process as well as the formula $\sigma_e = \langle i^2 \rangle_{Av} \sigma_e'$ becomes doubtful. Relatively good results are to be expected for high nuclear charges since, for these, collisions with nuclei are unimportant. The calculation of the nuclear stopping becomes unreliable when the energy drops to such a value that even the hardest collisions are strongly influenced by electronic shielding.

This paper grew out of a discussion, last summer, with Professor K. Lark-Horovitz about the difference between the appearance of tracks due to alpha-particles and fission particles. We are greatly indebted to him for his continued interest. Discussions in the early stages of our work with Professor H. A. Bethe have proved most helpful.

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The Intrinsic Inelasticity of Large Plates

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A theory is developed for the reaction of plates to forces of such short duration that the waves reflected from the boundary may be neglected. It is found that the velocity of the point of application of the force is proportional to the force, and hence the total displacement is proportional to the impulse of the force. The theory is applied to the impact of spheres with large plates. The coefficient of restitution is obtained as a function of the parameters of the impact. Complete agreement is obtained with previous experiments.

LARGE plates react in a curious manner to normal impulsive forces. A radial disturbance is, of course, propagated outwards. However, the center of the disturbance, the point where the impulse acted, remains stationary until the return of the disturbance reflected from the boundary of the plate. During the interim, the center behaves as if the plate were perfectly inelastic. The disturbance is shown diagrammatically in Fig. 1(a). An analogous phenomenon may be readily demonstrated with a loaded taut string. The disturbance initiated by an impulsive force is shown in Fig. 1(b) for this case. In each case the displacement U of the point of application is proportional to the impulse P,

$$U = \alpha P. \quad (1)$$

The theory of this effect is developed in §1. The proportionality constant α is given by Eqs. (9) and (10) for the plate and taut string, respectively.

This property of plates enables us to solve the problem of the bouncing of elastic spheres off large thin plates. The motion of the plate, and hence the energy it absorbs, could be calculated by standard methods if the force with which the sphere acts upon it were known. But this force, in turn, depends upon the motion of the plate. This dilemma is overcome by using the information contained in Eq. (1). The analysis is given in §2. The coefficient of restitution e of the sphere depends upon the various parameters of the collision, radius, mass, and initial velocity of sphere, etc., only through a single dimensionless parameter λ. We shall call this the inelasticity parameter. It is defined by Eq. (17a). The dependence of e upon λ is given by the graph of Fig. 2.

Raman published, in 1920, extensive experiments on the coefficients of restitution of hard elastic spheres rebounding off large thin glass

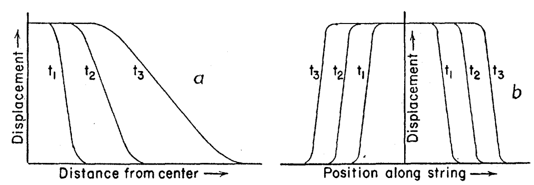

FIG. 1. (a) Propagation over a thin plate of a disturbance caused by an impulsive force. (b) Propagation along a string of a disturbance caused by an impulsive force.

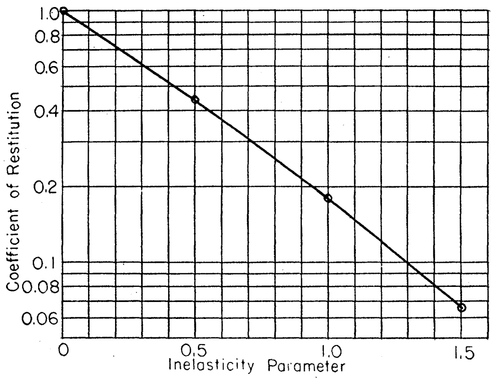

FIG. 2. Theoretical variation of coefficient of restitution with inelasticity parameter. The nearly straight line has been drawn through the four calculated points at $\lambda=0$, 0.5, 1.0, 1.5.

plates.[1] He gave a semi-empirical formula which fitted his data for values of e greater than 0.5, but which diverged rapidly as e became smaller. Recently this formula has been derived theoretically by an approximate method.[2] Raman's data are compared in §3 with the theory here developed. The coefficient of restitution obtained by this theory agrees completely with these data, as is shown in Fig. 4.

§1. ANALYSIS

In this analysis of the reaction of large plates to localized impulsive forces, we shall use the usual approximate theory of thin plates.[3] In this theory it is assumed that the radius of curvature of the plate is everywhere large compared with its thickness, and that the angle between the plate and the original plane is everywhere small. This approximate theory yields the following differential equation for the transverse displacement $U(x, y, t)$ of the plate:

$$(D\nabla^4+2\rho h\partial^2/\partial t^2)U=Z. \qquad (2)$$

In this equation, ρ is the density, $2h$ the thickness of the plate. The rigidity modulus D is defined by

$$D=(\tfrac{2}{3})h^3E', \quad E'=E/(1-\sigma^2), \qquad (3)$$

where E is Young's modulus, σ Poisson's ratio. The operator ∇^2 represents $\partial^2/\partial x^2+\partial^2/\partial y^2$. Finally, $Z(x, y, t)$ is the surface density of the normal force.

The formal solution of Eq. (2) will be obtained in terms of the eigenfunctions of the auxiliary equation

$$\{(D/2\rho h)^{\frac{1}{2}}\nabla^2+\omega\}W(x, y)=0, \qquad (2a)$$

and of the boundary conditions at the edge of the plate. The eigenvalues and normalized eigenfunctions of these equations will be denoted by ω_n and U_n, respectively. In order to obtain the coefficients in the expansion

$$U(x, y, t)=\textstyle\sum_n C_n(t)U_n(x, y), \qquad (4)$$

we substitute this expansion into Eq. (2), multiply by U_k, and integrate over the surface of the plate. Using the orthogonality property of the eigenfunctions, we obtain

$$(d^2/dt^2+\omega_k^2)C_k=(2\rho h)^{-1}\int U_kZdS. \qquad (5)$$

In the cases in which we are interested, the force is localized about a point, so that we may set

$$\int U_kZdS=U_k(0)F(t).$$

Here $U_k(0)$ is the value of U_k at the point of application of the force, which is denoted by $F(t)$. The force will be taken as first applied at $t=0$. Then the solution of Eq. (5), corresponding to the plate being initially stationary, is

$$C_k(t)=(2\rho h\omega_k)^{-1}U_k(0)\int_0^t F(t')\sin\omega_k(t-t')dt'.$$

The formal solution of our problem is now obtained by substituting this coefficient into the expansion (4). This substitution gives for the displacement at the point of application

$$U(0, t)=(2\rho h)^{-1}\textstyle\sum_n \omega_n^{-1}U_n^2(0) \times\int_0^t F(t')\sin\omega_n(t-t')dt'. \qquad (6)$$

This solution will now be simplified by performing the sum

$$\sum \omega_n^{-1}U_n^2(0)\sin\omega_n(t-t') \qquad (7)$$

before integrating with respect to t'. This sum will obviously not depend upon the shape of the plate, or upon the nature of the boundary conditions, as long as t is so small that the disturbance

[1] C. V. Raman, Phys. Rev. **15**, 277 (1920).
[2] C. Zener and H. Feshbach, J. App. Mech. **6**, A-67 (1939).
[3] See A. E. H. Love, *Mathematical Theory of Elasticity* (Cambridge, 1927), Fourth Edition, p. 487.

has not been reflected by the boundaries. By choosing a square plate, and by imposing the condition that the plate is free to pivot about fixed lines on its edges, we obtain the eigenfunctions

$$(2/L)\sin(\pi lx/L)\sin(\pi my/L),\quad 0<x,\, y<L \quad l,\, m=1,\, 2,\, \cdots.$$

We first sum over all states for which l^2+m^2 lies within a narrow range. The effect of this summation is to replace $U_n^2(0)$ by its average value $1/L^2$. The expression (7) may thus be replaced by

$$L^{-2}\sum_n \omega_n^{-1}\sin\omega_n(t-t').$$

This summation is now converted into an integration by making the plate indefinitely large. From the relation

$$(D/2\rho h)^{\frac{1}{2}}(2\pi/\lambda)^2=\omega$$

between wave-length λ and ω, which may be obtained from Eq. (2a), we find that the number of states associated with a range $d\omega$ is

$$\{L^2(2\rho h/D)^{\frac{1}{2}}/4\pi\}d\omega.$$

The combination of a two-dimensional medium with the dispersion associated with flexural vibrations has rendered the coefficient of $d\omega$ independent of ω, as in a taut string. This is why plates and taut strings react similarly to normal impulsive forces. Noting that

$$\int_0^\infty \omega^{-1}\sin\omega(t-t')d\omega=\pi/2,$$

we obtain for the summation of (7)

$$(2\rho h/D)^{\frac{1}{2}}/8.$$

Substituting back into Eq. (6), we obtain finally

$$U(0,t)=\alpha\int_0^t F(t')dt' \tag{8}$$

with

$$\alpha=(3\rho/E')^{\frac{1}{2}}/16\rho h^2. \tag{9}$$

In the case of a taut string, it may readily be shown that

$$\alpha=(2mc)^{-1}, \tag{10}$$

where m is the mass per unit length, and c is the velocity of transverse waves.

§2. Application to Impacts of Spheres with Plates

One application of the preceding theory is to the elastic normal impact of spheres with plates so large that the impact is over before the return of waves reflected from the boundaries.

In this application we must solve simultaneously the equations of motion of the sphere and of the plate. The first equation is

$$d^2z/dt^2=-m^{-1}F, \tag{11}$$

where z will be taken as the displacement of the center of the sphere from its position at contact, m is the mass of the sphere, and F is the reaction of plate on the sphere. The motion of the plate in the contact region is given by Eq. (1), which we shall write in the form

$$U=\alpha\int^t Fdt. \tag{12}$$

The displacement U refers strictly to the displacement of the mid-plane of plate.

The force F is a function only of the relative displacement of sphere and plate, namely of

$$s=z-U.$$

We may thus obtain a single equation in only the one dependent variable s by differentiating Eq. (12) twice with respect to time, and then subtracting this equation from Eq. (11).

$$d^2s/dt^2+m^{-1}F(s)+\alpha dF(s)/dt=0. \tag{13}$$

In order to obtain the appropriate boundary condition for this equation, we observe that U is constant when F is zero, that is, at the beginning and end of the collision. The appropriate boundary conditions are thus

$$\left.\begin{matrix} s=0 \\ ds/dt=v_0 \end{matrix}\right\} \quad \text{at} \quad t=0, \tag{14}$$

where v_0 is the incident velocity of the sphere. The velocity of rebound, ev_0, is then given by ds/dt where s has again returned to zero.

The reaction of the plate on the sphere is given explicitly as[4]

$$F(s)=ks^{\frac{3}{2}},$$

where

$$k=(4/3)r^{\frac{1}{2}}\{E_1'E_2'/(E_1'+E_2')\}.$$

[4] Reference 3, p. 198.

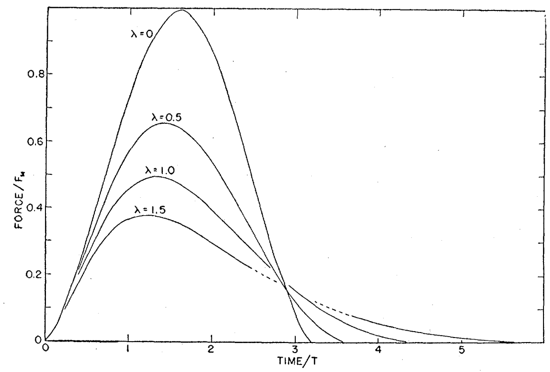

FIG. 3. Force exerted by a sphere upon a large plate during impact. F_M denotes the maximum force in the case of a plate of infinite thickness.

Here r is the radius of the sphere, E_1' and E_2' refer to the elastic moduli of Eq. (3) for the sphere and plate, respectively.

In order to introduce dimensionless variables, we make the transformation

$$s = Tv_0\sigma; \quad t = T\tau,$$

where T is a constant with dimensions of time. The boundary conditions then become

$$\left.\begin{array}{l}\sigma=0\\ d\sigma/d\tau=1\end{array}\right\} \text{ at } \tau=0.$$

The coefficient of restitution will be the value of $d\sigma/d\tau$ where σ returns to zero. The constant T will be so chosen that the first two coefficients in Eq. (13) are unity, that is, so that Eq. (13) becomes

$$d^2\sigma/d\tau^2 + (1+\lambda d/d\tau)\sigma^{\frac{3}{2}} = 0. \quad (15)$$

We find

$$T = (m/kv_0^{1/2})^{2/5}, \quad (16)$$

which is $0.311T_H$, where T_H is the duration, calculated by Hertz, of the impact in the case of a plate of infinite thickness.[4] The inelasticity parameter λ is given by

$$\lambda = \alpha m/T. \quad (17)$$

The dependence of λ upon the parameters of the impact is best seen from the following product of dimensionless factors:

$$\lambda = \frac{\pi^{3/5}}{3^{1/2}}\left(\frac{r}{2h}\right)^2\left(\frac{V_0}{V'}\right)^{1/5}\left(\frac{\rho_1}{\rho_2}\right)^{3/5}\left(\frac{E_1'}{E_1'+E_2'}\right)^{2/5}. \quad (17a)$$

Here ρ_1 and ρ_2 are the densities of the sphere and plate, respectively, while the velocity v' is defined by

$$v' = (E_2'/\rho_2)^{\frac{1}{2}}.$$

Equation (15) has been integrated numerically for several values of the inelasticity parameter. The coefficients of restitution, e, thus found are given in Table I. Intermediate values may be quite accurately found by interpolation on semi-log paper, as demonstrated in Fig. 2. In Fig. 3 the force F is shown as a function of time for these same values of λ. As λ increases, the force is seen to decrease with time in a nearly exponential manner. This behavior may be seen as a direct consequence of the equations of motion (11) and (12). For when the coefficient of restitution is small, the sphere and plate move almost with the same velocity during the latter part of the impact, or

TABLE I. *Coefficient of restitution.*

λ	0	0.5	1.0	1.5
e	1	0.44	0.18	0.067

$$d^2U/dt^2 \cong d^2z/dt^2.$$

From this it follows directly that

$$dF/dt \cong -(1/m\alpha)F.$$

§3. COMPARISON WITH EXPERIMENT

Raman's experiments with hard steel balls rebounding off glass plates are compared in Fig. 4 with his semi-empirical formula, and with

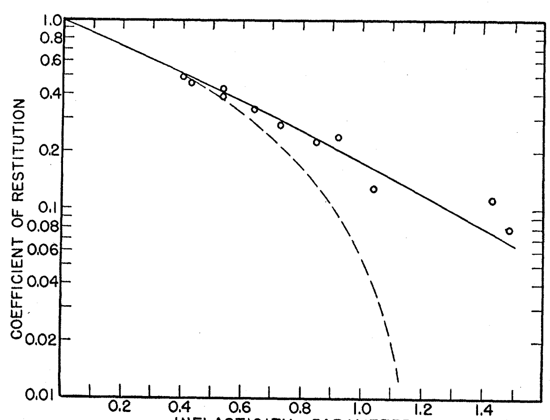

FIG. 4. Comparison of Raman's data with theory. The dashed line is given by Raman's formula, the full line by the present theory. Only those data are shown for which $e<0.5$. For $e>0.5$, the observations are in agreement with both theories.

the present theory. His formula agreed with experiment for values of e above 0.5. The present theory agrees with experiment over the entire investigated range. The present theory shows that e approaches zero only asymptotically as λ increases. The several cases of zero e recorded by Raman must be attributed to experimental difficulties in measuring small rebound velocities.

In our notation, Raman's formula is

$$e=(1-0.88\lambda)/(1+0.88\lambda).$$

In view of the success of this formula for small values of λ, it is of interest to see the relation between this formula and the present theory. This relation is given below.

According to the definition of e, the change in velocity of the sphere during impact is $(1+e)v_0$. It is also equal to $\int m^{-1}Fdt$. Hence

$$(1+e)v_0=\int m^{-1}Fdt. \qquad (18)$$

Likewise, according to the definition of e, the energy lost by the sphere during impact is $(1-e^2)mv_0^2/2$. But this is equal to the energy absorbed by the plate, namely, $\int F(dU/dt)dt$, or by Eq. (1), $\alpha\int F^2dt$. Hence

$$(1-e^2)mv_0^2/2=\alpha\int F^2dt. \qquad (19)$$

Upon dividing Eq. (19) by the square of Eq. (18), we obtain

$$(1-e)/(1+e)=R,$$

and hence

$$e=(1-R)/(1+R)$$

where

$$R=2m\alpha\int F^2dt/(\int Fdt)^2,$$

or, by Eq. (17),

$$R=2\lambda\{\int\sigma^3d\tau/(\int\sigma^{\frac{3}{2}}d\tau)^2\}.$$

When the inelasticity parameter λ is small, the bracketed ratio will be only slightly different from its value for $\lambda=0$. The value of this ratio has been calculated numerically for the case $\lambda=0$, and found to be 0.42. Hence in the limit of small λ,

$$R=0.84\lambda,$$

which is nearly the value used by Raman.

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On the Polar Vibrations of Alkali Halides

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The long wave-length, polar lattice vibrations of alkali halide crystals are discussed without making any specific assumptions about the detailed interactions between the ions. This is made possible by the introduction of the effective charge, e^*, of an ion defined as follows: All of the positive ions in a crystal slab are displaced by an equal amount in a direction perpendicular to the faces of the slab and all of the negative ions in the opposite direction. Then e^* is the ratio of the dipole moment per ion pair induced in the slab by this displacement to the relative displacement of the positive and the negative ions. Expressions are obtained for the frequency, ω_l, of the longitudinal vibration and the frequency, ω_t, of the transverse vibration in terms of the dielectric constant, k, of the crystal, the dielectric constant, k_0, obtained by extrapolating the square of the index of refraction of the crystal from high frequencies to zero frequency, and e^*. The ratio of the two frequencies is found to be independent of e^* and given by $\omega_l/\omega_t=(k/k_0)^{\frac{1}{2}}$.

THE calculation of a property of an ionic crystal which involves the lattice vibrations usually requires that detailed assumptions be made about the microscopic behavior of the crystal. However, statements that are independent of such a full knowledge can be made when the vibrations in question are of wave-length long compared to the lattice distance but short compared to the size of the crystal. Under

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