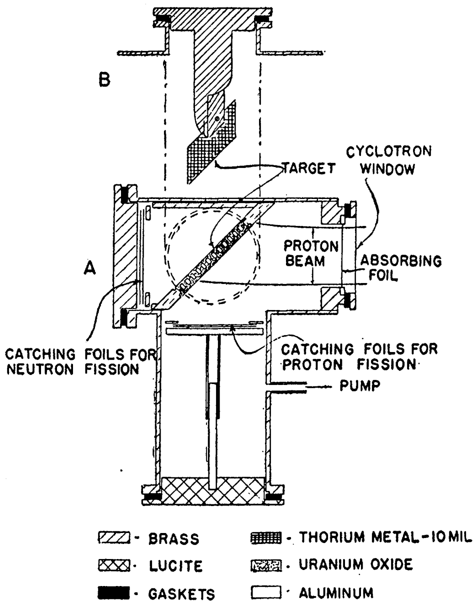

FIG. 1.

in any desired position. In a side tube (which was designed to accommodate a proportional counter) projecting at right angles from the body of the chamber, and well removed from the beam, foils were mounted so as to catch fission fragments emitted from the target surface. A 0.1-mil aluminum foil in front of the catching foils was found sufficient to retain all ordinary radioactive recoil atoms. Aluminum foils placed behind the catchers served to correct for radioactivities induced in them by neutrons or scattered protons.

A 10-mil sheet of thorium, which was found to be thick enough to stop the proton beam and proton-induced fission particles, was oriented so that its normal made angles of 45 degrees with both the beam and the catchers' normal. There are two such positions. In position I (Fig. 1B) the irradiated surface of the target faces the catcher; in position II, the other surface does. The amount of fission fragments caught in position II will be approximately the same as that fraction caught in position I that is due to fission induced by the neutrons from the cyclotron and the target. After a bombardment of 5 μAh in 1.5 hours, radioactivity was observed in a 1-mil aluminum catcher, whose decay could be followed for seven hours. A bombardment of 25 μAh in 8 hours yielded a sample which could be observed for over two and a half days. Identical bombardments with the target in position II gave no effect. When plotted on semi-logarithmic paper, the decay curves show the curvature characteristic of a mixture of many periods.

By replacing the 1-mil aluminum catcher with a stack of 0.1-mil aluminum foils it was found that the maximum range of fission fragments from ${}_{91}Pa^{233}$, the compound nucleus in question, lies between 0.5 and 0.6 mil of aluminum. Reducing the proton energy in successive steps by inserting aluminum absorbers just in front of the cyclotron window gave several points of the excitation function. The lowest proton energy for which we obtained an observable effect was 5.8 Mev. Had our thorium sheet been large enough to insure that no fraction of the proton beam would fail to strike it, we could have collected the background-induced fission fragments behind the target, without turning it into position II and repeating the bombardment.

In the case of uranium, however, we proceeded in this more direct manner. Figure 1A shows the position of a second catcher behind the target. It also shows the elliptic perforated brass plate which was packed and coated with uranium oxide embedded in pyroxylin. The brass provided sufficient cooling, and the oxide-filled channels gave the surface coats a higher stability. Thus we were able to find in a single run that uranium gives an effect (fission of 93^{239}) of the same order of magnitude as that found in thorium, and that the background effect is negligible.

Chemical investigations and experiments employing an ionization chamber for more accurate yield and range determinations are under way.

[1] I. C. Jacobsen and N. O. Lassen, Phys. Rev. **58**, 867 (1940).
[2] D. H. T. Gant, Nature **144**, 707 (1939).
[3] Niels Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).

Note on the Normalization of Dirac Functions

C. C. LIN
Department of Applied Mathematics, University of Toronto, Canada
April 18, 1941

A VERY simple method of normalizing the Dirac functions in the Kepler problem is given in this note. The normalization of Dirac functions in the Kepler problem was first done by Bechert[1] with the help of certain contour integrals in a complex plane. The calculation was, however, rather complicated. With the functions given in Infeld's form, a much simpler method of normalization leading to a neater result is proposed here. The result is, as I verified, the same as that given by Bechert, but it appears in a completely different form.

It has been shown[2] that the radial Dirac functions are given by

$$\chi_1 = C\left\{ (\epsilon\kappa-\gamma)^{\frac{1}{2}}\frac{\gamma_2+\gamma_1}{\gamma}\overset{0}{\mathfrak{F}}_{l,\gamma}(x) + (\epsilon\kappa+\gamma)^{\frac{1}{2}}\frac{\gamma_2-\gamma_1}{\gamma}\overset{1}{\mathfrak{F}}_{l,\gamma}(x)\right\} \quad (1)$$

$$\chi_2 = C\left\{ (\epsilon\kappa-\gamma)^{\frac{1}{2}}\frac{\gamma_2-\gamma_1}{\gamma}\overset{0}{\mathfrak{F}}_{l,\gamma}(x) + (\epsilon\kappa-\gamma)^{\frac{1}{2}}\frac{\gamma_2+\gamma_1}{\gamma}\overset{1}{\mathfrak{F}}_{l,\gamma}(x)\right\} \quad (2)$$

with the following definitions and formulae (C being the normalization factor):

$$\gamma_1=(\kappa-\alpha Z)^{\frac{1}{2}}, \quad \gamma_2=(\kappa+\alpha Z)^{\frac{1}{2}}, \quad \gamma=\gamma_1\gamma_2=(\kappa^2-\alpha^2Z^2)^{\frac{1}{2}}, \quad (3)$$

$$\epsilon=\frac{E}{E_0}=\left\{1+\frac{\alpha^2Z^2}{(l+\gamma)^2}\right\}^{-\frac{1}{2}}, \quad (4)$$

where l is a positive integer, or zero (only for $\kappa<0$), and κ is an integer, positive or negative, but not zero; $\overset{1}{\mathfrak{F}}_{l,\gamma}$ and $\overset{0}{\mathfrak{F}}_{l,\gamma}$ are the lowest two functions of a ladder of normalized functions defined by

$$\overset{l}{\mathfrak{F}}_{l,\gamma}(x)=\left(\frac{2a}{\gamma+l}\right)^{\gamma+l+\frac{1}{2}}\cdot\frac{1}{[\Gamma(2\gamma+2l+1)]^{\frac{1}{2}}}x^{l+\gamma}e^{-ax/(l+\gamma)},\tag{5}$$

$$\overset{m}{\mathfrak{F}}_{l,\gamma}(x)=\frac{(l+\gamma)(m+\gamma)}{a[(l+m+2\gamma)(l-m)]^{\frac{1}{2}}}\times\left(\frac{m+\gamma}{x}-\frac{a}{m+\gamma}+\frac{d}{dx}\right)\overset{m+1}{\mathfrak{F}}_{l,\gamma}(x),\tag{6}$$

where m is an integer, $0\leqslant m\leqslant l$, and

$$a=\frac{m_0c}{\hbar}\epsilon\alpha Z.\tag{7}$$

As C is the normalization factor and the functions $\overset{0}{\mathfrak{F}}_{l,\gamma}$, $\overset{1}{\mathfrak{F}}_{l,\gamma}$ are normalized, we obtain from (1), (2) and (3) the condition for C to make $\int_0^\infty(\chi_1^2+\chi_2^2)dx=1$:

$$1=\frac{8C^2}{\gamma^2}\{\epsilon\kappa^2+(\epsilon^2\kappa^2-\gamma^2)^{\frac{1}{2}}\alpha ZI_l(0,1)\},\tag{8}$$

where

$$I_l(0,1)=\int_0^\infty\overset{0}{\mathfrak{F}}_{l,\gamma}(x)\overset{1}{\mathfrak{F}}_{l,\gamma}(x)dx.\tag{9}$$

It will be shown below that

$$I_l(0,1)=-(\epsilon^2\kappa^2-\gamma^2)^{\frac{1}{2}}/\alpha Z\epsilon,\tag{10}$$

and hence, because of (8), we have

$$C=\epsilon^{\frac{1}{2}}/2\sqrt{2}.\tag{11}$$

This is the normalization factor required.

To calculate $I_l(0,1)$, let us consider

$$I_l(m,m+1)=\int_0^\infty\overset{m}{\mathfrak{F}}_{l,\gamma}\overset{m+1}{\mathfrak{F}}_{l,\gamma}\,dx.\tag{12}$$

By writing

$$\overset{m\mp}{\mathfrak{H}}_{l,\gamma}=\overset{m}{A}_{l,\gamma}\overset{m\mp}{H}_{l,\gamma}=\overset{m}{A}_{l,\gamma}\left(\frac{m+\gamma-1}{x}-\frac{a}{m+\gamma-1}\mp\frac{d}{dx}\right),$$
$$\overset{m}{A}_{l,\gamma}=\frac{(l+\gamma)(m+\gamma-1)}{a[(l+m-1+2\gamma)(l-m+1)]^{\frac{1}{2}}}\tag{13}$$

it is not difficult to verify that

$$H^{(m+1)-}=\xi_mH^{m-}+\eta_mH^{m+}+\zeta_m,\tag{14}$$

where ξ_m, η_m, ζ_m are constants given by

$$\xi_m+\eta_m=\frac{m+\gamma}{m+\gamma-1},\quad\xi_m-\eta_m=1,$$
$$\zeta_m=a(m+\gamma)\left\{\frac{1}{(m+\gamma-1)^2}-\frac{1}{(m+\gamma)^2}\right\}.\tag{15}$$

Furthermore, we have

$$\int_0^\infty fH^{m-}\varphi dx=\int_0^\infty\varphi H^{m+}fdx\tag{16}$$

provided f and φ vanish properly at $x=0$ and $x=\infty$. By making use of (13)–(16), we can easily see that (12) reduces to

$$I_l(m,m+1)=A^{m+1}\left\{(\xi_m+\eta_m)\frac{1}{A^m}I(m-1,m)+\zeta_m\right\},\tag{17}$$

or

$$\frac{I_l(m-1,m)}{A^m(m+\gamma-1)}+\frac{a}{(m+\gamma-1)^2}=\frac{I_l(m,m+1)}{A^{m+1}(m+\gamma)}+\frac{a}{(m+\gamma)^2}.\tag{18}$$

We note that this means that the quantity on either side of this equation is independent of m, and hence is equal to

$$a/(l+\gamma)^2,\tag{19}$$

since $I_l(l,l+1)$ is evidently zero (as $\overset{l+1}{\mathfrak{H}}_{l,\gamma}\overset{l}{\mathfrak{F}}_{l,\gamma}=0$). Hence by using the value of A^{m+1} given by (13), we have the value of $I_l(m,m+1)$. Putting $m=0$ in the result, we have

$$I_l(0,1)=-\{1-(\gamma^2/(l+\gamma)^2)\}^{\frac{1}{2}}.\tag{20}$$

Equation (10) is then obtained by solving (4) for $l+\gamma$ and substituting the result into (20).

The author wishes to thank Professor Infeld for suggesting the problem and for his help.

[1] Bechert, Ann. d. Physik **6**, 700 (1930). See also Bethe, *Handbuch der Physik* (1933), Vol. 24, p. 315.
[2] L. Infeld, Phys. Rev. **59**, 737 (1941). The notations used here are the same as those used in that paper.

Note on the "Kepler Problem" in a Spherical Space, and the Factorization Method of Solving Eigenvalue Problems

A. F. Stevenson
Department of Applied Mathematics, University of Toronto, Toronto, Canada
April 30, 1941

Schrödinger[1] has developed an elegant "factorization" method of solving certain eigenvalue problems, an improved version of which has recently been given by Infeld.[2] The only problem treated by these authors which had not previously been considered and solved by other more conventional methods is the "Kepler problem" in a spherical space. As Schrödinger stated that he found this problem "difficult to tackle in any other way," it may perhaps be of interest to indicate briefly how the solution may be obtained without too great complication by conventional methods. The differential equation can, in fact, easily be transformed into a standard type, but the nature of the singularities of this transformed equation makes the discussion a little different from usual (explicit use is made of the *continuity*, as well as the boundedness, of the solution). It may also be opportune to offer a few remarks on the applicability of the factorization method in general.

The problem referred to leads to the equation ((4.3) of reference 1, or (7.1) of reference 2)

$$\frac{d}{d\chi}\left(\sin^2\chi\frac{dy}{d\chi}\right)+[\lambda\sin^2\chi+2\mu\sin\chi\cos\chi-l(l+1)]y=0,\tag{1}$$

where λ is the eigenvalue parameter, μ a given constant, and $l=0, 1, 2, \cdots$.[3] A solution which is bounded and continuous in the interval $0\leqslant\chi\leqslant\pi$ is required. The substitution $x=\cot\chi$ transforms (1) into

$$\frac{d^2y}{dx^2}+\left[\frac{\lambda+2\mu x}{(1+x^2)^2}-\frac{l(l+1)}{1+x^2}\right]y=0\tag{2}$$

and the fundamental interval is now $-\infty\leqslant x\leqslant+\infty$. Equation (2) has regular singularities at the points