On a New Treatment of Some Eigenvalue Problems

L. INFELD
Department of Applied Mathematics, University of Toronto, Toronto, Canada
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A new method for treating the most important eigenvalue problems in quantum mechanics is developed. The solutions can be found immediately once the equations are factorized by means of linear operators. These operators acting on a normalized eigenfunction change it into a new normalized eigenfunction and all solutions can be found once the basic eigenfunction is known. This basic eigenfunction is a solution of a simple differential equation of the first order. The underlying theory is explained more fully on a special case (Section 1) and then the rules of procedure are formulated explicitly (Section 2). The rest of the paper contains applications including the Kepler problem treated according to Dirac's theory.

INTRODUCTION

THIS paper contains a very simple method of solving the most important eigenvalue problems in quantum mechanics. The method applies however only to the discontinuous spectrum. No power series or polynomial development is necessary. The difficulty of normalization is avoided; our formulae give the energy eigenvalues and the normalized eigenfunctions.

The basic ideas of the method are closely related to those recently developed by Schrödinger.[1] But there are also essential differences between these two methods. In Schrödinger's language the basic difference can be expressed as follows. Schrödinger uses a finite number of infinite ladders, whereas I use an infinite number of finite ladders. All problems on which Schrödinger illustrated his method can be treated directly by mine. But the opposite is not true. This is evident if we compare the two treatments of the Kepler problem (in a Euclidean, or spherical space). In this paper the Kepler problem furnishes a direct application of the method. But it cannot be treated directly by Schrödinger's method. What Schrödinger does, and what is usually done, is to use a transformation involving the coordinate and energy and to change the Kepler problem into a different, though related problem, accessible to his method.

The first section of this paper contains a typical eigenvalue problem which I have tried to treat slowly and carefully. The discussion of all other cases follows the same pattern and I present them in a tempo which accelerates with their increasing number.

1. THE FIRST CASE

When presented with a problem of the Sturm-Liouville type, we shall always change it into the form

$$d^2u/dx^2+ru+\lambda u=0; \quad u=u(x). \tag{1.1}$$

This is possible if in the original form

$$d(pv')/dy+qv+\lambda\rho v=0; \quad v=v(y), \tag{1.2}$$

the functions p, ρ are never negative and if ρ/p exists everywhere in the interval in which the solution is considered. The transformation leading from (1.2) to (1.1) is[2]

$$u=(p\rho)^{\frac{1}{4}}v; \quad x=\int_0^y\left(\frac{\rho}{p}\right)^{\frac{1}{2}}dy. \tag{1.3}$$

In particular if $\rho=p$, the transformation from (1.2) to (1.1) does not involve a change in the independent variable.[3] The boundary condition will always be the same: the vanishing of u at the ends of the interval. In the case in which the interval reaches infinity, the condition that $\int u^2dx$ (taken over the whole interval) is finite should be added.

We shall now formulate our first case:

$$u''-\frac{m(m+1)}{\sin^2 x}u+(\lambda+1)u=0. \tag{1.4}$$

[1] E. Schrödinger, Proc. Roy. Irish Acad. **A46**, 9–16 (May 1940).

[2] Courant-Hilbert, *Methoden der Mathematishen Physik* (Springer, Berlin, 1931), Vol. 1, p. 250.

[3] The choice of the form (1.1) is only to systematize our procedure. The method could be applied just as well to (1.2).

Indeed the equation (1.4) has the general form (1.1); m is assumed to be a positive integer[4] ($m=0, 1, 2\cdots$) and the interval for x is $(0, \pi)$. Putting $(\lambda+1)$ in (1.4) instead of λ seems to be unnecessary but it is connected with the history of this equation and with its physical meaning about which I shall say a word more in Sections 3 and 7.

The next step is to *factorize* the equation (1.4). Once this is done, the problem of finding the eigenvalues for λ and the eigenfunctions is easy and can be solved almost without calculation. To *factorize* (1.4) means to replace it by the following *two* equations:

$$\{(m+1)\cot x+d/dx\}\{(m+1)\cot x-d/dx\}\times u(\lambda, m)=\{\lambda-(m+1)^2+1\}u(\lambda, m), \quad (1.5a)$$

$$\{m\cot x-d/dx\}\{m\cot x+d/dx\}u(\lambda, m)=\{\lambda-m^2+1\}u(\lambda, m). \quad (1.5b)$$

After having performed the indicated linear operations we find that *both* (1.5a) and (1.5b) lead to (1.4). Therefore (1.4) can be written *either* in the form (1.5a) *or* (1.5b). In (1.5) the dependence of u on λ, m is expressed explicitly and that on x suppressed. The equation (1.5b) can be obtained from (1.5a) by interchanging the first two coefficients, and changing $(m+1)$ into m everywhere with the exception of $u(\lambda, m)$.

For later use we shall write (1.5) in the more general form:

$$H^{(m+1)+}H^{(m+1)-}u(\lambda, m)=(\lambda-L(m+1))u(\lambda, m), \quad (1.6a)$$

$$H^{m-}H^{m+}u(\lambda, m)=(\lambda-L(m))u(\lambda, m), \quad (1.6b)$$

where, in our special case

$$H^{m\pm}=m\cot x\pm d/dx; \qquad L(m)=m^2-1=(m-1)(m+1). \quad (1.7)$$

Theorem I

$$\int_0^\pi \varphi(H^{m-}f)dx=\int_0^\pi (H^{m+}\varphi)fdx, \quad (1.8)$$

if (φf) vanishes at the ends of the interval and $H^{m\pm}$ are given by (1.7). This means: the operators H^- and H^+ are *mutually adjoint*. The proof is self-evident.

Theorem II

Let us assume that for a particular value of $\lambda=\lambda_0$ and for $m\neq 0$, $u(\lambda_0, m)$ is *a* solution of (1.4). (We do not care, for the moment, whether $u(\lambda_0, m)$ satisfies our boundary condition.) We can prove that

$$u(\lambda_0, m+1)=H^{(m+1)-}u(\lambda_0, m), \quad (1.9a)$$

$$u(\lambda_0, m-1)=H^{m+}u(\lambda_0, m) \quad (1.9b)$$

are also solutions corresponding to the number pairs $(\lambda_0, m+1)$ and $(\lambda_0, m-1)$, respectively. This means: having a solution belonging to λ_0, m, we can find, by applying our linear operators, new solutions corresponding to λ_0, $m+1$ and λ_0, $m-1$. Thus we can go one step higher and one step lower in the m's. The argument can of course be prolonged and thus we obtain a *ladder* of solution in the m's, belonging to a definite λ_0.

For the proof we multiply (1.6a) by $H^{(m+1)-}$ and (1.6b) by H^{m+}. The result is

$$(H^{(m+1)-}H^{(m+1)+})(H^{(m+1)-}u(\lambda_0, m))=(\lambda_0-L(m+1))(H^{(m+1)-}u(\lambda_0, m)), \quad (1.10a)$$

$$(H^{m+}H^{m-})(H^{m+}u(\lambda_0, m))=(\lambda_0-L(m))(H^{m+}u(\lambda_0, m)). \quad (1.10b)$$

The comparison of (1.10) with (1.6) proves our theorem expressed in the equations (1.9). In our argument no use was made of the special form of the $H^\pm$.

We know that the ladder constructed in this way must end when the bottom, that is $m=0$, is reached since m is assumed to be $\geqslant 0$. The essential point to which we are driving is that the ladder in which we are interested has not only a bottom but also a top, that is, it is a finite ladder.

The equations (1.6a) and (1.6b) can be interpreted with the help of our ladder picture in the following way: by going one step up the ladder and one step down the ladder (or one step down and one step up) we arrive at the solution from which we started, but multiplied by $(\lambda_0-L(m+1))$ or by $(\lambda_0-L(m))$.

[4] We omit throughout, the case of $m<0$ since this does not lead to new eigenfunctions.

Theorem III

If $u(\lambda_0, m)$ is our solution, that is a regular solution vanishing at the ends of the interval, then the solutions obtained through (1.9) are also our solutions. The contrary is also true. Or: *our* and *not our* ladders do not mix. The ladder can be either entirely *ours* or entirely *not ours*.

If $u(\lambda_0, m)$ is our solution (for $m \neq 0$), then as a glance at (1.4) shows $u(\lambda_0, m)$ must go to zero at the end of the interval $(0, \pi)$, at least as rapidly as $\sin^2 x$. Therefore, $H^{(m+1)-}u(\lambda_0, m)$ and $H^{m+}u(\lambda_0, m)$ must be also our solution and thus the argument can be prolonged. This follows from the form of $H^{m\pm}$ expressed by (1.7). If, however, $u(\lambda_0, m)$ is not our solution then neither can be $u(\lambda_0, m+1) = H^{(m+1)-}(\lambda_0, m)$, because going one step up and one step down leads (except for a constant factor) to a solution from which we started.

Though in proving this theorem we made use of the special form of our equation (1.4) the argument is much the same in most of the cases considered later.

Theorem IV

A necessary condition that our solution $u(\lambda_0, m)$, belonging to λ_0, m should exist is that

$$\lambda_0 - L(m+1) \geqslant 0. \tag{1.11}$$

For the proof we multiply (1.6a) by $u(\lambda_0, m)$ and integrate from 0 to π. We obtain, because of Theorem I and (1.9a)

$$\int_0^\pi u^2(\lambda_0, m+1)dx = (\lambda_0 - L(m+1))\int_0^\pi u^2(\lambda_0, m)dx. \tag{1.12}$$

But this equation is not possible if $\lambda_0 - L(m+1) < 0$ and $u(\lambda_0, m)$ exists. In the important case

$$\lambda_0 - L(m+1) = 0, \tag{1.13}$$

we deduce

$$u(\lambda_0, m+1) = 0 \tag{1.14}$$

from (1.12) assuming that $u(\lambda_0, m)$ is our solution (and, therefore, not identically equal to zero). Thus our ladder has its top; it ends for the value of m which satisfies $\lambda_0 - L(m+1) = 0$, if such integer m exists. This theorem is again general and holds as long as (1.8) holds.

Theorem V

A necessary and sufficient condition for the existence of *our* ladder, belonging to (1.4) is:

$$\lambda = \lambda_l = L(l+1) = l(l+2); \quad l = 0, 1, 2, \cdots. \tag{1.15}$$

a. The condition is sufficient.—From now on let us write u_l^m instead of $u(\lambda_l, m)$, where l $(0, 1, 2 \cdots)$ is defined by (1.15). For a chosen l there may be various m's for which u_l^m is our solution. Let us choose

$$m = l, \tag{1.16}$$

which is always possible, as m is an integer. The equation (1.6a) takes then the simple form

$$H^{(l+1)+}H^{(l+1)-}u_l^l = 0. \tag{1.17}$$

This equation is fulfilled if

$$H^{(l+1)-}u_l^l = ((l+1)\cot x - d/dx)u_l^l = 0, \tag{1.18}$$

or

$$u_l^l = \alpha \sin^{l+1} x; \quad \alpha = \text{constant}. \tag{1.19}$$

This is *our* solution and as it contains an arbitrary constant it represents all the solutions belonging to $m = l$, $\lambda_l = L(l+1)$ and satisfying our boundary condition. Indeed we see: If there were any other solution $\bar{u}_l^l$, regular and satisfying our boundary condition and for which $H^{(l+1)-}\bar{u}_l^l \neq 0$, then we could go up the ladder to $\bar{u}_l^{l+1} \neq 0$ contrary to Theorem IV.

We cannot go higher towards greater m's, because the higher solutions are zero solutions. But we can go down the ladder, obtaining always *our* solutions, always different from zero, as it can be seen from (1.12) and from the fact that $L(l+1) - L(m+1)$ is always greater than zero for $m = 0, 1 \cdots l-1$. Thus we have for a given l a finite ladder with the top corresponding to $m = l$ and the bottom corresponding to $m = 0$. The solutions are:

$$u_l^0, u_l^1, u_l^2, \cdots, u_l^l, \tag{1.20}$$

defined explicitly by

$$\begin{aligned} u_l^l &= \alpha \sin^{l+1} x; \\ u_l^{m-1} &= H^{m+}u_l^m = (m \cot x + d/dx)u_l^m. \end{aligned} \tag{1.21}$$

b. The condition is necessary.—We shall prove now that these are the only solutions satisfying our boundary condition. Let us assume that this is not so. First we see that our solutions do not exist for $\lambda_0<0$. Indeed in this case $\lambda_0-L(m+1)=\lambda_0-(m+1)^2+1$ is always negative (Theorem IV). The case $L(l+1)=(l+1)^2-1$ for $l=0, 1\cdots$ was discussed before. Let us now assume $\lambda=L(l'+1)=(l'+1)^2-1=l'(l'+2)$ where l' is *not* an integer but greater than zero, and investigate the corresponding ladder which we assume to exist. If l and $l+1$ are two integers satisfying

$$l<l'<l+1,$$

then $u_{l'}^{l}$ must exist as it was gained by a legitimate climbing up from the given solution $u_{l'}^{m}$. But $u_{l'}^{l+1}$ cannot be our solution because $L(l'+1)-L(l+2)<0$. Thus from *our* solution we could obtain a solution by going up the ladder which is not *ours*, contrary to Theorem III.

Summarizing we can say: To each $\lambda_l=l(l+2)$ $(l=0, 1\cdots)$ and only to such λ's, there belongs a finite ladder of solutions the top of which and the successive steps down to $m=0$ are described by (1.21).

Although our argument applied to the special form (1.15) of $L(l+1)$ it will be much the same in all other cases. There is, however, one special feature of (1.4) which will not appear later. In the case of (1.4) we could equally well characterize the ladder by starting from the bottom. Indeed for $m=0$ the solution is simply

$$u_l^0=\alpha\sin(l+1)x \tag{1.22}$$

and we could go along the same ladder up to the top instead of down to the bottom.

There is one more problem: that of *normalization*. We shall now introduce the normalized functions U_l^m and the normalized operators $\mathcal{H}^-$ and $\mathcal{H}^+$. We define

$$\mathcal{H}_l^{m\pm}=(L(l+1)-L(m))^{-\frac{1}{2}}H^{m\pm}. \tag{1.23}$$

(The new operators have *two* indices: l and m.)

$$U_l^{m+1}=\mathcal{H}_l^{(m+1)-}U_l^m;\quad U_l^{m-1}=\mathcal{H}_l^{m+}U_l^m. \tag{1.24}$$

Theorem VI

If U_l^m is *our* normalized solution then U_l^{m+1}, U_l^{m-1} are also *our* normalized solutions. Indeed instead of (1.6) we can write:

$$\mathcal{H}_l^{(m+1)+}\mathcal{H}_l^{(m+1)-}U_l^m=U_l^m, \tag{1.25a}$$

$$\mathcal{H}_l^{m-}\mathcal{H}_l^{m+}U_l^m=U_l^m \tag{1.25b}$$

and multiplying (1.25) by U_l^m and integrating we have:

$$\int_0^\pi(U_l^{m+1})^2dx=\int_0^\pi(U_l^{m-1})^2dx$$
$$=\int_0^\pi(U_l^m)^2dx=1. \tag{1.26}$$

By using the operators $\mathcal{H}^\pm$ instead of $H^\pm$ we thus obtain a normalized ladder if the solution from which we start, that is the top, is normalized. Thus the final, normalized solution of our problem can be written:

$$\lambda=l(l+2);$$

$$U_l^l=\frac{1}{\sqrt{\pi}}\left(\frac{2.4\cdots(2l+2)}{1.3\cdots(2l+1)}\right)^{\frac{1}{2}}\sin^{l+1}x, \tag{1.27}$$

$$U_l^{m-1}=\mathcal{H}_l^{m+}U_l^m=((l+1+m)(l+1-m))^{-\frac{1}{2}}$$
$$\times(m\cot x+d/dx)U_l^m, \tag{1.28}$$
$$l, m=0, 1, 2\cdots;\quad m\leqslant l.$$

Although the reasoning leading to this result was fairly long, the mechanical calculations are practically non-existent, once the equation is factorized. Equation (1.28) and $\lambda_l=L(l+1)=l(l+2)$ are known, once the factorization is performed and U_l^l is a normalized solution of an elementary differential equation of the first order.

2. Generalization

We shall now summarize in general terms the procedure which leads to a solution of all the eigenvalue problems to be considered.

The equation of the type

$$u''+r(x, m)u+\lambda u=0;\quad m=0, 1\cdots \tag{2.1}$$

has to be brought into the form

$$(k(x, m+1)+d/dx)(k(x, m+1)-d/dx)$$
$$\times u(\lambda, m)=(\lambda-L(m+1))u(\lambda, m), \tag{2.2a}$$

$$(k(x, m)-d/dx)(k(x, m)+d/dx)u(\lambda, m)$$
$$=(\lambda-L(m))u(\lambda, m). \tag{2.2b}$$

We assume that this is possible and we assume further that $L(l+1)-L(m+1)>0$, for $m=0\cdots$

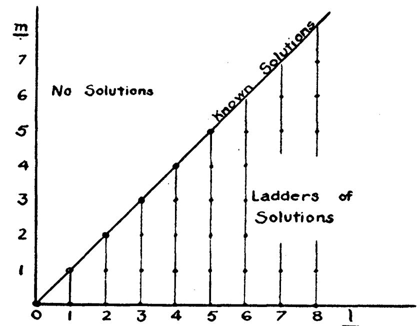

FIG. 1.

$l-1$. Then the eigenvalues for λ are $\lambda_0, \lambda_1, \lambda_2, \cdots$ where

$$\lambda_l = L(l+1); \quad l = 0, 1, \cdots. \tag{2.3}$$

To each value of λ_l there belong solutions

$$u_l^0, u_l^1, u_l^2 \cdots u_l^l. \tag{2.4}$$

The top of this ladder, that is u_l^l is determined by the equation

$$(k(x, l+1) - d/dx)u_l^l = 0, \tag{2.5}$$

from which follows

$$u_l^l = C \exp\left[\int_{x_0}^{x} k(x, l+1)dx\right], \tag{2.6}$$

where the constant C is determined by the condition

$$\int (u_l^l)^2 dx = 1.$$

The other expressions in (2.4) are given by the recurrence formula

$$u_l^{m-1} = (L(l+1) - L(m))^{-\frac{1}{2}} \times (k(x, m) + d/dx)u_l^m = \mathcal{H}_l^{m+}u_l^m. \tag{2.7}$$

The eigenfunctions obtained in this way are normalized.

Figure 1 will perhaps make the situation clearer. Once having the equations factorized we know λ_l as function of l and can therefore regard the solutions of (2.1) as functions of x, and two positive integers l, m. The solutions along the line bisecting the (l, m) plane are given immediately by a simple quadrature. From each of these solutions a ladder leads down to the solutions corresponding to the same value of λ_l and obtained by successive application of the $\mathcal{H}^+$ operation. No solutions lie above the bisecting line.[5]

The questions which have so far been omitted and which are important for the application of this method are: 1.) When can an equation of the type (2.1) be factorized? 2.) How can the factors k and L be found if the factorization is possible? Here is the answer to these questions:[6] from (2.1) and (2.2) follows:

$$k^2(x, m+1) + k'(x, m+1) + L(m+1) = -r(x, m), \tag{2.8a}$$

$$k^2(x, m) - k'(x, m) + L(m) = -r(x, m). \tag{2.8b}$$

Replacing $m+1$ by m, in (2.8a) we have:

$$k^2(x, m) + k'(x, m) + L(m) = -r(x, m-1), \tag{2.9a}$$

$$k^2(x, m) - k'(x, m) + L(m) = -r(x, m). \tag{2.9b}$$

We introduce:

$$-2\alpha = -2\alpha(x, m) = r(x, m) + r(x, m-1), \tag{2.10a}$$

$$2\beta = 2\beta(x, m) = r(x, m) - r(x, m-1). \tag{2.10b}$$

Therefore (2.9) can be written

$$k^2 + L = \alpha, \tag{2.11}$$

$$k' = \beta \tag{2.12}$$

or

$$2kk' = \alpha'; \quad k' = \beta \tag{2.13}$$

and therefore

$$k = \alpha'/2\beta. \tag{2.14}$$

Thus the last equation gives us k, if the factorization is possible. Now the answer to our first question follows immediately. We introduce (2.14) into (2.11) and we have

$$L(m) = \alpha - \alpha'^2/4\beta^2. \tag{2.15}$$

If $L(m)$ calculated by (2.15) is a function of m alone, then the factorization is possible and (2.14) with (2.15) give all the data necessary for factorization, since α and β are known functions.

Let us take as an example that considered in Section 1. We have:

$$r(x, m) = -(m(m+1)/\sin^2 x) + 1;$$
$$\alpha = (m^2/\sin^2 x) - 1;$$
$$\beta = -m/\sin^2 x.$$

[5] This diagram was suggested to me by Professor Synge.

[6] This problem was investigated along different lines by my students, Coleman and Lin.

Therefore:

$$k(x, m)=m \cot x; \quad L(m)=m^2-1,$$

exactly as before. This method can be used to obtain the factorization in all the cases treated later.

3. Spherical Harmonics

We shall now solve the differential equation for associated spherical harmonics:

$$\frac{1}{\sin x}\frac{d}{dx}\left(\sin x\frac{d\eta}{dx}\right)-\frac{m^2\eta}{\sin^2 x}+\lambda\eta=0. \quad (3.1)$$

We shall bring this equation in the form (1.1) by a transformation

$$y=(\sin^{\frac{1}{2}} x)\eta. \quad (3.2)$$

Thus η are the ordinary associated spherical harmonics and y the density functions belonging to them. Equation (3.1) takes the form

$$y''-\frac{m^2-\frac{1}{4}}{\sin^2 x}y+(\lambda+\tfrac{1}{4})y=0. \quad (3.3)$$

This equation can be factorized:

$$\begin{aligned}&\{(m+\tfrac{1}{2})\cot x+d/dx\}\\ &\quad\times\{(m+\tfrac{1}{2})\cot x-d/dx\}y\\ &\qquad=\{\lambda-m(m+1)\}y, \quad (3.4a)\\ &\{(m-\tfrac{1}{2})\cot x-d/dx\}\\ &\quad\times\{(m-\tfrac{1}{2})\cot x+d/dx\}y\\ &\qquad=\{\lambda-(m-1)m\}y. \quad (3.4b)\end{aligned}$$

Therefore we have, in this case, comparing (3.4) with (1.6)

$$\left.\begin{aligned}&L(m+1)=m(m+1);\\ &H^{m\mp}=(m-\tfrac{1}{2})\cot x\mp d/dx.\end{aligned}\right\} \quad (3.5)$$

The solutions for which η is regular in the interval $(0, \pi)$ exist only if

$$\lambda_l=L(l+1)=l(l+1); \quad l=0, 1, 2, \cdots. \quad (3.6)$$

The functions y_l^0, y_l^1, $y_l^2\cdots y_l^l$ form a ladder the top of which is determined by

$$H^{(l+1)-}y_l^l=0, \quad (3.7)$$

therefore, because of (3.5)

$$\left.\begin{aligned}&y_l^l=\alpha \sin^{l+\frac{1}{2}}x;\\ &y_l^{m-1}=H^{m+}y_l^m=\{(m-\tfrac{1}{2})\cot x+d/dx\}y_l^m\end{aligned}\right\} \quad (3.8)$$

determines our unique solution. By normalizing the top solution and introducing the $\mathcal{H}$'s instead of the H's we obtain a normalized ladder of Y_l^m functions characterized finally by:

$$\left.\begin{aligned}&\lambda_l=l(l+1);\\ &Y_l^l=\left(\frac{1.3\cdots(2l+1)}{2.2.4\cdots 2l}\right)^{\frac{1}{2}}\sin^{l+\frac{1}{2}}x;\\ &Y_l^{m-1}=[(l+m)(l+1-m)]^{-\frac{1}{2}}\\ &\qquad\times\{(m-\tfrac{1}{2})\cot x+d/dx\}y_l^m.\end{aligned}\right\} \quad (3.9)$$

We can formulate a more general problem characterized by the equation

$$\varphi''+2\varphi'\gamma\cot x-\frac{m^2+2m\gamma-m}{\sin^2\alpha}\varphi+\lambda\varphi=0, \quad (3.10)$$

where m, as before, is an integer ($m\geqslant 0$) and γ is an arbitrary positive parameter. Again this equation takes the form (1.1) by a substitution

$$u=(\sin^{\gamma} x)\varphi \quad (3.11)$$

and we obtain

$$u''-\frac{(m+\gamma)(m+\gamma-1)}{\sin^2 x}u+(\lambda+\gamma^2)u=0. \quad (3.12)$$

We see that this equation is a generalization of the two cases considered up to now. Indeed putting $\gamma=1$, we have the equation (1.4) and $\gamma=\frac{1}{2}$ leads to (3.3). The functions u are *generalized* spherical harmonics and the condition that φ are regular everywhere in the interval $(0, \pi)$ leads to the determination of normalized functions $U^m{}_{l,\gamma}$ satisfying (3.12). The result is:

$$\left.\begin{aligned}&\lambda=l(l+2\gamma);\\ &U^l{}_{l,\gamma}=\pi^{-\frac{1}{2}}\left[\frac{\Gamma(l+\gamma+1)}{\Gamma(l+\gamma+\frac{1}{2})}\right]^{\frac{1}{2}}\sin^{l+\gamma}x,\\ &\mathcal{H}_l{}^{(m+1)\mp}=[(l+m+2\gamma)(l-m)]^{-\frac{1}{2}}\\ &\qquad\times\{(m+\gamma)\cot x\mp d/dx\},\\ &U_{l,\gamma}{}^{m-1}=\mathcal{H}_l{}^{m+}U^m{}_{l,\gamma}.\end{aligned}\right\} \quad (3.13)$$

The equations (1.27) and (1.28) follow from (3.13) for $\gamma=1$, and the equations (3.9) for $\gamma=\frac{1}{2}$.

4. The Kepler Problem

The application of this method to the Kepler problem is straightforward and does not require the usual transformation which changes the original eigenvalue problem into a different one. The differential equation is:

$$\psi''+\frac{2}{x}\psi'+\frac{2\psi}{x}-\frac{m(m+1)}{x^2}\psi+\lambda\psi=0, \quad (4.1)$$

where, in the obvious notation:

$$\lambda=(2\hbar^2/\mu Z^2e^4)E; \quad x=(\mu Ze^2/\hbar^2)r. \quad (4.2)$$

(The usual notation is to use l instead of our m in (4.1) and, later n instead of our $l+1$. But in order to make the comparison with Section 1 simpler, we use here this unconventional notation.)

We perform the transformation

$$\rho=x\psi(x) \quad (4.3)$$

and obtain in place of (4.1) the equation which has the form of (1.1):

$$\rho''+(2/x)\rho-[m(m+1)/x^2]\rho+\lambda\rho=0. \quad (4.4)$$

The usual way to treat (4.4) is to perform the transformation

$$\xi=(-\lambda)^{\frac{1}{2}}x. \quad (4.5)$$

Then (4.4) takes the form

$$\bar{\rho}''+(2\bar{\rho}/\xi)\bar{\lambda}-[m(m+1)/\xi^2]\bar{\rho}-\bar{\rho}=0; \quad \bar{\lambda}=(-\lambda)^{-\frac{1}{2}} \quad (4.6)$$

and $\bar{\rho}$ are functions of ξ. But although we can find the solution of(4.4) from (4.6) these two are quite different eigenvalue problems. This can be seen if we compare the orthogonality conditions for these two equations. If ρ_1, ρ_2 belong to different values of λ and similarly $\bar{\rho}_1$, $\bar{\rho}_2$ to different values of $\bar{\lambda}$, we have

$$\int_0^\infty \rho_1\rho_2 dx=0 \quad\text{and}\quad \int_0^\infty (\bar{\rho}_1\bar{\rho}_2/\xi)d\xi=0. \quad (4.7)$$

Our method gives us, however, the direct solution of (4.4) and the use of the transformation (4.5) is not necessary.

The factorization of (4.4) gives:

$$\left\{\frac{m+1}{x}-\frac{1}{m+1}+\frac{d}{dx}\right\}\left\{\frac{m+1}{x}-\frac{1}{m+1}-\frac{d}{dx}\right\}\rho = \left(\lambda+\frac{1}{(m+1)^2}\right)\rho, \quad (4.8a)$$

$$\left\{\frac{m}{x}-\frac{1}{m}-\frac{d}{dx}\right\}\left\{\frac{m}{x}-\frac{1}{m}+\frac{d}{dx}\right\}\rho = \left(\lambda+\frac{1}{m^2}\right)\rho, \quad (4.8b)$$

which has again the general form of (2.2). Putting

$$\lambda_l=-1/(l+1)^2,$$

or

$$E_l=-\mu Z^2e^4/2\hbar^2(l+1)^2, \quad (4.9)$$

we obtain regular solutious vanishing at the end of the interval $(0, \infty)$. To each λ_l of the form (4.9) we have $\rho_l^0, \rho_l^1\cdots\rho_l^l$, where

$$\rho_l^l=\alpha x^{l+1}e^{-x/(l+1)} \quad (4.10)$$

is a solution of

$$\left(\frac{l+1}{x}-\frac{1}{l+1}-\frac{d}{dx}\right)\rho_l^l=0. \quad (4.11)$$

From this top of the ladder we can reach the bottom by the recurrence formula:

$$\rho_l^{m-1}=H^+\rho_l^m=\left(\frac{m}{x}-\frac{1}{m}+\frac{d}{dx}\right)\rho_l^m. \quad (4.12)$$

If we try to repeat the uniqueness proof of Section 1 we encounter one difficulty: for $\lambda>0$ the expression $\lambda-L(m+1)=\lambda+1/(m+1)^2$ is *always* positive. The uniqueness theorem can be proved only under the assumption $\lambda<0$, and only then does a discrete spectrum exist.

The final result for normalized functions R_l^m can be expressed now:

$$\left.\begin{aligned}&\lambda=-1/(l+1)^2;\\ &R_l^l=\left(\frac{2}{l+1}\right)^{l+1}\frac{1}{(l+1)[(2l+1)!]^{\frac{1}{2}}}\times x^{l+1}e^{-x/l+1}\\ &R_l^{m-1}=(l+1)m[(l+1-m)(l+1+m)]^{-\frac{1}{2}}\times\left(\frac{m}{x}-\frac{1}{m}+\frac{d}{dx}\right)R_l^m.\end{aligned}\right\} \quad (4.13)$$

5. The Generalized Kepler Problem and the Oscillating Rotator

We shall now generalize the Kepler problem, as before we generalized the equation for spherical harmonics. This generalization will later be applied to the solution of two problems: the oscillating rotator and the Kepler problem treated by Dirac's equations.

We start from the equation

$$f''+\frac{2a}{x}f-\frac{(m+\gamma)(m+\gamma-1)}{x^2}f+\lambda f=0, \quad (5.1)$$

assuming $\gamma>0$ and looking for solutious which vanish at the ends of the interval $(0, \infty)$ and for which $\int_0^\infty f^2dx$ exists. (For $\gamma=a=1$ Eq. (5.1) goes over into (4.1).)

The last equation can be factorized:

$$\left\{\frac{m+\gamma}{x}-\frac{a}{m+\gamma}+\frac{d}{dx}\right\}\times\left\{\frac{m+\gamma}{x}-\frac{a}{m+\gamma}-\frac{d}{dx}\right\}f=\left\{\lambda+\frac{a^2}{(m+\gamma)^2}\right\}f, \quad (5.2a)$$

$$\left\{\frac{m+\gamma-1}{x}-\frac{a}{m+\gamma-1}-\frac{d}{dx}\right\}\times\left\{\frac{m+\gamma-1}{x}-\frac{a}{m+\gamma-1}+\frac{d}{dx}\right\}f=\left\{\lambda+\frac{a^2}{(m+\gamma-1)^2}\right\}f. \quad (5.2b)$$

If $\lambda<0$ we have a discrete eigenvalue problem for

$$\lambda_l=L(l+1)=-a^2/(l+\gamma)^2 \quad (5.3)$$

and the ladder $f^0{}_{l,\gamma}, f^1{}_{l,\gamma}\cdots f^l{}_{l,\gamma}$ is determined by

$$f^l{}_{l,\gamma}=\alpha x^{l+\gamma}\exp\left(-xa/(l+\gamma)\right);$$

$$f^m{}_{l,\gamma}=\left(\frac{m+\gamma}{x}-\frac{a}{m+\gamma}+\frac{d}{dx}\right)f_{l,\gamma}{}^{m+1}=H_{,\gamma}{}^{m+1}f_{l,\gamma}{}^{m+1}. \quad (5.4)$$

Finally, the normalized solutions are

$$\left.\begin{aligned}F^l{}_{l,\gamma}&=(2a/\gamma+l)^{\gamma+l+\frac{1}{2}}(\Gamma(2l+2\gamma+1))^{-\frac{1}{2}}\\&\quad\times x^{l+\gamma}\exp-[xa/(l+\gamma)];\\F^m{}_{l,\gamma}&=(l+\gamma)(m+\gamma)[(l+m+2\gamma)(l-m)]^{-\frac{1}{2}}\\&\quad\times\left(\frac{m+\gamma}{x}-\frac{a}{m+\gamma}+\frac{d}{dx}\right)F_{l,\gamma}{}^{m+1},\end{aligned}\right\} \quad (5.5)$$

which again are a generalization of (4.13).

The solution of (5.1) enables us so solve the problem of an oscillating rotator. Its equation is:[7]

$$\varphi''+\frac{2a}{x}\varphi-\frac{m(m+1)+a}{x^2}\varphi+\lambda\varphi=0, \quad (5.6)$$

where a is a given, positive parameter.

But, for a given m, the equations (5.6) and (5.1) are identical if

$$\gamma=-(m-\tfrac{1}{2})+[(m+\tfrac{1}{2})^2+a]^{\frac{1}{2}}. \quad (5.7)$$

Therefore: $\varphi_l{}^m$ is a normalized solution of (5.6) if

$$\varphi_l{}^m=F^m{}_{l,\gamma},$$

where γ is defined by (5.7) and must be kept constant on the way down the ladder from $F^l{}_{l,\gamma}$ to $F^m{}_{l,\gamma}$. Introducing (5.7) into (5.3) we have for λ the known expression:

$$(-\lambda_l)^{\frac{1}{2}}=\frac{a}{n+\frac{1}{2}+[(m+\frac{1}{2})^2+a]^{\frac{1}{2}}} \quad (5.8)$$

with $n=l-m$. This example shows that our method can sometimes be used even if a direct factorization is not possible.

6. The Kepler Problem Treated by Dirac's Equations

The Kepler problem treated by Dirac's equations leads, for the radial functions, to the following differential equations:[8]

$$\frac{d\chi_1}{dr}-\kappa\frac{\chi_1}{r}=\left\{\frac{\mu c}{\hbar}\left(1-\frac{E}{E_0}\right)-\alpha\frac{Z}{r}\right\}\chi_2, \quad (6.1a)$$

$$\frac{d\chi_2}{dr}+\kappa\frac{\chi_2}{r}=\left\{\frac{\mu c}{\hbar}\left(1+\frac{E}{E_0}\right)+\alpha\frac{Z}{r}\right\}\chi_1;\quad \alpha=\frac{e^2}{\hbar c}. \quad (6.1b)$$

[7] A. Sommerfeld, *Atombau und Spektrallinien* (Vieweg Sohn, 1929), Vol. II, p. 24–32.

[8] *Handbuch der Physik* (Berlin, Springer, 1933), XXIV/1, p. 312.

Here χ_1, χ_2 are the function densities, as the normalization condition is

$$\int_0^\infty (\chi_1^2+\chi_2^2)dr=1. \tag{6.2}$$

κ is an integer, positive or negative, but not equal to zero. E is the total energy and $E_0=\mu c^2$ the rest energy.

We introduce the following notation:

$$\left.\begin{aligned} \gamma_1&=(\kappa-\alpha Z)^{\frac{1}{2}};\quad \gamma_2=(\kappa+\alpha Z)^{\frac{1}{2}} \\ \gamma_1\gamma_2&=(\kappa^2-\alpha^2Z^2)^{\frac{1}{2}}=\gamma \\ \epsilon&=E/E_0;\quad b=mc/h \end{aligned}\right\} \tag{6.3}$$

and replace χ_1, χ_2, by φ_1, φ_2 in (6.1) defined by:

$$\varphi_1=\gamma_1(\chi_1+\chi_2);\quad \varphi_2=\gamma_2(\chi_1-\chi_2). \tag{6.4}$$

The result is:

$$\varphi_1'-b\varphi_1=(\gamma/r+\epsilon b\gamma_1/\gamma_2)\varphi_2, \tag{6.5a}$$

$$\varphi_2'+b\varphi_2=(\gamma/r-\epsilon b\gamma_2/\gamma_1)\varphi_1. \tag{6.5b}$$

Now, replacing φ_1, φ_2 by ψ_1, ψ_2 defined through:

$$\psi_1=\varphi_1+\varphi_2;\quad \psi_2=\varphi_1-\varphi_2, \tag{6.6}$$

we obtain

$$\left(\frac{\gamma}{r}-\frac{a}{\gamma}-\frac{d}{dr}\right)\psi_1=b\left(\frac{\epsilon\kappa}{\gamma}-1\right)\psi_2, \tag{6.7a}$$

$$\left(\frac{\gamma}{r}-\frac{a}{\gamma}+\frac{d}{dr}\right)\psi_2=b\left(\frac{\epsilon\kappa}{\gamma}+1\right)\psi_1, \tag{6.7b}$$

where

$$a=b\epsilon\alpha Z. \tag{6.8}$$

From (6.7) we easily obtain two equations of the second order in which the function ψ_1 and ψ_2 are separated. These are:

$$\left\{\frac{\gamma}{r}-\frac{a}{\gamma}+\frac{d}{dr}\right\}\left\{\frac{\gamma}{r}-\frac{a}{\gamma}-\frac{d}{dr}\right\}\psi_1=b^2\left(\frac{\epsilon^2\kappa^2}{\gamma^2}-1\right)\psi_1, \tag{6.9a}$$

$$\left\{\frac{\gamma}{r}-\frac{a}{\gamma}-\frac{d}{dr}\right\}\left\{\frac{\gamma}{r}-\frac{a}{\gamma}+\frac{d}{dr}\right\}\psi_2=b^2\left(\frac{\epsilon^2\kappa^2}{\gamma^2}-1\right)\psi_2. \tag{6.9b}$$

But these are exactly the equations (5.2a) and (5.2b) for $m=0$ and $m=1$. Therefore:

$$\psi_1=AF^0{}_{l,\gamma};\quad \psi_2=BF^1{}_{l,\gamma}, \tag{6.10}$$

$$b^2\left(\frac{\epsilon^2\kappa^2}{\gamma^2}-1\right)=\lambda_l+\frac{a^2}{\gamma^2}. \tag{6.11}$$

The last equation, together with (6.8) and (5.3), gives the known formula:

$$\epsilon=E/E_0=(1+\alpha^2Z^2/(l+\gamma)^2)^{-\frac{1}{2}}. \tag{6.12}$$

In (6.10) A and B are constants. The functions $F^0{}_{l,\gamma}$ and $F^1{}_{l,\gamma}$ are defined by (5.5) and gained by descending from $F^l{}_{l,\gamma}$ to the lowest level $(F^0{}_{l,\gamma})$ and to the level above $(F^1{}_{l,\gamma})$.

But the ratio A/B is not arbitrary. We can find it by going back to the equations (6.7) of the first order. In terms of the H's belonging to the generalized Kepler problem (Section 5) they can be written

$$AH^{1-}F^0{}_{l,\gamma}=Bb(\epsilon\kappa/\gamma-1)F^1{}_{l,\gamma}, \tag{6.13a}$$

$$BH^{1+}F^1{}_{l,\gamma}=Ab(\epsilon\kappa/\gamma+1)F^0{}_{l,\gamma}. \tag{6.13b}$$

Let us multiply these equations by the normalization factor N which changes the H's into the $\mathcal{H}$'s. Then we obtain from (6.13):

$$\begin{aligned} A\mathcal{H}_l{}^{1-}F^0{}_{l,\gamma}&=AF^1{}_{l,\gamma} \\ &=NBb(\epsilon\kappa/\gamma-1)F^1{}_{l,\gamma}, \end{aligned} \tag{6.14a}$$

$$\begin{aligned} B\mathcal{H}_l{}^{1+}F^1{}_{l,\gamma}&=BF^0{}_{l,\gamma} \\ &=NAb(\epsilon\kappa/\gamma+1)F^0{}_{l,\gamma}. \end{aligned} \tag{6.14b}$$

Therefore:

$$\begin{aligned} N^{-2}&=(b^2/\gamma^2)(\epsilon^2\kappa^2-1); \\ A/B&=[(\epsilon\kappa-\gamma)/(\epsilon\kappa+\gamma)]^{\frac{1}{2}}. \end{aligned} \tag{6.15}$$

Thus the solution is

$$\psi_1=C(\epsilon\kappa-\gamma)^{\frac{1}{2}}F^0{}_{l,\gamma};\quad \psi_2=C(\epsilon\kappa+\gamma)^{\frac{1}{2}}F^1{}_{l,\gamma}. \tag{6.16}$$

The constant C is to be determined by the normalization condition (6.12).

Is $l=0$ permissible? For $l=0$, ψ_2 vanishes, because the upper index is greater than the lower. From (6.12), and because $\epsilon>0$, we have $\epsilon_{(l=0)}=\gamma/|\kappa|$. Thus $\psi_1=0$ if $\kappa>0$ and $\psi_1\neq 0$ if $\kappa<0$. This result usually gained in a complicated way, follows immediately from our formulae.

Going back to the functions χ_1, χ_2 from which we started, we have, because of (6.16), (6.6) and (6.4)

$$\begin{aligned} \chi_1\sim(\epsilon\kappa-\gamma)^{\frac{1}{2}}&[(\gamma_1+\gamma_2)/\gamma]F^0{}_{l,\gamma} \\ &-(\epsilon\kappa+\gamma)^{\frac{1}{2}}[(\gamma_1-\gamma_2/\gamma]F^1{}_{l,\gamma}, \end{aligned} \tag{6.17a}$$

$$\begin{aligned} \chi_2\sim(\epsilon\kappa-\gamma)^{\frac{1}{2}}&[(\gamma_2-\gamma_1)/\gamma]F^0{}_{l,\gamma} \\ &+(\epsilon\kappa+\gamma)^{\frac{1}{2}}[(\gamma_1+\gamma_2)/\gamma]F^1{}_{l,\gamma}. \end{aligned} \tag{6.17b}$$

7. The Kepler Problem in a Hypersphere

Schrödinger considered in his paper (quoted above) a very interesting case: the Kepler problem in a spherical space. This problem is of theoretical interest because its solution gives, unlike that in a Euclidean space, only a discrete spectrum. The equation which Schrödinger derives is:

$$\frac{d}{dx}\left(\sin^2 x\frac{d\sigma}{dx}\right)+2\nu\sin x\cos x\sigma-m(m+1)\sigma+\lambda\sin^2 x\sigma=0, \quad (7.1)$$

where

$$\nu=(\mu/\hbar^2)RZe^2;\quad \lambda=(2\mu/\hbar^2)ER^2 \quad (7.2)$$

and R is the radius of the hypersphere.

A word about how (7.1) was obtained: The quadratic form of the space is assumed to be

$$dr^2+R^2\sin^2(r/R)(d\vartheta^2+\sin^2\vartheta d\varphi^2)=R^2dx^2+R^2\sin^2 x(d\vartheta^2+\sin^2\vartheta d\varphi^2), \quad (7.3)$$

if

$$x=r/R. \quad (7.4)$$

After having written the Schrödinger equation

$$\Delta\psi+(2m/\hbar^2)(E-V)\psi=0 \quad (7.5)$$

for this space, we introduce into (7.5)

$$V=-(Ze^2/R)\cot x, \quad (7.6)$$

because this V is a harmonic function in a spherical space, that is, it satisfies the equation:

$$\frac{d}{dx}\left(\sin^2 x\frac{dV}{dx}\right)=0. \quad (7.7)$$

The V in (7.6) corresponds for a small x, to the Coulomb energy $-Ze^2/r$. To solve (7.5) we must express ψ as the product of the ordinary spherical harmonics and $\sigma(x)$ defined by (7.1), this equation replacing the old radial equation (3.1) for the Kepler problem in a Euclidean space.

The transformation

$$s=(\sin x)\sigma \quad (7.8)$$

brings (7.1) into the desired form:

$$s''+(2\nu\cot x)s-\frac{m(m+1)}{\sin^2 x}s+(\lambda+1)s=0. \quad (7.9)$$

Let us assume $\nu=0$. Evidently the term $(2\nu\cot x)s$ owes its existence to V, represented by (7.6). Therefore $\nu=0$ means a free particle in a spherical space. But then the equation (7.9) (for $\nu=0$) is identical with (1.4). Thus the physical interpretation of (1.4) is given. It is the radial part of the wave equation of a free particle in a spherical space.

We can now turn towards solving, by our method, the more complicated equation (7.9). The factorization gives:

$$\left\{(m+1)\cot x-\frac{\nu}{m+1}+\frac{d}{dx}\right\}\times\left\{(m+1)\cot x-\frac{\nu}{m+1}-\frac{d}{dx}\right\}s=\left\{\lambda-m(m+2)+\frac{\nu^2}{(m+1)^2}\right\}s, \quad (7.10a)$$

$$\left\{m\cot x-\frac{\nu}{m}-\frac{d}{dx}\right\}\left\{m\cot x-\frac{\nu}{m}+\frac{d}{dx}\right\}s=\left\{\lambda-(m-1)(m+1)+\frac{\nu^2}{m^2}\right\}s. \quad (7.10b)$$

A regular solution vanishing at the end of the interval $(0,\pi)$ exists, if

$$\lambda_l=L(l+1)=l(l+2)-\nu^2/(l+1)^2, \quad (7.11)$$

or

$$E_l=\frac{l(l+2)\hbar^2}{2\mu R^2}-Z^2\frac{e^4\mu}{2\hbar^2(l+1)^2}. \quad (7.12)$$

For small l's and increasing R, (7.12) tends towards Bohr's formula and for very large l's towards the very dense spectrum of a free particle. The uniqueness can be proved similarly as in Section 1, and therefore the continuous spectra must be absent.[9]

The top of the ladder $s_l^0, s_l^1\cdots s_l^l$ is:

$$s_l^l=\alpha\sin^{l+1}x\exp-(\nu x/l+1), \quad (7.13)$$

satisfying the differential equation

$$\left\{(l+1)\cot x-\frac{\nu}{l+1}-\frac{d}{dx}\right\}s_l^l=0 \quad (7.14)$$

[9] For a given λ, positive or negative, we can always find such a positive l' that $=L(l'+1)$. But *our* solutions exist only if $L(l'+1)-L(m+1)$ goes through zero for some integer m. Therefore l' must be an integer if our solutions exist.

and eigenfunctions of all other levels are given by:

$$s_l^{m-1} = \left(m \cot x - \frac{\nu}{m} + \frac{d}{dx}\right) s_l^m. \qquad (7.15)$$

8. Other Examples

In our treatment of the most fundamental equations in quantum mechanics, one was left out, that of a harmonic oscillator. It seems not to fit into the developed pattern, because of its equation

$$\psi'' - x^2\psi + \lambda\psi = 0, \qquad (8.1)$$

the parameter m does not appear explicitly. I shall show briefly how to treat this case though the recurrence formulae to which it leads are well known. Instead of (8.1) we write:

$$\psi'' - x^2\psi - 2m\psi + \lambda'\psi; \quad \lambda' = \lambda + 2m \qquad (8.2)$$

and the factorization gives:

$$(x - d/dx)(x + d/dx)\psi = (\lambda' - (2m+1))\psi, \qquad (8.2a)$$

$$(x + d/dx)(x - d/dx)\psi = (\lambda' - (2m-1))\psi, \qquad (8.2b)$$

therefore

$$\left.\begin{aligned} &\lambda' = 2l+1; \quad \psi_l^l = \pi^{-\frac{1}{4}} \exp(-x^2/2); \\ &\psi_l^{m-1} = [2(l-m+1)]^{-\frac{1}{2}}(x - d/dx)\psi_l^m \end{aligned}\right. \qquad (8.3)$$

or putting $n = l - m$ we have the normalized eigenfunctions for $\lambda = 2n+1$:

$$\left.\begin{aligned} &\psi_0 = \pi^{-\frac{1}{4}} \exp(-x^2/2); \\ &\psi_{n+1} = [2(n+1)]^{-\frac{1}{2}}(x - d/dx)\psi_n \\ &\psi_{n-1} = (2n)^{-\frac{1}{2}}(x + d/dx)\psi_n. \end{aligned}\right\} (8.4)$$

From (8.4) we have

$$2x\psi_n = (2(n+1))^{\frac{1}{2}}\psi_{n+1} + (2n)^{\frac{1}{2}}\psi_{n-1}, \qquad (8.5)$$

the application of which allows us quickly to calculate the intensities. Indeed we have

$$q_{n',n} = \int_{-\infty}^{+\infty} \psi_{n'}(x\psi_n)dx \qquad (8.6)$$

and therefore

$$q_{n+1,n} = [(n+1)/2]^{\frac{1}{2}}; \quad q_{n-1,n} = (n/2)^{\frac{1}{2}}. \qquad (8.7)$$

A few words now about Bessel's equation: It can easily be factorized, but, of course, it cannot be treated in the described way. Indeed

$$Z'' - [(m^2 - \tfrac{1}{4})/x^2]z + \lambda z = 0 \qquad (8.8)$$

can be written:

$$\left\{\frac{m+\frac{1}{2}}{x} + \frac{d}{dx}\right\}\left\{\frac{m+\frac{1}{2}}{x} - \frac{d}{dx}\right\} z = \lambda z, \qquad (8.9a)$$

$$\left\{\frac{m-\frac{1}{2}}{x} - \frac{d}{dx}\right\}\left\{\frac{m-\frac{1}{2}}{x} + \frac{d}{dx}\right\} z = \lambda z. \qquad (8.9b)$$

The described method breaks down, however, because $L(m) = L(m+1) = 0$.[10]

The fact that all the important quantum-mechanical eigenvalue problems can be treated by this method seems to suggest that it is something more than merely a mathematical trick. Unfortunately, I was unable to find a deeper reason for this.

My thanks are due to Dr. Griffith, Professors Synge and Stevenson for reading the manuscript and for helpful criticism.

[10] It may be noted in passing that we can deduce quickly the known recurrence formulae from (8.9):

$$\left(\frac{m+\frac{1}{2}}{x} - \frac{d}{dx}\right) Z_m = Z_{m+1}; \quad \left(\frac{m-\frac{1}{2}}{x} + \frac{d}{dx}\right) Z_m = Z_{m-1}.$$