# Calculation of the Torque on a Ferromagnetic Single Crystal in a Magnetic Field

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When a disk cut from a cubic crystal of a ferromagnetic material is placed in a magnetic field parallel to its plane, the torque exerted on the disk by the field generally increases continually with the strength of the field, approaching a finite limit (saturation) in very high fields. An "anomaly" for one orientation of a crystal of iron-silicon alloy has recently been observed by Tarasov. We have confirmed his findings and observed a similar effect for a series of orientations of disks cut parallel to the (110) and (100) planes. As the field strength is increased, the torque passes through a well-defined maximum before beginning the final approach to saturation. This anomalous peak is now explained in terms of the domain theory; no new concepts are necessary. A graphical method is described for calculating torque and magnetization curves for any direction of the applied field. Torque curves depend markedly on the demagnetizing factor of the disk and, in many cases, a high demagnetizing factor prevents observation of a maximum in the curve of torque *vs.* field strength.

WHEN a disk cut from a single crystal of a ferromagnetic material is placed in a magnetic field, it is generally subject to a torque on account of the anisotropic character of the crystal, even when the crystal is cubic. The anisotropy constant of the crystal is probably best determined by measurement of this torque. In most experiments hitherto performed the torque increases continually with field strength and approaches a finite value simply related to a single anisotropy constant, and the observed relation between torque and field strength agrees with calculation.

Recently, however, Tarasov[1] has found that under certain conditions the torque passes through a maximum before declining to its limiting value in high fields. The purpose of this note is to show that this supposedly anomalous behavior is in accord with theory, and to give a method of calculating torque curves for any orientation of the crystal with respect to the field. Theory predicts, and our experiments show, that a maximum occurs in the torque *vs.* field strength curve for most crystal orientations provided the demagnetizing factor of the specimen is small. In order to have a small demagnetizing factor our experiments were performed with a single crystal disk 22 mm in diameter and 0.22 mm thick.

[1] L. P. Tarasov, Phys. Rev. **56**, 1224 (1939).

## SIMPLE TORQUE CURVES

When the specimen of magnetic material, disk or oblate ellipsoid, is cut with its principal plane parallel to the (100) plane of a single crystal, the equations for calculating the torque per unit volume $L$, and the magnetization $I$, in the presence of a field of strength $H$, are derived from the expression for the energy density $E$. The energy density is composed of two parts, $E_K$ due to the magnetic anisotropy of the crystal and $E_H$ due directly to the presence of the field.

$$E = E_K + E_H = K_1(s_1^2 s_2^2 + s_2^2 s_3^2 + s_3^2 s_1^2) - H I_s \cos\theta. \quad (1)$$

Here $K_1$ is the anisotropy constant of the cubic crystal, the $s$'s are the direction cosines of the local magnetization, with respect to the three crystal axes and $\theta$ is the angle between the direction of the field and that of the local magnetization. In accordance with the domain theory it is assumed that each domain is magnetized to saturation ($I = I_s$); the torque acting on any domain is then

$$L = -dE_K/d\theta \quad (2)$$

ergs/cm³ and the torque on the whole specimen is the resultant of the torques on all of the domains. The strength of the field is obtained by minimizing $E$ and is

$$H = -\frac{dE_K/d\theta}{I_s \sin\theta}. \quad (3)$$

When the specimen is demagnetized the resultant is zero. When the field is increased in strength the directions of magnetization in most of the domains are changed by displacement of the boundaries between domains, until they are directed along that direction of easy magnetization that is nearest to the direction of the field; when this process is near completion the torque first attains appreciable magnitude, and the calculations of this article begin at this point.

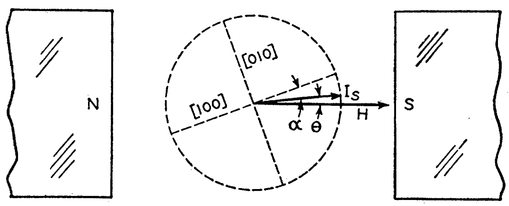


FIG. 1. Single crystal disk in a magnetic field showing directions of easy magnetization in the (001) plane, and nonparallelism of $H$ and $I_s$.

When all of the domains are similarly oriented Eqs. (1) to (3) are easily applied. In the (001) plane (see Fig. 1) Eqs. (2) and (3) reduce to

$$L=K_1\,(\sin 4\alpha)/2,$$
$$H=K_1\,\sin 4\alpha/2I_s\,\sin\theta$$

in which $\alpha$ is the angle between $I_s$ and the nearest crystal axis [100].

The torque in this plane is greatest when $\alpha$ is 22.5°. $H$ and $L$ have been calculated using the above equations with $\alpha+\theta=22.5°$, the field being applied always 22.5° from a crystal axis. In any experiment the strength of the true field $H$ is not equal to that of the applied field, $H_a$, because of the demagnetizing field, $\Delta H$, which is determined by the magnetization and dimensions of the specimen. For comparison with experiment 160 oersteds (the demagnetizing field for saturation) were added to $H$ to obtain $H_a$, and $L$ was plotted against $H_a$ as shown in Fig. 2. The experimental curve shown for comparison was obtained using a very thin specimen cut and ground parallel to a (001) plane of a crystal containing 3.6 percent silicon and the remainder iron, prepared as described by Williams.[2] The ratio, $m$, of diameter (22 mm) to thickness (0.22 mm) was 100, and the corresponding demagnetizing factor is $N=0.10$.

[2] H. J. Williams, Phys. Rev. **52**, 747–751 (1937).

The demagnetizing field when the magnetization is near saturation is thus $NI_s=160$. Both experimental and theoretical curves are consistent with an anisotropy constant $K_1=350{,}000$ ergs cm$^{-3}$, in fair agreement with results of Tarasov.[3]

The agreement between theory and experiment indicates that for this simplest case the theory is good.

## General Case

Lines of constant crystal energy, $E_K$, are plotted in stereographic projection in Fig. 3, using Eq. (1). The angles $\phi$ and $\psi$ are latitude and longitude and the spherical triangle shown includes all points of the complete sphere that are nonequivalent crystallographically in a cubic crystal. Each point on the projection represents a direction determined by the corresponding point on the sphere, and the center of the sphere.

This diagram can be used to find the torque on a crystal when placed in a field having the direction indicated by any point $H$ on the diagram, as shown in Fig. 4. When the field is weak the point representing the magnetization of the domain $I_s$ will be at [100], the nearest direction of easy magnetization, and when the field strength is increased the rotation of $I_s$ will be represented by some line going from [100] to $H$. The path followed by $I_s$ will be straight if $H$ lies on the line between [100] and [110], or on the line between [100] and [111], and the

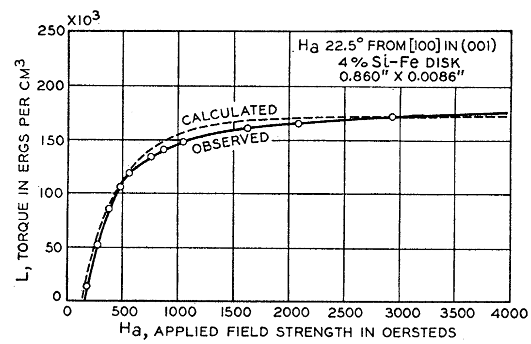


FIG. 2. Torque/volume *vs.* strength of applied field. Solid line is observed relation, dashed line the relation calculated from simple domain theory using the previously known value of the anisotropy constant $K_1=350{,}000$. The intercept on the $H_a$ axis is the demagnetizing field determined by the dimensional ratio of the disk ($NI_s=160$).

[3] L. P. Tarasov, Phys. Rev. **56**, 1231–1240 (1939).

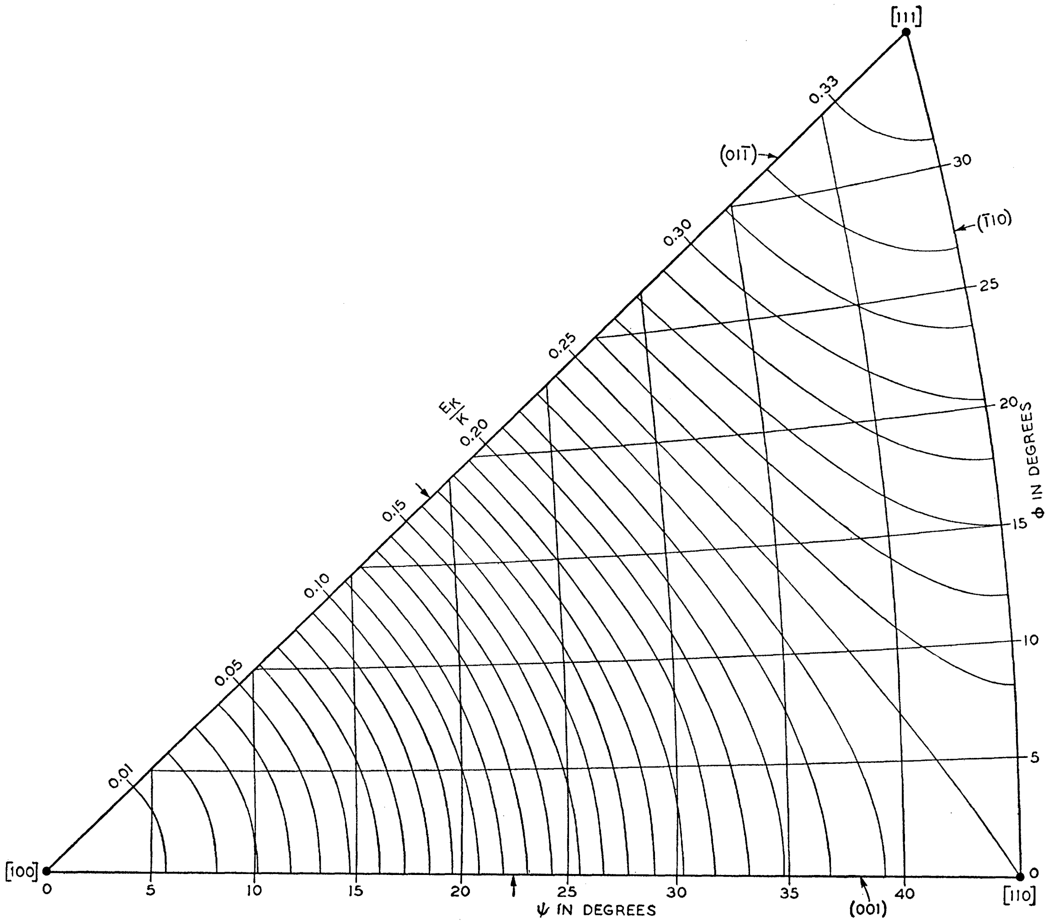


FIG. 3. Lines along which the density of the energy of magnetic anisotropy $E_k$ is constant. [See Eq. (1).] Stereographic projection of a cubic crystal with scales of latitude $\varphi$ and longitude $\psi$.

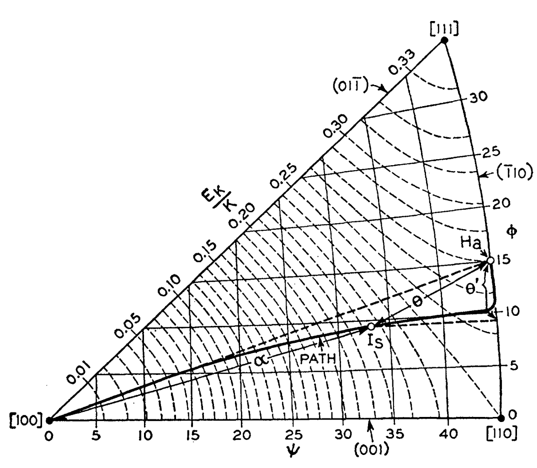


FIG. 4. The position of the magnetization vector $I_s$ of a domain is determined graphically as shown here for a given orientation of the field $H$ and angle $\theta$ between $H$ and $I_s$.

calculation of the torque can be made most easily in these cases by using the equations that have been used repeatedly for this purpose.[4] If $H$ is applied in any other direction the path followed by $I_s$ will be curved and may be determined graphically in a way to be described.

Whatever the path that the end of the $I_s$ vector follows, it is evident that a maximum in $L$ may be expected when $I_s$ lies in the region in which the lines of constant crystal energy are closest together, because $L$ is the derivative of $E_K$ with respect to $\theta$. In Fig. 3 the region in which the equipotential lines are closest together is marked with small arrows in the (001) and (01$\bar{1}$) planes. A maximum in the $L$ *vs.* $H$ curve

[4] E.g. by R. M. Bozorth, Phys. Rev. **50**, 1076–1081 (1936).

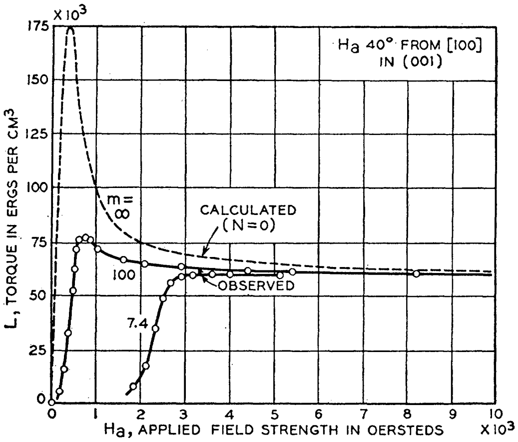


FIG. 5. Curves showing torque *vs.* field strength for disk parallel to (001) plane. Theoretical curve is for infinitely thin disk; observed curves are for disks having ratio of diameter to thickness of 100 and 7.4.

is thus to be expected when $H$ and [100] lie on opposite sides of this region. For example, in the (001) plane the field must "pull" the $I_s$ vector over the place ($\psi=22.5°$) where $L$ is a maximum, and the same maximum value of $L$ occurs in the $L$ *vs.* $H$ curves for each position of $H$ between $\psi=22.5°$ and $\psi=45°$. It may thus be determined that for over 80 percent of all of the possible orientations of $H$ with respect to the crystal axes, $L$ passes through a maximum. As mentioned below, the existence of too large a demagnetizing factor may in some cases cause the maximum to disappear.

To illustrate the determination of the path and of the $L$ *vs.* $H$ curves, two cases will be considered for each of which the torque passes through a maximum and declines to its final value. In the first case $H$ is applied in the (001) plane 40° from the [100] direction ($\phi=0°$, $\psi=40°$), in the second it is applied in the ($\bar{1}10$) plane 75°, 70° or 65° from the [001] direction ($\phi=15°$, 20° or 25°, $\psi=45°$). In the first case the path is obviously a straight line, in the second case the way in which the path is to be determined may be outlined as follows. Draw circles of various diameters about the point $H$ and note the points at which these circles are tangent to the equipotential lines. At these points the energy $E_K+E_H$ is a minimum and the locus of such points is the required path. Along this path $E_K$ is plotted against $\theta$, the angle between $I_s$ and $H$, and the slope of this curve is $-L$ as given by Eq. (2). $H$ is found by dividing $L$ by $I_s \sin\theta$ in accordance with Eq. (3). Paths and curves so determined are shown in what follows:

## (001) PLANE

In the (001) plane, with $H$ directed 40° from [001], the appropriate equations are Eqs. (2) and (3) with

$$\theta=40°-\alpha.$$

The curve so calculated with $K_1=350{,}000$ is shown in Fig. 5.

For comparison with theory data were taken on the thin crystal described above. The observed curve shows the expected maximum and a limiting value in high fields that agrees with the calculated limit. However, the height of the observed maximum lies considerably below the theoretical one. This may be attributed to the action of the demagnetizing field, and this point of view is substantiated by the fact that no maximum at all was observed when a much thicker disk was used having a ratio of diameter to thickness of $m=7.4$.

The way in which the demagnetizing field lowers the value of $L$ in intermediate fields is shown in Fig. 6. The applied field $H_a$ has turned the magnetization vector $I_s$ through the angle $\alpha$ from the nearest direction of easy magnetization [100]. In this position $I_s$ gives rise to an antiparallel demagnetizing field, $\Delta H$, which adds vectorially to $H_a$ to form the true field, $H$. Now $H$ will be nearer to the [110] direction than will $H_a$; consequently, if $\Delta H$ or the angle between $H_a$ and [100] is large enough, the true

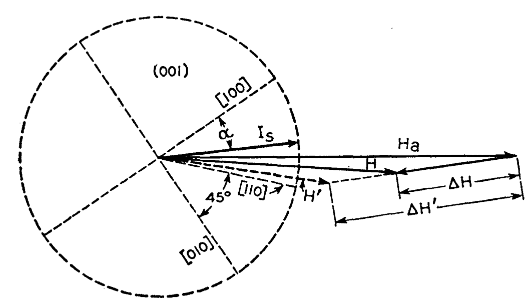


FIG. 6. Orientations of vectors $I_s$ and $\mathbf{H}=\mathbf{H}_a-\mathbf{\Delta H}$ in (001) plane, to illustrate effect of demagnetizing field $\Delta H$ on the direction of $H$ and the magnitude of the torque.

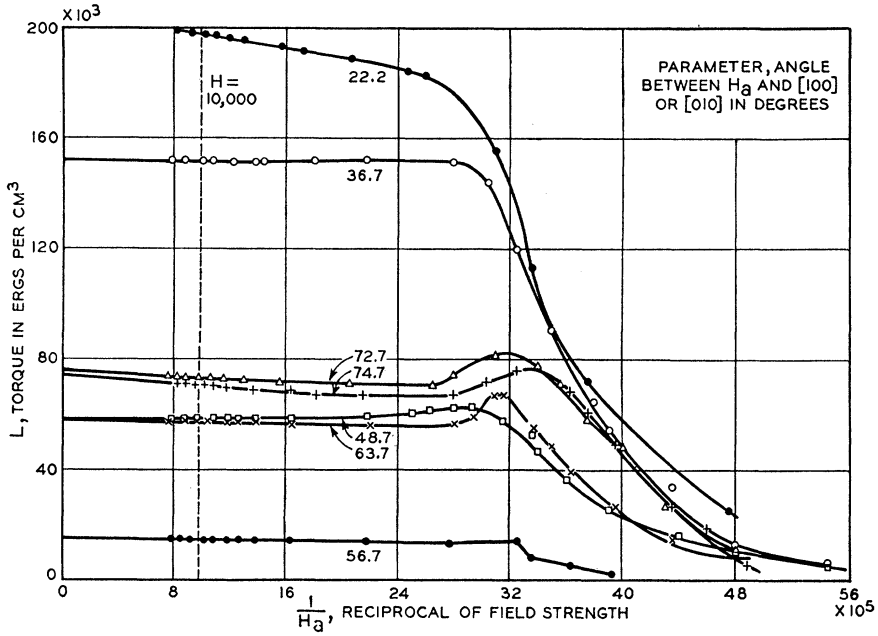


FIG. 7. Curves of torque *vs.* reciprocal of applied field strength for various directions of the field applied in the (001) plane, showing ease of extrapolation to $H_a = \infty$.

field will lie almost parallel to [110] and some of the domains will spring to another equally stable position lying between [110] and [010]. These domains will have a torque opposite in sign to those lying on the other side of [110], and the resultant of the torques of all the domains will be considerably lowered. The thicker the crystal or the smaller its diameter the greater will be $\Delta H$ and the smaller the torque in all except very intense fields. Since the crystal used had a ratio of diameter to thickness as high as 100, it is not surprising that such a maximium has never been observed before.

Other data for this specimen for other positions of the field with respect to the crystal axes are shown in Fig. 7. Here $L$ is plotted *vs.* the reciprocal of $H_a$ in the way suggested by Schlechtweg[5] and used by him and Tarasov.[1] It may be noted that, in general, the portions of the curves for which $H_a$ is largest are not as free from curvature as those published by Tarasov, but this method of plotting is nevertheless a convenient one for extrapolating to infinite field strength.

## (110) PLANE

When a disk is cut parallel to a (110) plane, and magnetized in a direction lying between adjacent [100] and [111] directions, the calculation of $L$ and $H$ can be made by the method often described.[4] But when $H$ lies between adjacent [110] and [111] directions the problem is complicated by two factors: The path followed by $I_s$ is curved, and there are two equidistant directions of easy magnetization. Heretofore the torque in the (110) plane has been calculated for all azimuths by assuming that the magnetization vector of each domain is confined to the plane by the high demagnetizing field in the direction perpendicular to the plane; but this assumption is invalid because the magnetization in half of the domains may have a considerable component perpendicular to the plane in one direction and the other half an equal component in the opposite direction so that the over-all magnetization is confined to the plane.

In the illustrative example to follow it is assumed that in the weaker fields the magnetization is equally divided between the two directions [100] and [010]. Calculation of the path of the vector rotating from just one of these positions is, of course, sufficient because the path of the other is the mirror image of the first.

Figure 4 shows the graphical construction of the path of the vector, $I_s$, as it rotates from [100] to $H$ which is located at $\phi = 15°$, $\psi = 45°$. The contour lines are the same lines of constant

[5] H. Schlechtweg, Ann. d. Physik [5] 27, 573 (1936).

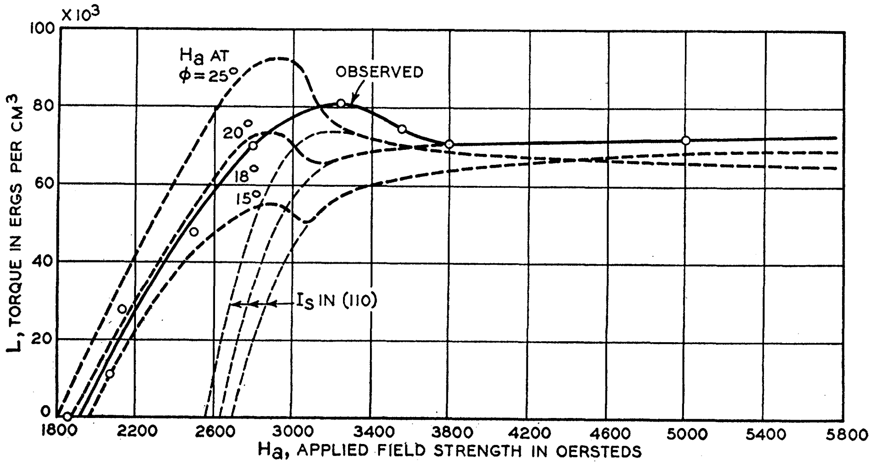


FIG. 8. The observed relation between torque and applied field strength (solid line) as compared with the calculations based on straight-forward domain theory (heavy dashed lines) or on the older method of calculation (light dashed lines), using a demagnetizing factor $N=1.75$ appropriate for the crystal measured ($m=4.0$).

crystal energy as those of Fig. 3. Arcs of circles are drawn, each point on which represents a vector making the same angle $\theta$ with the field, $H$. The point of tangency of this circle with a line of constant crystal energy is a point on the required path followed by $I_s$, for the crystal energy is constant along the one line and the energy due to the field, $HI_s\cos\theta$, is constant along the circle and the sum of the two energies is thus a minimum at the point of tangency. Arcs for other values of $\theta$ determine points of minimum energy for other positions of $I_s$ corresponding to other field strengths, and the locus of such points is the required path of the vector, $I_s$.

Lines of constant $\theta$ are circles on the stereographic projection as well as on the sphere, but the centers of the circles on the projection are not at the projection of the center of the circles. The center of the projection can be found simply in the following way. Draw a straight line from [100] through $H$. Determine the angular distance, $\alpha_0$, from [100] to $H$ by laying off from [100] along the projection of the (001) plane a linear distance equal to that from [100] to $H$ and reading the angle on the $\psi$ scale. On the straight line passing through [100] and $H$, mark off the angles $\alpha_0+\theta$ and $\alpha_0-\theta$, using the same scale. These two points are at opposite ends of a diameter of the required circle which may then be easily drawn. The distance $l$ on the projection from [100] to any point $\psi=\psi_1$, $\phi=0$, is given by

$$l=l_0\tan(\psi_1/2),$$

where $l_0$ is the radius of the projected sphere.

The next procedure is to plot the crystal energy $E_K$ as a function of $\theta$, the angle between $I_s$ and $H$, for all points on the path of $I_s$. In accordance with Eqs. (2) and (3) the slope of this function determines the torque $L_1$ and the field strength $H$:

$$L_1=-dE_K/d\theta,$$

$$H=-\frac{dE_K/d\theta}{I_s\sin\theta}=\frac{L_1}{I_s\sin\theta}$$

for all domains originally parallel to [100]. The torque due to all of the domains, those originally parallel to [100] and to [010], is best determined by finding the magnitude and direction of the resultant $I$ of all of the domains and multiplying this by the sine of the angle, $\theta'$, between this and $H$. Thus, the torque on each cubic centimeter of specimen is

$$L=HI_s\cos\beta\sin\theta',$$

where $\beta$ is the angle between $I_s$ and the (110) plane measured along the great circle perpendicular to the (110) plane. Approximate values of the angles $\beta$ and $\theta'$ are easily read from Fig. 7; for the case under consideration calculations have shown that the angle $\beta$ is very nearly equal to the difference between 45° and the value of $\psi$ for the position of $I_s$, and $\theta'$ is nearly equal to the difference in the values of $\phi$ for $I_s$ and for $H$. If necessary a more accurate evaluation of $\beta$ and $\theta'$ can be made analytically, or graphically using a stereographic net.

Finally, the component of the resultant $I$ that is parallel to $H$ is

$$I = I_s \cos\theta.$$

This quantity is useful in determining the demagnetizing fields that must be calculated in order to compare the theoretical torque curves with the results of experiment.

The results of the calculation are shown in Fig. 8. Three different directions were chosen for the application of the field, those at 15°, 20° and 25° from the [110] direction in the $(\bar{1}10)$ plane ($\psi = 45°$, $\phi = 15°$, 20° and 25°). The heavy dashed lines were calculated as described above; the light dashed lines show the relation expected on the former assumption according to which the magnetization in each domain is confined to the plane of the disk. For comparing the theory with experiment the applied field, $H_a$, is calculated by adding to the true field, $H$, the demagnetizing field, $\Delta H$, appropriate for a disk having a ratio of diameter to thickness of 4.0. The experimental curve is shown in the same figure as a heavy solid line, and was obtained using a disk 0.8 cm in diameter and 0.2 cm thick which had its edges cut away to make the demagnetizing field more nearly that of an oblate ellipsoid of the same dimensional ratio. The demagnetizing factor used in the calculations was $N = 1.75$ and the demagnetizing field strength at saturation is therefore 2800. The other constants used were $I_s = 1600$, $K = 350{,}000$.

The calculated curves bear a close resemblance to those observed. It is to be expected that the calculations will not be exact, for no account has been taken of the deviation of the direction of the true field from that of the applied field, a deviation similar to that illustrated in Fig. 6 for the (100) plane.

It should be pointed out that the magnitude of the demagnetizing field, or the field at which the torque curve leaves the $H$ axis, is in itself strong confirmation of the truth of the assumption that the domains are directed initially along [100] and [010], rather than along [110] as formerly assumed. In the latter case the calculated torque curve begins to rise at 2800 cos20° or 2600 oersteds, in the former case at 2800 cos46° or 1900 oersteds, when $H_a$ is applied at $\phi = 20°$. The experimental curve shows an initial rise at 1900 to 2000 oersteds.

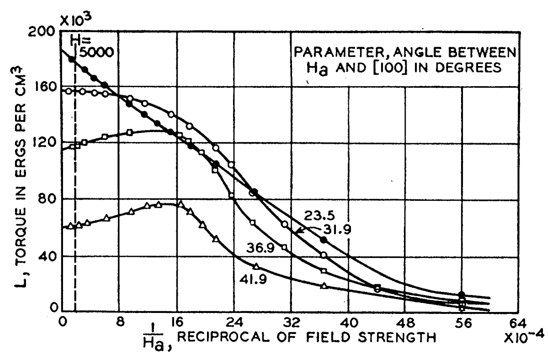


FIG. 9. Torque *vs.* reciprocal of applied field strength for various directions of the applied field in the $(\bar{1}10)$ or $(1\bar{1}0)$ plane (see Fig. 4).

Curves showing $L$ as dependent on $1/H$ are given in Fig. 9 for other directions of the applied field in the (110) plane.

It would appear that the simple linear relation observed by Tarasov[1] to exist between $L$ and $1/H_a$ at high values of $H_a$, has no general experimental or theoretical validity, but refers only under limited conditions of crystal orientation and specimen form.