ments were made by the method of Bancroft and Jacobs.[8] It will be seen that, within the experimental error, the observed velocity as computed from the assumed wave-length is independent of the length of the specimen, and is a function of the wave-length alone.

Except for the effects of the above-mentioned discrepancy, which we have shown experimentally to be slight in the case of longitudinal waves, it is felt that the velocities determined from Pochhammer's solutions may be used with perfect confidence.

In conclusion, it appears that the work of Shear and Focke[1] is explained on a quantitative basis. Table I reproduces the behavior of the longitudinal vibrations which they observed with remarkable fidelity. The flexural vibrations behave in much the way one would expect in the light of Eq. (9), and it is noteworthy that the experimental data suggest that a common asymptote for the flexural and longitudinal velocities at high frequency is not improbable. The observed torsional vibrations may possibly belong to one of the sheets of Eq. (7). It also seems likely that the unexplained points at high frequency lie upon one of the higher sheets of Eqs. (2), (7), or (9), for in a qualitative way they lie in the region associated with these more complicated vibrations.

It is a pleasure to acknowledge our indebtedness to Professor Francis Birch for his encouragement and help, particularly in checking most of the rather tedious algebraic work.

[8] D. Bancroft and R. B. Jacobs, Rev. Sci. Inst. **9**, 279 (1938).

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Non-Uniform Particle Density in Nuclear Structure

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The Coulomb repulsion between protons must give rise to a tendency for the proton density to vary within a nucleus from a minimum value at the center to a maximum near the boundary. A non-uniform proton density may be expected to create forces which distort the neutron distribution and tend to make the two distributions vary in the same manner. If surface effects are neglected, it is possible to calculate the energy correction associated with the non-uniform densities without making special assumptions about the nuclear forces. The neglect of surface effects permits the assumption that the variations in density are small departures from essentially constant distributions. It is found that the departure from uniform density is appreciable in heavy nuclei, but the energy correction is negligible.

1. INTRODUCTION

THE Coulomb repulsion between protons must give rise to a tendency for the proton density to vary within a nucleus from a minimum value at the center to a maximum near the boundary. A non-uniform proton density may be expected to create forces which distort the neutron distribution and tend to make the two particle densities vary in the same manner. If surface effects are neglected, it is possible to calculate the energy correction associated with the non-uniform densities as well as the densities themselves in a comparatively rigorous and simple manner. The neglect of surface effects permits the assumption that the variations in density are small departures from essentially constant distributions. A suitable model for the systematic neglect of surface effects is provided by supposing the nuclear system enclosed in a box of radius R. At the boundary the radial derivative of the wave function with respect to any radial coordinate must vanish:

$$\frac{\partial}{\partial r_l}\psi(x_1y_1z_1\cdots x_Ay_Az_A)=0,\ r_l=R. \qquad (1)$$

This boundary condition makes the particle densities have zero slope at $r=R$ in contrast with the correct boundary condition which makes the particle densities vanish at $r=\infty$. When the Coulomb interaction is omitted, the boundary condition implies essentially constant particle densities throughout the nucleus. We postulate the existence of a volume energy density, $\mathcal{E}(\rho_\nu, \rho_\pi)$, which depends only on the particle densities. The total energy is

$$E(RNZ)=E_v(RNZ)+E_w(RNZ)+E_c(RNZ) \quad (2)$$

with

$$E_v=\int\int\int \mathcal{E}(\rho_\nu, \rho_\pi)d\tau, \quad (3)$$

$$E_c=e^2(Z-1)/2Z\int\cdot\int \rho_\pi(1)\rho_\pi(2)/r_{12}\cdot d\tau_1 d\tau_2$$

$$-e^2(81/64\pi)^{1/3}\int\int\int \rho_\pi{}^{4/3}d\tau, \quad (4)$$

$$E_w=\hbar^2/8M\cdot\int\int\int\{(\nabla\rho_\pi)^2/\rho_\pi+(\nabla\rho_\nu)^2/\rho_\nu\}d\tau. \quad (5)$$

The first term in Eq. (2) is the volume energy which includes the major part of the kinetic energy and all the potential energy derived from the specifically nuclear forces. No special assumption will be made in this paper about the nuclear forces or the energy density $\mathcal{E}$. An exchange term, calculated by the statistical method, is included in the Coulomb energy.[1] The third term, E_w, is Weizsäcker's formula which gives a lower limit for the additional kinetic energy associated with non-uniform particle densities.[1]

To determine the nuclear radius, use is made of the auxiliary quantity

$$R_v=A^{\frac{1}{3}}\Re((N-Z)/A) \quad (6)$$

obtained by solving the equation

$$\frac{\partial}{\partial R}E_v(RNZ)=0. \quad (7)$$

From the minimum condition,

$$\frac{\partial}{\partial R}E(RNZ)=0, \quad (8)$$

we obtain

$$R=R_v(1+E_{vc}/E_v''), \quad (9)$$

$$E_v(RNZ)=E_v(R_vNZ)+E_{vc}{}^2/2E_v''. \quad (10)$$

E_{vc} is the Coulomb energy for the radius R_v and

$$E_v''=R_v{}^2(\partial^2E_v/\partial R^2)_v. \quad (11)$$

In this discussion we take[2]

$$R_v=0.5A^{\frac{1}{3}}(e^2/mc^2), \quad E_v''=100Amc^2. \quad (12)$$

In Eqs. (6)–(12), the particle densities are assumed to be constant. After the energy correction for non-uniform densities has been calculated, it will be clear that the associated correction to the radius is negligible.

The perturbation method developed in Section 2 requires knowledge of the second partial derivatives of $E_v(RNZ)$ with respect to N and Z. Section 3 contains the derivation of relations expressing the partial derivatives in terms of total derivatives which can be evaluated from empirical or semi-empirical data. Numerical results are obtained and discussed in Section 4.

2. The Pertubation Method

The equations

$$4\pi R^3\rho_{\nu 0}=3N, \quad 4\pi R^3\rho_{\pi 0}=3Z, \quad (13)$$

$$\rho_\nu=\rho_{\nu 0}(1+v_\nu(r/R)), \quad \rho_\pi=\rho_{\pi 0}(1+v_\pi(r/R)), \quad (14)$$

express the particle densities in terms of functions v_ν and v_π. Consistency requires that v_ν and v_π satisfy the orthogonality conditions

$$\int_0^1 v_\nu x^2dx=\int_0^1 v_\pi x^2dx=0, \quad (15)$$

while the additional condition

$$dv_\nu(x)/dx=dv_\pi(x)/dx=0, \quad x=0, 1, \quad (16)$$

follows from Eq. (1) and the continuity properties of the wave function.

It is now possible to express the energy as a

[1] C. F. v. Weizsäcker, Zeits. f. Physik **96**, 431 (1935). Equation (50) of this reference lacks a numerical factor $(3/4\pi)^{\frac{1}{3}}$ in the right-hand member of the equation.

[2] E. Feenberg, Phys. Rev. **59**, 149 (1941).

power series in v_ν and v_π with the result

$$E_v=\frac{4\pi}{3}R^3\mathcal{E}(\rho_{\nu 0},\ \rho_{\pi 0})$$
$$+\frac{3}{2}\Big[\Big(\int_0^1 v_\nu{}^2x^2dx\Big)N^2\frac{\partial^2}{\partial N^2}$$
$$+\Big(\int_0^1 v_\pi{}^2x^2dx\Big)Z^2\frac{\partial^2}{\partial Z^2}$$
$$+2\Big(\int_0^1 v_\nu v_\pi x^2dx\Big)NZ\frac{\partial}{\partial N}\frac{\partial}{\partial Z}\Big]E_v(RNZ), \tag{17}$$

$$E_c=3e^2Z(Z-1)/5R$$
$$\cdot\Big[1-0.76/Z^{\frac{2}{3}}-2.5\int_0^1 v_\pi x^4dx$$
$$+15\int_0^1 v_\pi x\int_0^x v_\pi(x')x'^2dx'dx$$
$$-0.51Z^{-\frac{2}{3}}\int_0^1 v_\pi{}^2x^2dx\Big], \tag{18}$$

$$E_w=3h^2/8MR^2\cdot\Big[Z\int_0^1(\partial v_\pi/\partial x)^2x^2dx$$
$$+N\int_0^1(\partial v_\nu/\partial x)^2x^2dx\Big]. \tag{19}$$

All terms in v_ν and v_π of order higher than the second have been dropped in keeping with the assumption that the particle densities do not depart greatly from their average values.

Now we introduce a set of polynomials which satisfy the conditions (15) and (16) and are normalized and orthogonal on the interval $0\leqq x\leqq 1$ with the weight factor x^2. The functions v_ν and v_π are written as linear combinations of these polynomials with coefficients determined by the condition that the energy be made as small as possible. Actual calculations have been carried through with only two members of the set:

$$v_1=6.1675\{1-3.75x^2+2.5x^3\},$$
$$v_2=11.028\{1-13.7879x^2+24.1819x^3-11.2424x^4\}. \tag{20}$$

It should be clear from the following table that any reasonable variable particle density can be represented rather well by a constant plus a suitable linear combination of v_1 and v_2.

x	0	0.2	0.4	0.6	0.8	1.0
v_1	6.167	5.366	3.454	1.172	−0.740	−1.542
v_2	11.028	6.881	0.603	−2.121	−0.349	1.672

A number of integrals involving v_1 and v_2 are required. These are

$$\int_0^1 v_1x^4dx=-0.1432,$$
$$\int_0^1 v_2x^4dx=0.0428,$$
$$\int_0^1 v_1x\int_0^x v_1(x')x'^2dx'dx=0.02470,$$
$$\int_0^1 v_2x\int_0^x v_2(x')x'^2dx'dx=0.00827,$$
$$\int_0^1 v_1x\int_0^x v_2(x')x'^2dx'dx=0.05400, \tag{21}$$
$$\int_0^1 v_2x\int_0^x v_1(x')x'^2dx'dx=-0.05201,$$
$$\int_0^1(\partial v_1/\partial x)^2x^2dx=20.378,$$
$$\int_0^1(\partial v_2/\partial x)^2x^2dx=61.302,$$
$$\int_0^1(\partial v_1/\partial x)(\partial v_2/\partial x)x^2dx=-2.014.$$

We write

$$v_\nu=a_\nu v_1+b_\nu v_2,$$
$$v_\pi=a_\pi v_1+b_\pi v_2, \tag{22}$$

and obtain

$$E_v=\frac{4\pi}{3}R^3\mathcal{E}(\rho_{\nu 0},\ \rho_{\pi 0})$$
$$+\frac{3}{2}\Big[(a_\nu{}^2+b_\nu{}^2)N^2\frac{\partial^2}{\partial N^2}$$
$$+(a_\pi{}^2+b_\pi{}^2)Z^2\frac{\partial^2}{\partial Z^2}$$
$$+2(a_\nu a_\pi+b_\nu b_\pi)NZ\frac{\partial}{\partial N}\frac{\partial}{\partial Z}\Big]E_v(RNZ), \tag{23}$$

TABLE I. *Partial derivatives of* $E_v(R_vNZ)/mc^2$.

Z	A	$\partial^2E_v/\partial N^2$	$\partial^2E_v/\partial Z^2$	$\partial^2E_v/\partial N\partial Z$
24	50	1.070	1.384	−0.773
44	100	0.481	0.760	−0.379
62	150	0.295	0.543	−0.248
80	200	0.212	0.421	−0.183

$$E_c=3e^2Z(Z-1)/5R\cdot[1-0.76/Z^{\frac{2}{3}}+0.3580a_\pi-0.1070b_\pi+0.3705a_\pi{}^2+0.1241b_\pi{}^2+0.0299a_\pi b_\pi-0.51Z^{-\frac{2}{3}}(a_\pi{}^2+b_\pi{}^2)], \quad (24)$$

$$E_v=156.2A^{\frac{1}{3}}(R/R_v)^2\times[(2N/A)(a_\nu{}^2+3.01b_\nu{}^2-0.20a_\nu b_\nu)+(2Z/A)(a_\pi{}^2+3.01b_\pi{}^2-0.20a_\pi b_\pi)]. \quad (25)$$

3. Partial Derivatives of E_v with Respect to N and Z

The only assumption required in this section is that E_v be given as a function of R, N and Z. There is no need for the special assumption (3) and the surface energy need not be neglected since it does not appear explicitly in the calculation. To avoid any implication of dependence on the assumption (3) new symbols E_0, R_0 are introduced here in place of E_v, R_v. Equation (8) is required in the form

$$\frac{\partial}{\partial R}E_0(RNZ)=E_c/R. \quad (26)$$

We use a number of relations of the type

$$\frac{\partial}{\partial N}\frac{\partial}{\partial R}E_0(RNZ)=\left(\frac{d}{dN}-\frac{dR}{dN}\frac{\partial}{\partial R}\right)\frac{\partial}{\partial R}E_0(RNZ),$$
$$\left(\frac{\partial}{\partial N}+\frac{dR}{dN}\frac{\partial}{\partial R}\right)\frac{\partial}{\partial N}E_0(RNZ)=\frac{d}{dN}\left(\frac{d}{dN}-\frac{dR}{dN}\frac{\partial}{\partial R}\right)E_0(RNZ). \quad (27)$$

These relations when combined with Eq. (26) yield

$$\frac{\partial^2E_0}{\partial N^2}=\frac{d^2E_0}{dN^2}+\left(\frac{dR}{dN}\right)^2\frac{\partial^2E_0}{\partial R^2}-\frac{d}{dN}\left(\frac{E_c}{R}\frac{dR}{dN}\right)-\frac{dR}{dN}\frac{d}{dN}\frac{E_c}{R},$$
$$\frac{\partial^2E_0}{\partial Z^2}=\frac{d^2E_0}{dZ^2}+\left(\frac{dR}{dZ}\right)^2\frac{\partial^2E_0}{\partial R^2}-\frac{d}{dZ}\left(\frac{E_c}{R}\frac{dR}{dZ}\right)-\frac{dR}{dZ}\frac{d}{dZ}\frac{E_c}{R}, \quad (28)$$
$$\frac{\partial}{\partial N}\frac{\partial}{\partial Z}E_0=\frac{d}{dN}\frac{d}{dZ}E_0+\frac{dR}{dN}\frac{dR}{dZ}\frac{\partial^2E_0}{\partial R^2}-\left(\frac{dR}{dN}\frac{d}{dZ}+\frac{dR}{dZ}\frac{d}{dN}\right)\frac{E_c}{R}-\frac{E_c}{R}\frac{d}{dN}\frac{d}{dZ}R.$$

To obtain the corresponding equations for $E_0(R_0NZ)$ we replace R by R_0 in Eq. (28) and omit the terms containing the Coulomb energy. There results

$$\frac{\partial^2E_0}{\partial N^2}=\frac{d^2E_0}{dN^2}+\frac{E_0''}{R_0{}^2}\left(\frac{dR_0}{dN}\right)^2,$$
$$\frac{\partial^2E_0}{\partial Z^2}=\frac{d^2E_0}{dZ^2}+\frac{E_0''}{R_0{}^2}\left(\frac{dR_0}{dZ}\right)^2, \quad (29)$$
$$\frac{\partial}{\partial N}\frac{\partial}{\partial Z}E_0=\frac{d}{dN}\frac{d}{dZ}E_0+\frac{E_0''}{R_0{}^2}\frac{dR_0}{dN}\frac{dR_0}{dZ}.$$

If $\partial^2/\partial R^2[E_0(RNZ)]$ is replaced by $E_0''/R_0{}^2$ and only terms containing E_0'' to the first and zeroth powers are retained, the right-hand members of the Eqs. (28) become identical with the corresponding right-hand members in the Eqs. (29). This means that the second partial derivatives of $E_v(RNZ)$ in Eq. (23) may be replaced by those of $E_v(R_vNZ)$ without appreciable error.

TABLE II. *Energy correction and density coefficients.*

Z	A	a_π	b_π	a_ν	b_ν	δE	$\rho_\pi(R)/\rho_\pi(0)$	$\rho_\nu(R)/\rho_\nu(0)$
24	50	−0.0215	0.0031	−0.0092	0.0006	$-0.7\,mc^2$	1.15	1.07
44	100	−0.0368	0.0063	−0.0167	0.0016	$-3.4\,mc^2$	1.27	1.12
62	150	−0.0481	0.0092	−0.0222	0.0026	$-7.8\,mc^2$	1.35	1.16
80	200	−0.0580	0.0119	−0.0269	0.0037	$-14.3\,mc^2$	1.43	1.20

To evaluate the total derivatives we use the energy formula

$$E_v(R_vNZ)/mc^2=-\alpha A+[96|N-Z|+25(N-Z)^2]/A, \quad (30)$$

which contains Wigner's linear term in $|N-Z|$. The quadratic term comes partly from the kinetic energy and partly from the potential energy. Wigner's more general formula[3] reduces to Eq. (30) for even values of N and Z. The major contributions to the second derivatives come from the quadratic term $25(N-Z)^2/A$. If the numerical coefficient 25 is too small, which seems likely, our results for the energy correction and the deviations from uniformity will come out too large. Another approximation tending in the same direction is that involved in the use of Weizsäcker's E_w for the extra kinetic energy term. Thus the energy correction and the deviations from uniform density given by this calculation must be interpreted as upper limits on the true values. Table I contains numerical values of the partial derivatives computed from Eqs. (29) and (30).

4. Numerical Results

The energy correction produced by the quadratic and linear terms in a_ν, a_π, b_ν, b_π can be written as

$$\delta E=c_a a_\pi-c_b b_\pi+p_\pi a_\pi^2+q_\pi b_\pi^2+p_\nu a_\nu^2+q_\nu b_\nu^2-2t_{\nu\pi}(a_\nu a_\pi+b_\nu b_\pi)-2t_\nu a_\nu b_\nu-2t_\pi a_\pi b_\pi \quad (31)$$

with coefficients in mc^2 units defined as follows:

$$c_a=0.430Z(Z-1)/A^{\frac{1}{3}},$$

$$c_b=0.128Z(Z-1)/A^{\frac{1}{3}},$$

$$p_\pi=-\frac{3}{2}Z^2\frac{\partial^2E_v}{\partial Z^2}+156A^{\frac{1}{3}}(2Z/A)+0.441Z(Z-1)/A^{\frac{1}{3}}-0.611Z^{\frac{1}{3}}(Z-1)/A^{\frac{1}{3}},$$

$$q_\pi=-\frac{3}{2}Z^2\frac{\partial^2E_v}{\partial Z^2}+470A^{\frac{1}{3}}(2Z/A)+0.149Z(Z-1)/A^{\frac{1}{3}}-0.611Z^{\frac{1}{3}}(Z-1)/A^{\frac{1}{3}}, \quad (32)$$

$$p_\nu=-\frac{3}{2}N^2\frac{\partial^2E_v}{\partial N^2}+156A^{\frac{1}{3}}(2N/A),$$

$$q_\nu=-\frac{3}{2}N^2\frac{\partial^2E_v}{\partial N^2}+470A^{\frac{1}{3}}(2N/A),$$

$$t_{\nu\pi}=-\frac{3}{2}NZ\frac{\partial}{\partial N}\frac{\partial}{\partial Z}E_v,$$

$$t_\nu=15.4A^{\frac{1}{3}}(2N/A),$$

$$t_\pi=15.4A^{\frac{1}{3}}(2Z/A)-0.018Z(Z-1)/A^{\frac{1}{3}}.$$

Equation (31) can be reduced to diagonal form by a repeated application of the process of completing squares. The numerical results of minimizing δE are listed in Table II.

Certain features of the results in Table II can be understood without calculation by examination of Eq. (31), remembering that t_ν and t_π are almost negligible. It is evident that a_π must be negative and b_π positive. Also $a_\nu a_\pi$ and $b_\nu b_\pi$ must both be positive which makes a_ν negative and b_ν positive. The negative value of $\partial^2E_v/\partial N\partial Z$ requires the neutron and proton particle densities to vary in a parallel manner resulting in a tendency toward the formation of a hollow center within the nucleus. The dependence of δE on N, Z is very similar to that of the Coulomb "expansion" energy $E_{vc}^2/2E_v''$, but the latter correction is three to four times the size of the former. Finally it must be stressed that approximations in this calculation make the computed magnitude of δE somewhat greater than the true value. Also the coefficients a_π, b_π, a_ν, b_ν are all somewhat too large in absolute value. Satisfactory convergence is indicated by the small values of the b_ν/a_ν, b_π/a_π ratios.

E. Wigner has estimated δE by another method with results similar to those stated in Table II. The writer wishes to thank Professor Wigner for information about his calculation in advance of publication.

[3] E. Wigner, Phys. Rev. **51**, 947 (1937).