varied about 50°F. It should also be mentioned that no "absolute" calibration of the apparatus by comparison with one whose residual ionization has been measured under an enormous shield has ever been made. However, the cumulative observations made with the apparatus and the magnitudes of the several "coefficients" obtained from the data it yields, have satisfied the writer that the radioactive contamination of the chamber must be very small.

While Monk and Compton consider the pulses likely due to a solar influence, they consider that the persistence of the pulse amplitude of the cosmic-ray fluctuations and the decrease in Chree's magnetic pulses with distance from the primary pulse indicate the two varieties of disturbance are of different origin. However, the similarities of the frequencies of the pulses obtained by the same method of analysis combined with Graziadei's[8] association of his approximately 0.4-percent cosmic-ray intensity fluctuations of 27.2-day periodicity with solar disturbances, indicate the desirability of further examination of this point.

[8] H. T. Graziadei, Akad. Wiss. Wien. [IIa] 145, 495 (1936).

MAY 15, 1941 PHYSICAL REVIEW VOLUME 59

Scattering in the Pair Theory of Nuclear Forces

J. W. WEINBERG
Department of Physics, University of California, Berkeley, California

(Received March 12, 1941)

The scattering of particles of spin $\frac{1}{2}$ and mesotronic mass m by heavy nuclear constituents is investigated with the pair theory of Critchfield and Teller, in which a nucleus appears as an extended source of mesotrons. A rigorous formula, (19), is found for the cross section, by obtaining explicitly the normal coordinates of the perturbed mesotron field. When the "constants" are adjusted to give a model of nuclear forces with approximately the right range and saturation properties, $\sigma \simeq 32\pi(mc^2/E)^4(\hbar/mc)^2$. Although, in other models, σ may decrease more rapidly with the energy E, it is always about 10^{-24} cm^2 at low energies, as contrasted with the cross section of about 10^{-28} cm^2 observed for cosmic-ray mesotrons. These results are discussed in relation to other accounts of the pair field theory.

CRITCHFIELD, Teller, Wigner and Lamb[1–4] have developed a field theory which explains, in a qualitative way, the saturation and spin dependence of nuclear forces. The field, when unperturbed by nuclei, is assumed to be a Fermi gas of charged particles with the mass of the mesotron and described by the Dirac equation. A proton or neutron interacts with this field by emission and absorption of neutral pairs of "mesotrons," and such processes can occur not only at the point which specifies the position of the nucleus, but over a finite region, spherically symmetrical about that point, and of, roughly, the nuclear radius.

Past discussions have been concerned with bound states of the pair field and with nuclear forces; but here we shall find how mesotrons are scattered by nuclei. It is of some interest to compare our conclusions with the remarkably small experimental cross section for the scattering of cosmic-ray mesotrons by nuclei: for mesotron energies of $\lesssim$350 Mev, Wilson[5] has obtained an upper limit of 10^{-28} cm^2 for the cross section. Our results seem to be about a hundred times larger.[6]

[1] C. L. Critchfield and E. Teller, Phys. Rev. 51, 289 (1937).
[2] Wigner, Critchfield and Teller, Phys. Rev. 56, 530 (1939).
[3] C. L. Critchfield, Phys. Rev. 56, 540 (1939).
[4] C. L. Critchfield and W. E. Lamb, Jr., Phys. Rev. 58, 46 (1940).

[5] J. G. Wilson, Proc. Roy. Soc. 174, 73 (1940).
[6] R. E. Marshak and V. F. Weisskopf [Phys. Rev. 59, 130 (1941)], have found agreement with experiment with a similar pair-field theory, which, however, pictures the nucleus as a point source. The divergent nuclear interaction characteristic of such a model is fitted to experiment by

Although it is not certain that mesotrons of half-integer spin are present in cosmic rays or that they intervene in β-processes, it is likely, indeed, that a nucleus is better described as an extended source of charged particles, rather than a point source. In theories embodying such a description, the formal methods of this note may find further application.

I

The notation used is that of Lamb and Critchfield;[4] the coupling constant is η, the nuclear source distribution is given by $u(x)$ satisfying

$$\int u(x)^2 d\mathbf{x}=1; \tag{1}$$

and the total field Hamiltonian may be expressed in terms of the quantized amplitude $\psi(\mathbf{x})$, a 4-rowed one-column matrix in the spin-variables:

$$H=\int \psi^\dagger(\mathbf{x})[\beta-i(\boldsymbol{\alpha},\boldsymbol{\nabla})]\psi(\mathbf{x})d\mathbf{x} -\eta\left[\int \psi^\dagger(\mathbf{x})u(x)d\mathbf{x}\right]\beta\left[\int u(x')\psi(\mathbf{x}')d\mathbf{x}'\right]. \tag{2}$$

β, $\boldsymbol{\alpha}$ are the usual Dirac matrices, $\hbar=\mu=c=1$ in these units; and the commutation laws characteristic of the Fermi statistics are assumed to hold among the components of ψ and $\psi^\dagger$.

By resolving $\psi(\mathbf{x})$ into its normal vibrations $\psi_E(\mathbf{x})$ we may transform H to diagonal form. The equation governing $\psi_E(\mathbf{x})$ is self-adjoint;

$$[E+i(\boldsymbol{\alpha},\boldsymbol{\nabla})-\beta]\psi_E(\mathbf{x}) +\eta u(x)\beta\int u(x')\psi_E(\mathbf{x}')d\mathbf{x}'=0; \tag{3}$$

and, therefore, the spectrum of E is real, and the solutions of ψ_E are orthogonal in the sense:

$$(E-E')\int \psi_E^\dagger(\mathbf{x})\psi_{E'}(\mathbf{x})d\mathbf{x}=0. \tag{4}$$

"cutting off" the nuclear forces. The smallness of the coupling constant obtained in this way is chiefly responsible for their small cross section. We wish to take this opportunity to thank Drs. Marshak and Weisskopf for their kindness in sending us their manuscript before publication.

Dividing (3) by the operator $[E+i(\boldsymbol{\alpha},\boldsymbol{\nabla})-\beta]$,

$$\psi_E(\mathbf{x})+\frac{\eta u(x)}{E+i(\boldsymbol{\alpha},\boldsymbol{\nabla})-\beta}\beta\int u(x')\psi_E(\mathbf{x}')d\mathbf{x}' =\psi_0(\mathbf{x}), \tag{5}$$

where $\psi_0(\mathbf{x})$ must satisfy the Dirac equation and the boundary conditions appropriate to a wave function,

$$[E+i(\boldsymbol{\alpha},\boldsymbol{\nabla})-\beta]\psi_0(\mathbf{x})=0. \tag{6}$$

(5) may be rewritten in the form

$$\psi_E(\mathbf{x})+[E+\beta-i(\boldsymbol{\alpha},\boldsymbol{\nabla})][\Delta+q^2]^{-1}u(x)\zeta_E =\psi_0(\mathbf{x}), \tag{7}$$

where

$$\zeta_E\equiv\eta\beta\int u(x)\psi_E(\mathbf{x})d\mathbf{x},\quad E^2\equiv 1+q^2. \tag{8}$$

By multiplication with $\eta\beta u(x)$ and integration over all $\mathbf{x}$,

$$[1+\eta(1+\beta E)F(q)]\zeta_E=\eta\beta\int u(x)\psi_0(\mathbf{x})d\mathbf{x}. \tag{9}$$

In this equation,

$$F(q)\equiv\int u(x)[\Delta+q^2]^{-1}u(x)d\mathbf{x} =4\pi\int_0^\infty dp\,p^2 v(p)^2/[q^2-p^2], \tag{10}$$

where $v(\mathbf{p})=v(p)$ is the Fourier component of $u(x)$; i.e.,

$$v(\mathbf{p})=(2\pi)^{-\frac{3}{2}}\int d\mathbf{x}u(x)\exp[-i(\mathbf{p},\mathbf{x})],\qquad 4\pi\int_0^\infty v(p)^2p^2dp=1. \tag{11}$$

For every value of E, such that q is real, $|E|\geqslant 1$, $\psi_0(\mathbf{x})$ is a free-particle solution of the Dirac equation with propagation vector $\mathbf{q}$, $q^2=(\mathbf{q},\mathbf{q})$:

$$\psi_0(\mathbf{x})=\zeta(\mathbf{q})\exp[i(\mathbf{q},\mathbf{x})], \tag{12}$$

where $\zeta(\mathbf{q})$ is a matrix depending on $\mathbf{q}$ and the sign of E. If $|E|<1$, on the other hand, ψ_0 vanishes identically; and then (9) becomes a secular equation for E. Normal vibrations corresponding to the discrete spectrum of solutions are quadratically integrable and cannot contribute to the scattering problem.

In (10), therefore, q must be taken real. The integrand of $F(q)$ has a pole at $p=q$; and $F(q)$ is defined by deforming the contour into the lower half-plane. This makes $u(x)/(\Delta+q^2)$ well-defined and asymptotic to an *outgoing* wave, a condition necessary for scattering:

$$F(q)=f(q)-2\pi^2 iqv(q)^2,$$
$$f(q)\equiv 4\pi\int_0^\infty dp[p^2v(p)^2-q^2v(q)^2]/(q^2-p^2). \tag{13}$$

A useful form of these relations is obtained by direct application of the Green's function for outgoing waves, and explicit integration over angles:

$$F(q)=-\int d\mathbf{x}\int d\mathbf{x}'\frac{u(x)u(x')}{4\pi|\mathbf{x}-\mathbf{x}'|}\exp[iq|\mathbf{x}-\mathbf{x}'|]$$
$$=\frac{2\pi i}{q}\int_0^\infty dx\int_0^\infty dx'xx'u(x)u(x')$$
$$\times[\exp(iq|x+x'|)-\exp(iq|x-x'|)]. \tag{14}$$

The real part of $F(q)$ is given as :

$$f(q)=+4\pi\int_0^\infty dx\int_{-\infty}^{+\infty}dx'xx'u(x)$$
$$\times u(x')(2q)^{-1}\sin(q|x-x'|). \tag{15}$$

We may now solve (9) for ζ_E and (7) for ψ_E:

$$\zeta_E=\frac{(2\pi)^{\frac{3}{2}}\eta v(q)}{1+2\eta F-(\eta Fq)^2}[\beta-\eta(E-\beta)F(q)]\zeta(\mathbf{q}), \tag{16}$$

$$\psi_E(\mathbf{x})-\zeta(\mathbf{q})\exp[i(\mathbf{q},\mathbf{x})]=[E+\beta-i(\boldsymbol{\alpha},\boldsymbol{\nabla})]\zeta_E$$
$$\times\int d\mathbf{x}'\frac{u(x')}{4\pi|\mathbf{x}-\mathbf{x}'|}\exp[iq|\mathbf{x}-\mathbf{x}'|]$$
$$\sim[E+\beta-i(\boldsymbol{\alpha},\boldsymbol{\nabla})]\zeta_E(e^{iqx}/4\pi x)$$
$$\times\int d\mathbf{x}'u(x')\exp[-iq(\mathbf{n},\mathbf{x}')]$$
$$\sim(\pi/2)^{\frac{1}{2}}(e^{iqx}/x)v(q)[E+\beta+q(\boldsymbol{\alpha},\mathbf{n})]\zeta_E, \tag{17}$$

where $\mathbf{n}=\mathbf{x}/x$.

The expectation value of the particle-density operator in a stationary state of the field is a sum of terms of the form $\psi_E{}^\dagger(\mathbf{x})\psi_E(\mathbf{x})$ over all occupied normal vibrations; and therefore each normal vibration has its own scattering coefficient per unit solid angle:

$$\sigma_E=(\pi/2)v(q)^2\zeta_E{}^\dagger[E+\beta+q(\boldsymbol{\alpha},\mathbf{n})]^2\zeta_E$$
$$=\frac{8\pi^4E\eta^2v(q)^4}{|1+2\eta F-(\eta qF)^2|^2}\zeta^\dagger(\mathbf{q})$$
$$\times[\beta-\eta(E-\beta)F^*(q)][E+\beta+q(\boldsymbol{\alpha},\mathbf{n})]$$
$$\times[\beta-\eta(E-\beta)F(q)]\zeta(\mathbf{q}). \tag{18}$$

We may set

$$\zeta(\mathbf{q})=\exp[-\beta(\boldsymbol{\alpha},\mathbf{q})(2q)^{-1}\tan^{-1}q]\zeta(0),$$

and average over the spin orientation of $\zeta(0)$ to obtain a result, independent of the sign of E:

$$\sigma_E=\frac{16\pi^4\eta^2q^2v(q)^4}{|1+2\eta F-(\eta qF)^2|^2}$$
$$\times[1+q^{-2}+(|\eta qF|^2-2\eta f-1)\cos^2\tfrac{1}{2}\vartheta], \tag{19}$$

where $\cos\vartheta\equiv(\mathbf{q},\mathbf{x})/qx$. The cross section becomes independent of η as $\eta\to\infty$:

$$\lim_{\eta\to\infty}\sigma_E=\frac{16\pi^4v(q)^4\cos^2\frac{1}{2}\vartheta}{f(q)^2+4\pi^4q^2v(q)^4}. \tag{20}$$

The functions which appear in the foregoing analysis may be discussed generally in terms of a mean width a of the source-distribution function u:

$$a^2\equiv-\int u(x)\Delta^{-1}u(x)d\mathbf{x}=4\pi\int_0^\infty v(p)^2dp. \tag{21}$$

Thus,

$$f(0)=-a^2,\quad f(a^{-1})\simeq 0,\quad f(q)\sim q^{-2}\quad\text{for } q\gg a^{-1}; \tag{22}$$

and

$$[qv(q)^2]=0 \text{ for } q=0,\quad a^{-1}v(a^{-1})^2\simeq\text{max.},\quad qv(q)^2\ll q^{-2}, \tag{23}$$

for $q\gg a^{-1}$. In order that a be approximately the nuclear radius it should be taken $\simeq 1$ in our units; and one may observe that σ_E is measured in units of 5.13×10^{-26} cm^2.

With a particular choice of $u(x)$, it is not difficult to obtain explicit formulae and numerical values for the cross section; e.g. for

$$\begin{aligned} u(x) &= (2\pi a)^{-\frac{1}{2}} x^{-1} e^{-x/a}, \\ v(p) &= \pi^{-1} a^{\frac{3}{2}} (1+a^2p^2)^{-1}, \\ f(q) &= -a^2(1-a^2q^2)(1+a^2q^2)^{-2}, \\ F(q) &= -a^2(1-iaq)^{-2}, \end{aligned} \tag{24}$$

$$\lim_{\eta\to\infty} \sigma_E = 16a^2(1+a^2q^2)^{-2}\cos^2\tfrac{1}{2}\vartheta. \tag{25}$$

A more appropriate choice of u might be

$$\begin{aligned} u(x) &= (\pi a^2/2)^{-\frac{3}{4}} \exp(-x^2/a^2), \\ v(p) &= (a^2/2\pi)^{\frac{3}{4}} \exp(-a^2p^2/4), \\ f(q) &= -a^2\,{}_1F_1(1, \tfrac{1}{2}, -a^2q^2/2); \end{aligned} \tag{26}$$

and

$$\lim_{\eta\to\infty} \sigma_E = 2\pi a^2[{}_1F_1(-\tfrac{1}{2}, \tfrac{1}{2}, a^2q^2/2)^2 + \pi a^2q^2/2]^{-1}\cos^2\tfrac{1}{2}\vartheta. \tag{27}$$

The experimental values, which make $E \lesssim 5$, can be inserted, to obtain the total cross sections for the respective functions (24) and (26):

$$5\times10^{-24}\ \text{cm}^2 > \sigma > 10^{-26}\ \text{cm}^2,$$
$$2\times10^{-24}\ \text{cm}^2 > \sigma > 10^{-32}\ \text{cm}^2,$$

the larger value corresponding to $E=1$.

In the second case, the cross section is too large for $E \lesssim 4$, a large fraction of the experimental range in these experiments.

We conclude that this theory does not account for the small scattering of cosmic-ray mesotrons.

II

The equation (19) for σ_E shows what is apparently resonance whenever

$$\text{Real}\ \{1+2\eta F(q) - \eta^2 q^2 F(q)^2\} = 0 \tag{28}$$

and when this condition obtains,

$$[\sigma_E]_{\text{Res}} = q^{-2}[1+(1+E/2\pi^2\eta q^3 v^2)^{-1}\cos\vartheta],$$

which, when integrated over all angles gives the expected $[\bar{\sigma}_E]_{\text{Res}} = 4\pi q^{-2}$. To satisfy (28) one must take

$$\eta a^2 \backsimeq \tfrac{1}{2} \text{ for } q \ll a^{-1}, \quad \eta \backsimeq q \text{ for } q \gg a^{-1}; \tag{29}$$

but for $q \backsimeq a^{-1}$ one would need imaginary values of η. The apparent resonance, in both cases, actually makes the value of σ_E very large compared to that when q is displaced by an amount $\backsimeq a^{-1}$.

We may understand the origin of the low energy resonance by studying the discrete spectrum of our fundamental equation (3), which has been defined by $\psi_0 \equiv 0$, $|E|<1$, $q \equiv iQ$ in (9):

$$\left[1-4\pi\eta(1+\beta E)\int_0^\infty dp p^2 v(p)^2/(p^2+Q^2)\right]\zeta_E = 0.$$

Thus ζ_E must be an eigenvector of β, satisfying $\beta\zeta_E = \pm\zeta_E$, which gives rise to the secular equation for Q:

$$4\pi\eta[1\pm(1-Q^2)^{\frac{1}{2}}]\int_0^\infty dp p^2 v(p)^2/(p^2+Q^2) = +1. \tag{30}$$

The alternative sign results in there being always but one solution for Q as a function of η, and two solutions for $E = \pm|(1-Q^2)^{\frac{1}{2}}|$. (In the particular cases of (24) and (26) the secular equation has the respective forms:

$$\eta a^2[1\pm(1-Q^2)^{\frac{1}{2}}] = [1+a|Q|]^2,$$

and

$$\eta a^2[1\pm(1-Q^2)^{\frac{1}{2}}][aQ\Phi + \Phi' - aQ/2] = \Phi',$$

with

$$\Phi = \Phi(aQ) = (2\pi)^{-\frac{1}{2}}\int_0^{aQ} \exp(-t^2/2)dt.)$$

As η is increased from zero, a solution of the secular equation first becomes possible when $Q=0$, $E=\pm1$ and $\eta a^2 = \frac{1}{2}$ according to (30) and (21). This means that there exist two discrete levels, each doubly degenerate because of the spin, one of which separates itself from the continuum at $E=+1$, the other at $E=-1$. As η increases, they approach, cross at $E=0$; and then each moves asymptotically toward the point of origin of the other. The condition for low energy resonance in σ_E is precisely that for the appearance of discrete levels at the edges of the continuum.

The pair-field theory of nuclear forces would require η to be quite large compared to a^{-1}, however, and the foregoing resonance could never occur. Instead, the high energy resonance for $q \backsimeq \eta$ would be present; and, for this, there seems to be no familiar physical picture. Its presence appears to be connected with the existence of the bound, lowest, state of the pair-field considered in references (2), (4).

To see how this comes about, and also to bring our account of the behavior of the pair-field into closer connection with that given in references (2) and (4), we present a phase-shift analysis of the scattering process.

In polar coordinates, following Dirac,[7] we may rewrite (3):

$$[E+(i\epsilon/x)(d/dx)x+\beta(i\epsilon/x)j-\beta]\psi_k(\mathbf{x}) + \eta u(x)\beta\int d\mathbf{x}'u(x')\psi_k(\mathbf{x}')=0, \quad (31)$$

where $j\equiv\beta[(\boldsymbol{\sigma},\mathbf{L})+1]$, the "angular momentum" operator, $\epsilon\equiv(\boldsymbol{\alpha},\mathbf{x})/x$, $\mathbf{L}=-i[\mathbf{x}\times\boldsymbol{\nabla}]$, and $j\psi_k\equiv k\psi_k$ with k any nonvanishing integer. The perturbation term affects only those functions with spherically symmetric parts, i.e., those with $k=\pm1$, because of the spherical symmetry of $u(x)$. For all values of $|k|>1$, the integral vanishes, and we have to deal with the equation for a free particle, the solutions of which show no scattering. In what follows, therefore, we shall confine ourselves to $k=\pm1$.

The angular dependence of $\psi_k(\mathbf{x})$ now occurs only in the form of the first spherical harmonic; and we may eliminate the angular coordinates through the formal introduction of ϵ:

$$\psi_k(\mathbf{x})\equiv[1-(i\epsilon/q^2)(E-\beta)(d/dx)]\zeta_k[S_k(x)/x],$$

where $S_k(x)$ is a scalar function of the radius alone, and ζ_k is a 4-rowed, one-column matrix satisfying the characteristic equation $\beta\zeta_k=k\zeta_k$. There results for $S_k(x)$, the relation:

$$[q^2+(d/dx)^2]S_k(x)+4\pi\eta(1+kE)\times\left[xu(x)\int_0^\infty x'u(x')S_k(x')dx'\right]=0. \quad (32)$$

This is quite comparable to (3) and may be solved in similar fashion with the aid of the Green's function

$$[q^2+(d/dx)^2]^{-1}=\int_{-\infty}^{+\infty}dx'(2q)^{-1}\sin q|x-x'|. \quad (33)$$

The solutions of (3) and of (31) are brought into agreement by observing that

$$f(q)=4\pi\int_0^\infty xu(x)[q^2+(d/dx)^2]^{-1}xu(x)dx,$$

in accordance with (33) and (15). In this way, one obtains:

$$S_k(x)\sim\sin qx-\frac{2\pi^2qv(q)^2\eta(1+kE)}{1+\eta(1+kE)f(q)}\cos qx, \quad (34)$$

and the phase shift δ_k is defined by

$$\tan\delta_k\equiv-\frac{2\pi^2qv(q)^2\eta(1+kE)}{1+\eta(1+kE)f(q)}. \quad (35)$$

When δ_k/π is given the customary interpretation of the fractional momentum shift of a vibration in the continuum, due to the perturbation, this becomes exactly Eq. (9) of reference 4;[8] and it leads directly to the value obtained therein, for the energy of the bound state of the field:

$$\Delta E=-(2/\pi)\int_0^\infty dq(q^2+1)^{-\frac{1}{2}}q(\delta_{+1}+\delta_{-1}). \quad (36)$$

(This is, of course, not quite the *lowest* energy of the field; for one could also fill the two discrete states of energy $\backsimeq -1$. Their contribution to ΔE would be small, however, when η is taken $\gg a^{-1}$.)

For values of $E<0$, with $q\gg a^{-1}$, δ_{+1} remains $\backsimeq 0$; but $\delta_{-1}\backsimeq\pi$ for $q<\eta$ and $\backsimeq 0$ for $q>\eta$, varying through the resonance value of $\pi/2$ in the neighborhood of $q=\eta$. The resonance of the cross section at $q=\eta$ can thus be connected with the limitation of ΔE to finite value, of order $-\eta$.

The problem of scattering of mesotrons when the coupling between the nucleus and the pair-field is spin dependent, cannot be solved in the same manner as the problem treated here, because the reaction of emission and absorption processes on the nuclear spin plays an essential part.

I wish to thank Professor J. R. Oppenheimer for suggesting this problem and for many stimulating discussions concerning its solution. It is a pleasure, too, to recall Dr. Critchfield's helpful and very cordial letters.

[7] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, 1935), §73.

[8] Dr. Critchfield has independently arrived at the formula (19) by applying this interpretation to his Eq. (9), reference 4. With his friendly agreement, this note appears in its original form.