gamma+l}\right)^{\gamma+l+\frac{1}{2}}\cdot\frac{1}{[\Gamma(2\gamma+2l+1)]^{\frac{1}{2}}}x^{l+\gamma}e^{-ax/(l+\gamma)},\tag{5}$$

$$\overset{m}{\mathfrak{F}}_{l,\gamma}(x)=\frac{(l+\gamma)(m+\gamma)}{a[(l+m+2\gamma)(l-m)]^{\frac{1}{2}}}\times\left(\frac{m+\gamma}{x}-\frac{a}{m+\gamma}+\frac{d}{dx}\right)\overset{m+1}{\mathfrak{F}}_{l,\gamma}(x),\tag{6}$$

where m is an integer, $0\leqslant m\leqslant l$, and

$$a=\frac{m_0c}{\hbar}\epsilon\alpha Z.\tag{7}$$

As C is the normalization factor and the functions $\overset{0}{\mathfrak{F}}_{l,\gamma}$, $\overset{1}{\mathfrak{F}}_{l,\gamma}$ are normalized, we obtain from (1), (2) and (3) the condition for C to make $\int_0^\infty(\chi_1^2+\chi_2^2)dx=1$:

$$1=\frac{8C^2}{\gamma^2}\{\epsilon\kappa^2+(\epsilon^2\kappa^2-\gamma^2)^{\frac{1}{2}}\alpha ZI_l(0,1)\},\tag{8}$$

where

$$I_l(0,1)=\int_0^\infty\overset{0}{\mathfrak{F}}_{l,\gamma}(x)\overset{1}{\mathfrak{F}}_{l,\gamma}(x)dx.\tag{9}$$

It will be shown below that

$$I_l(0,1)=-(\epsilon^2\kappa^2-\gamma^2)^{\frac{1}{2}}/\alpha Z\epsilon,\tag{10}$$

and hence, because of (8), we have

$$C=\epsilon^{\frac{1}{2}}/2\sqrt{2}.\tag{11}$$

This is the normalization factor required.

To calculate $I_l(0,1)$, let us consider

$$I_l(m,m+1)=\int_0^\infty\overset{m}{\mathfrak{F}}_{l,\gamma}\overset{m+1}{\mathfrak{F}}_{l,\gamma}\,dx.\tag{12}$$

By writing

$$\overset{m\mp}{\mathcal{H}}_{l,\gamma}=\overset{m}{A}_{l,\gamma}\overset{m\mp}{H}_{l,\gamma}=\overset{m}{A}_{l,\gamma}\left(\frac{m+\gamma-1}{x}-\frac{a}{m+\gamma-1}\mp\frac{d}{dx}\right),$$
$$\overset{m}{A}_{l,\gamma}=\frac{(l+\gamma)(m+\gamma-1)}{a[(l+m-1+2\gamma)(l-m+1)]^{\frac{1}{2}}}\tag{13}$$

it is not difficult to verify that

$$H^{(m+1)-}=\xi_mH^{m-}+\eta_mH^{m+}+\zeta_m,\tag{14}$$

where ξ_m, η_m, ζ_m are constants given by

$$\xi_m+\eta_m=\frac{m+\gamma}{m+\gamma-1},\quad\xi_m-\eta_m=1,$$
$$\zeta_m=a(m+\gamma)\left\{\frac{1}{(m+\gamma-1)^2}-\frac{1}{(m+\gamma)^2}\right\}.\tag{15}$$

Furthermore, we have

$$\int_0^\infty fH^{m-}\varphi dx=\int_0^\infty\varphi H^{m+}fdx\tag{16}$$

provided f and φ vanish properly at $x=0$ and $x=\infty$. By making use of (13)–(16), we can easily see that (12) reduces to

$$I_l(m,m+1)=A^{m+1}\left\{(\xi_m+\eta_m)\frac{1}{A^m}I(m-1,m)+\zeta_m\right\},\tag{17}$$

or

$$\frac{I_l(m-1,m)}{A^m(m+\gamma-1)}+\frac{a}{(m+\gamma-1)^2}=\frac{I_l(m,m+1)}{A^{m+1}(m+\gamma)}+\frac{a}{(m+\gamma)^2}.\tag{18}$$

We note that this means that the quantity on either side of this equation is independent of m, and hence is equal to

$$a/(l+\gamma)^2,\tag{19}$$

since $I_l(l,l+1)$ is evidently zero (as $\overset{l+1}{\mathcal{H}}_{l,\gamma}\overset{l}{\mathfrak{F}}_{l,\gamma}=0$). Hence by using the value of A^{m+1} given by (13), we have the value of $I_l(m,m+1)$. Putting $m=0$ in the result, we have

$$I_l(0,1)=-\{1-(\gamma^2/(l+\gamma)^2)\}^{\frac{1}{2}}.\tag{20}$$

Equation (10) is then obtained by solving (4) for $l+\gamma$ and substituting the result into (20).

The author wishes to thank Professor Infeld for suggesting the problem and for his help.

[1] Bechert, Ann. d. Physik 6, 700 (1930). See also Bethe, *Handbuch der Physik* (1933), Vol. 24, p. 315.

[2] L. Infeld, Phys. Rev. 59, 737 (1941). The notations used here are the same as those used in that paper.

Note on the "Kepler Problem" in a Spherical Space, and the Factorization Method of Solving Eigenvalue Problems

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April 30, 1941

Schrödinger[1] has developed an elegant "factorization" method of solving certain eigenvalue problems, an improved version of which has recently been given by Infeld.[2] The only problem treated by these authors which had not previously been considered and solved by other more conventional methods is the "Kepler problem" in a spherical space. As Schrödinger stated that he found this problem "difficult to tackle in any other way," it may perhaps be of interest to indicate briefly how the solution may be obtained without too great complication by conventional methods. The differential equation can, in fact, easily be transformed into a standard type, but the nature of the singularities of this transformed equation makes the discussion a little different from usual (explicit use is made of the *continuity*, as well as the boundedness, of the solution). It may also be opportune to offer a few remarks on the applicability of the factorization method in general.

The problem referred to leads to the equation ((4.3) of reference 1, or (7.1) of reference 2)

$$\frac{d}{d\chi}\left(\sin^2\chi\frac{dy}{d\chi}\right)+[\lambda\sin^2\chi+2\mu\sin\chi\cos\chi-l(l+1)]y=0,\tag{1}$$

where λ is the eigenvalue parameter, μ a given constant, and $l=0,1,2,\cdots$.[3] A solution which is bounded and continuous in the interval $0\leqslant\chi\leqslant\pi$ is required. The substitution $x=\cot\chi$ transforms (1) into

$$\frac{d^2y}{dx^2}+\left[\frac{\lambda+2\mu x}{(1+x^2)^2}-\frac{l(l+1)}{1+x^2}\right]y=0\tag{2}$$

and the fundamental interval is now $-\infty\leqslant x\leqslant+\infty$. Equation (2) has regular singularities at the points

$x=i,\ -i,\ \infty$, and since these are the *only* singularities, it may be solved in terms of hypergeometric functions.[4] A solution, valid for large $|x|$, which remains bounded at ∞, is

$$y=(x-i)^{-l-\alpha}(x+i)^{\alpha}F(l+\alpha+\alpha^*,\ l+1+\alpha-\alpha^*,\ 2l+2,\ -2i/(x-i)),\quad (3)$$

where F denotes the hypergeometric function, α is a root of the equation

$$\alpha^2-\alpha=(\lambda-2i\mu)/4 \quad (4)$$

and α^* is the conjugate complex of α. The second solution behaves like $|x|^{l+1}$ at ∞, and must be discarded.

The series occurring in (3) is convergent if x lies within the circle C, defined by $|x-i|=2$. Because of the singularities at $\pm i$, however, the analytic continuation of (3) along the real axis *inside* C is not in general a continuous function. A *sufficient* condition for the existence of an acceptable solution is that the series in (3) should terminate. This requires the relations

$$l+\alpha+\alpha^*=-n',\quad n'=0,\ 1,\ 2,\ \cdots$$

or, from (4),

$$\lambda=n^2-1-\mu^2/n^2,\quad n=n'+l+1$$

in agreement with the eigenvalues found by Schrödinger and Infeld. To show that the condition is also necessary, we must examine the explicit form of the analytic continuation of (3) inside C. This may readily be done, since relations between the various solutions of the hypergeometric equation are completely known.[4] We thus find that the solution defined by (3) is, indeed, discontinuous at $x=0$ along the real axis, unless the above condition is satisfied.

The (un-normalized) eigenfunctions may be written

$$y_{n'}=\sin^l\chi e^{-\mu\chi/n-in'\chi}F(-n',\ l+1+i\mu/n\ \ 2l+2,\ 1-e^{2i\chi}).$$

These eigenfunctions are real in spite of their apparent complex form as may be shown by utilizing known relations of the hypergeometric function.

The general theory underlying the factorization method has been investigated by Coleman.[5] Although the analysis is not quite complete, it is very probable that the method is, in practice, restricted to certain equations of hypergeometric or confluent hypergeometric[6] type, or equations reducible to these by simple substitutions. But this does not, of course, detract from the elegance of the method in those cases where it *is* applicable (which include most of the soluble problems of quantum mechanics). It is, indeed, remarkable that eigenvalue problems associated with equations whose complete analysis is fairly complicated, can be solved in such a simple manner.

I wish to thank Professor Infeld for letting me see his paper in advance of publication, and for interesting discussion. I am also indebted to Mr. Coleman for communicating to me his results before publication.

[1] E. Schrödinger, Proc. Roy. Irish Acad. **46A**, 9 (1940).
[2] L. Infeld, Phys. Rev. **59**, 737 (1941).
[3] The results hold, however, for *any* $l \geq 0$.
[4] See, for instance, E. T. Whittaker, and G. N. Watson, *Modern Analysis* (Cambridge, 1927), Chapter 14.
[5] A. J. Coleman, not yet published.
[6] Reference 4, Chapter 16.

Distinction between Longitudinal and Transverse Mesons

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COSMIC-RAY evidence shows that the cross section for the scattering of mesons by nucleons is much smaller than that calculated from the meson theory of nuclear force fields, while the interaction of mesons of unit spin with photons of high energy seems also to be too strong to account for the observed soft secondaries of the hard component. We want to propose a way of removing these discrepancies in the vector meson theory.

If one adopts the single force hypothesis proposed by Bethe[1] and leaves out the contribution from the processes in which Heisenberg's condition[2]

$$|(\mathbf{p}_1-\mathbf{p}_2)^2-\frac{1}{c^2}(E_1-E_2)^2|\leqq(a\mu c)^2 \quad (1)$$

(μ being the mass of a meson, a a number of order unity) for the applicability of the present quantum theory is not satisfied, the cross section for the scattering of the longitudinally polarized mesons by nucleons is at most of the order of 10^{-28} cm^2, while the cross section for the transversely polarized mesons of high energy is roughly

$$a^2\pi(g^2/\mu c^2)^2\sim a^2\times10^{-27}\ \text{cm}^2. \quad (2)$$

Moreover, the transition probability for mesons from the longitudinal to the transverse state, which is mostly due to the process in which charged mesons are scattered in the electrostatic field of atoms, is very small, the cross section being given by

$$2\pi(e^2/\mu c^2)^2Z^2\log(137)^2\sim Z^2\times10^{-28}\ \text{cm}^2. \quad (3)$$

The above results lead to the conclusion that transverse mesons, if created, are largely scattered in the upper atmosphere, and that the greater part of the hard component found at sea level consists of longitudinal mesons. If we adopt the symmetrical theory of nuclear forces, the cross section for the transition of a transverse meson from the charged to the neutral state is also given by (2). This behavior of transverse mesons seems to favor the interpretation of the cloud-chamber experiments[3] made in the upper atmosphere. The very penetrating component which persists under thickness of matter might consist primarily of longitudinally polarized neutral mesons.

We have also examined the radiative processes of longitudinal and transverse mesons separately. We have calculated the differential cross section for the collision of a longitudinal meson with a photon of equal and opposite momentum and found that its highest term in powers of the energy E of the incident photon is given by

$$\frac{5}{96}\left(\frac{e^2}{\mu c^2}\right)^2\left(\frac{E}{\mu c^2}\right)^2(1-\cos\theta)^2d\Omega \quad (4)$$

where θ is the scattering angle. This term has a very small value in the region where condition (1) is satisfied, whereas such a term as $(1-\cos\theta)^2$ does not appear in the corresponding expression for the transverse mesons or in the cross section for the creation of meson pairs by γ-rays.