On the Theory of Dielectric Loss

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When a sinusoidal voltage is applied to a condenser containing an absorptive dielectric the absorption current can be split up into two components, one in phase and another in quadrature with the voltage. In the present paper, direct relations, suitable for numerical calculation, are established between these components, and a general relationship between the dielectric constant and the dielectric loss is given.

IF one admits the validity of the Hopkinson principle of superposition, the current $I(t)$ in an absorptive condenser of geometric capacitance C_0 and leakage resistance $R=1/G_0$ is given by[1]

$$I(t)=G_0U+C_0\frac{dU}{dt}+\int_{-\infty}^{t}\frac{dU(\tau)}{d\tau}\varphi(t-\tau)d\tau, \quad (1)$$

where $U(t)$ is the applied voltage and $\varphi(t)$ the dielectric relaxation function ($\varphi(\infty)=0$).

When the voltage is sinusoidal,

$$U(t)=U_0e^{i(\omega t-\delta)}, \quad (2)$$

it is easily shown, that the expressions for steady state current and for dielectric loss can be written formally in the same way as in the case of a non-absorptive condenser by introducing a conductance function $G(\omega)$ and a capacitance function $C(\omega)$.

The steady state current is given by

$$I(\omega)=[G(\omega)+j\omega C(\omega)]U(\omega); \quad (3)$$

the dielectric loss factor is given by the tangent of the loss angle, δ

$$\text{tg } \delta=G(\omega)/\omega C(\omega), \quad (4)$$

where

$$G(\omega)=G_0+\omega\int_0^{\infty}\varphi(\sigma)\sin\omega\sigma d\sigma, \quad (5a)$$

and

$$C(\omega)=C_0+\int_0^{\infty}\varphi(\sigma)\cos\omega\sigma d\sigma. \quad (5b)$$

Let us put

$$A(\omega)=G(\omega)-G_0, \quad (6a)$$

$$B(\omega)=C(\omega)-C_0. \quad (6b)$$

The functions $A(\omega)$ and $B(\omega)$ characterize the absorption current. AU is the part of this current in phase, and BU the part in quadrature with the applied voltage.

Within the whole range of frequencies, up to about 10^{10} cycles per second, concerned here $A(\omega)$ and $B(\omega)$ are often obtained by *direct* measurements. But it is impossible to measure directly $\varphi(\sigma)$ within the corresponding range of values of its arguments because this involves d.c. measurements for extremely short times.[2] The relations (5) are, therefore, of no use for experimental work at high frequencies. An examination of these formulae shows, however, that they can be inverted and that $\varphi(\sigma)$ can even be eliminated from them. The purpose of this paper is to establish a direct relation between the functions A and B.

The validity of this relation is not restricted to the theory of anomalous dielectrics. It also represents the expression of a new theorem in the theory of linear networks. And we believe it to be of interest even from a more general point of view, as it will prove to be an application of a very general integral transformation mentioned until now only in purely mathematical papers.

The theory of linear networks starts formally from the same equation as does the theory of dielectric after-effects, i.e. from Eq. (1), if written in a slightly different form by introducing the

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[1] J. B. Whitehead, *Dielectric Theory and Insulation* (McGraw-Hill, 1928); M. F. Manning and M. E. Bell, Rev. Mod. Phys. **12**, 215 (1940).

[2] Most measurements extend only to 10^{-3} or 10^{-4} sec. Only in the case of highly anomalous dielectrics does one succeed in extending the range of measurements up to 10^{-6} sec. (B. Gross, Zeits. f. Physik **108**, 598 (1938)).

indicial admittance $h(\sigma)$ and omitting the term $C_0(dU/dt)$. But by separating the d.c. component, $h(\sigma)$ can be written always in the form $f(\sigma)$ $+$const where $f(\infty)=0$; and the theory remains more general retaining a term $C_0(dU/dt)$, applying then to impulsive networks too. The main difference between the two theories consists in the shape of $\varphi(\sigma)$ and $h(\sigma)$, respectively; $\varphi(\sigma)$ does not change its sign and is continuously decreasing, whereas $h(\sigma)$ may, or may not be oscillating, but for dissipative networks every oscillating term contains a damping factor $e^{-p\sigma}$. In both cases, therefore, the integrals over $|\varphi(\sigma)|$ and $|h(\sigma)-\text{const}|$, respectively, exist from 0 to ∞. Hence

$$\int_0^\infty |\varphi(\sigma)|d\sigma=Q. \tag{7}$$

Q is the total residual charge of an anomalous condenser after application of a unit voltage during infinite time. A corresponding equation is valid for $|h(\sigma)-\text{const}|$.

All calculations based on Eq. (1), so far as they do not use any particular analytical expression for $\varphi(\sigma)$, but only the general property (7), are therefore valid both for anomalous dielectrics and linear dissipative networks. Only the names have to be changed. For the theory of networks we search for an expression which gives a relation between the real and the imaginary components of the steady state admittance.

$\varphi(\sigma)$ is certainly continuous for $0<\sigma\leq\infty$. Equation (7) shows that the integral over $|\varphi(\sigma)|$ exists from 0 to ∞. This behavior enables us to consider (5) as Fourier integrals which can be inverted by Fourier's transform, thus giving

$$\varphi(\sigma)=\frac{2}{\pi}\int_0^\infty \frac{A(\omega)}{\omega}\sin\omega\sigma d\omega, \tag{8a}$$

$$\varphi(\sigma)=\frac{2}{\pi}\int_0^\infty B(\omega)\cos\omega\sigma d\omega. \tag{8b}$$

The knowledge of any one of the functions $A(\omega)$ or $B(\omega)$ is sufficient for the calculation of $\varphi(\sigma)$. The relations (8) suggest the possibility of determining the dielectric relaxation function for values $\sigma\ll 1$ by means of a.c. measurements. This method would not suffer from the experimental difficulties inherent in d.c. methods. It may furnish this function over a much wider range of σ than known hitherto, but it must be said that the problem of evaluating the integrals (8) for numerically given functions $A(\omega)$ or $B(\omega)$ is difficult.

The relations (8) need a further comment. The function $\varphi(\sigma)$ is equal to zero for all values $\sigma<0$. This is not the behavior of the functions defined by the integrals (8), which are, respectively, an odd and an even function. These integrals represent $\varphi(\sigma)$ only for positive values of σ, but this does not limit the validity of our calculations, because the particular form of Eq. (1) excludes the occurrence of negative arguments in $\varphi(\sigma)$.

In order to eliminate $\varphi(\sigma)$, we substitute it in (5a) by its value (8b) and in (5b) by the expression (8a). There results

$$\frac{A(\omega)}{\omega}=\frac{2}{\pi}\int_0^\infty \sin\omega\sigma d\sigma\int_0^\infty B(\alpha)\cos\sigma\alpha d\alpha, \tag{9a}$$

$$B(\omega)=\frac{2}{\pi}\int_0^\infty \cos\omega\sigma d\sigma\int_0^\infty \frac{A(\alpha)}{\alpha}\sin\sigma\alpha d\alpha. \tag{9b}$$

These relations would be of no use for numerical applications if it were not possible to simplify them. Fortunately it is possible to do so. One obtains finally (see Appendix)

$$\frac{A(\omega)}{\omega}=\frac{2}{\pi}\int_0^\infty B(\alpha)\frac{\omega}{\omega^2-\alpha^2}d\alpha, \tag{10a}$$

$$B(\omega)=\frac{2}{\pi}\int_0^\infty \frac{A(\alpha)}{\alpha}\frac{\alpha}{\alpha^2-\omega^2}d\alpha. \tag{10b}$$

The integrals are *principal values.*

These are the desired relations between anomalous capacitance and conductance. Equation (10a) gives $G(\omega)-G_0$ as a function of $C(\omega)-C_0$; Eq. (10b) gives $C(\omega)-C_0$ as a function of $G(\omega)-G_0$. In this form, the integrals are suitable for numerical application. In consequence of the factor $1/(\alpha^2-\omega^2)$, these are mainly the values of the integrand in the vicinity of the point $\alpha=\omega$, which contribute to the value of the integrals for a given ω. The graphical or numerical evaluation of the integrals is therefore not difficult.

The expression for the dielectric loss is now obtained easily. Taking into account that

$$\int_0^\infty \frac{d\alpha}{\alpha^2-\omega^2}=0 \tag{11}$$

and observing that for technical, and higher, frequencies

$$G_0 \ll G(\omega), \tag{12}$$

there results

$$\operatorname{tg}\delta=-\frac{2}{\pi}\frac{\omega}{C(\omega)}\int_0^\infty \frac{C(\alpha)}{\omega^2-\alpha^2}d\alpha. \tag{13}$$

The knowledge of the capacitance function $C(\omega)$ is therefore sufficient for calculating the dielectric loss.

The form of the relations (10) is remarkable. These equations are a pair of integral equations of the first kind, each one of them inverting the other one. It can be shown that they apply to very general periodic or nonperiodic functions, which can be represented by these integrals. For periodic functions, the integrals may be simplified and result finally in Hilbert's inversion formula.[3] But the investigation of the general properties of the system (10) seems to be a matter of purely mathematical interest and will be given elsewhere.[4]

Appendix

The relations (10) can be obtained directly from (9). Here we prefer to verify them by substitution.

If we substitute in Eq. (10a) the value of $B(\alpha)$ given by Eq. (5b), that is

$$B(\alpha)=\int_0^\infty \varphi(\sigma)\cos\alpha\sigma d\sigma \tag{14}$$

and write the result explicitly, we have

$$\frac{A(\omega)}{\omega}=\frac{2}{\pi}\lim_{\delta=0}\left[\int_0^{\omega-\delta}\frac{\omega}{\omega^2-\alpha^2}d\alpha\int_0^\infty\varphi(\sigma)\cos\alpha\sigma d\sigma+\int_{\omega+\delta}^\infty\frac{\omega}{\omega^2-\alpha^2}d\alpha\int_0^\infty\varphi(\sigma)\cos\alpha\sigma d\sigma\right]. \tag{15}$$

But

$$\left|\int_0^\infty\varphi(\sigma)\cos\alpha\sigma d\sigma\right|\leq\int_0^\infty|\varphi(\sigma)|d\sigma=Q. \tag{16}$$

[3] D. Hilbert, *Grundzuege einer allgemeinen Theorie der Integralgleichungen* (Leipzig, 1912), p. 75.
[4] B. Gross, Ann. Acad. Bras. Sci. **13**, 31 (1941).

The integral (16) is absolutely and uniformly convergent. Now consider the second of the repeated integrals in (15). There exists the corresponding double integral L

$$L=\int_{\omega+\delta}^\infty\int_0^\infty\varphi(\sigma)\frac{\omega}{\omega^2-\alpha^2}\cos\alpha\sigma d\alpha d\sigma, \tag{17a}$$

because the functions under the integral are all continuous and because

$$|L|\leq\int_{\omega+\delta}^\infty\int_0^\infty\left|\varphi(\sigma)\frac{\omega}{\omega^2-\alpha^2}\cos\alpha\sigma\right|d\alpha d\sigma\leq\frac{Q}{2}\ln\frac{2\omega+\delta}{\delta}. \tag{17b}$$

The order of integration in Eq. (15) can therefore be inverted:[5]

$$\frac{A(\omega)}{\omega}=\frac{2}{\pi}\lim_{\delta=0}\int_0^\infty\varphi(\sigma)d\sigma\times\left[\int_0^{\omega-\delta}\frac{\omega\cos\alpha\sigma}{\omega^2-\alpha^2}d\alpha+\int_{\omega+\delta}^\infty\frac{\omega\cos\alpha\sigma}{\omega^2-\alpha^2}d\alpha\right]. \tag{18}$$

Designating by F the expression in parentheses, one sees at once that it can be written in the form

$$\begin{aligned}2F&=\int_{-\infty}^{-\delta}\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha+\int_\delta^\infty\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha-\int_{2\omega-\delta}^{2\omega+\delta}\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha\\&=\sin\sigma\omega\left[\int_{-\infty}^{-\delta}\frac{\sin\sigma\alpha}{\alpha}d\alpha+\int_\delta^\infty\frac{\sin\sigma\alpha}{\alpha}d\alpha\right]+\cos\sigma\omega\left[\int_{-\infty}^{-\delta}\frac{\cos\sigma\alpha}{\alpha}d\alpha+\int_\delta^\infty\frac{\cos\sigma\alpha}{\alpha}d\alpha\right]-\int_{2\omega-\delta}^{2\omega+\delta}\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha\\&=\sin\sigma\omega[\pi-2\mathrm{Si}(\sigma\delta)]-\int_{2\omega-\delta}^{2\omega+\delta}\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha,\end{aligned} \tag{19}$$

where Si is the sine integral. In consequence of the absolute convergence of the integral $\int_0^\infty\varphi(\sigma)\sin\omega\sigma d\sigma$ and because $|\mathrm{Si}(\sigma\delta)|<\mathrm{const}$ and $\left|\int_{2\omega-\delta}^{2\omega+\delta}\right|<\mathrm{const}$, for $0\leq\sigma\leq\infty$, $0\leq\delta\leq\omega$, we have[6]

$$\lim_{\delta\to0}\int_0^\infty\varphi(\sigma)\left[2\sin\omega\sigma\,\mathrm{Si}(\sigma\delta)+\int_{2\omega-\delta}^{2\omega+\delta}\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha\right]d\sigma=\int_0^\infty\lim_{\delta\to0}\varphi(\sigma)\left[2\sin\omega\sigma\,\mathrm{Si}(\sigma\delta)+\int_{2\omega-\delta}^{2\omega+\delta}\frac{\cos\sigma(\alpha-\omega)}{\alpha}d\alpha\right]d\sigma=0, \tag{20}$$

and therefore finally

$$A(\omega)=\omega\int_0^\infty\varphi(\sigma)\sin\omega\sigma d\sigma, \tag{21}$$

in accordance with Eq. (5a). In a similar manner, one verifies (10b).

[5] R. Courant, *Differential-und Integralrechnung II* (Berlin, 1931), second edition, p. 251, 252.
[6] E. W. Hobson, *Theory of Functions of a Real Variable* (Cambridge, 1907), first edition, p. 597.