# The Velocity of Longitudinal Waves in Cylindrical Bars*

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The velocity of longitudinal waves in cylindrical bars may be expressed as the velocity at infinite wave-length times a function of two variables: Poisson's ratio, and the ratio of the diameter of the bar to the wave-length. This function is computed over the domain of the arguments which is of physical interest. Asymptotic values for the velocities at very short wave-lengths are deduced, and the variation of the displacement as a function of the radius is discussed. It is found that a similar analysis can be applied to torsional and flexural waves.

IN the light of recent experiments dealing with the propagation of ultrasonic elastic disturbances in cylindrical bars,[1] it seems desirable to carry out a more complete numerical development of the theory than is at present available. Existing treatments of the subject are open to the common criticism that, as they approach rigor, they become too unwieldy to be of practical value, and the numerical computations based upon them become prohibitively long. It is the purpose of the present article to provide a concise and accurate method of analyzing the results of experiments dealing with longitudinal waves in slender cylindrical bars. Certain facts pertaining to the propagation of other types of disturbances will be touched upon. The solutions upon which the present work is based are the familiar ones due to L. Pochhammer,[2] presented in detail by Love,[3] and discussed by several other authors.[4, 5]

We deal first with the velocity of propagation of longitudinal waves in an infinitely long rod. The frequency equation for this case is suggested by Love.[6] This equation asserts a functional relation between wave-length, frequency, radius of the bar, density, and the two elastic moduli of the isotropic material under consideration. We thus appear to be dealing with an equation involving six variables, which we would like to be able to solve explicitly for any particular variable as a function of the other five. The situation is further complicated by the transcendental nature of the functions involved. It is important to notice, however, that, by a suitable choice of parameters, the equation may be reduced to a function of three variables, each of which is to be determined as a function of the other two. Such a functional relationship is readily expressed in tabular form by computing a comparatively small number of roots. The simplification thus introduced is doubtless to be anticipated from the standpoint of dimensional analysis.

We define the following quantities, and tabulate their equivalents in terms of Love's notation:

| SYMBOL | DESCRIPTION | LOVE'S NOTATION |
|---|---|---|
| $v$ | Velocity of the wave | $p/\gamma$ |
| $v_0$ | "Bar velocity" | $(E/\rho)^{\frac{1}{2}}$ |
| $\beta$ | $(1-2\sigma)/(1-\sigma)$; $\sigma$ = Poisson's ratio | — |
| $x$ | $(v/v_0)^2(1+\sigma)$ | — |
| $L$ | Wave-length | $2\pi/\gamma$ |
| $d$ | Diameter of the bar | $2a$ |
| $h$ | $\gamma(\beta x-1)^{\frac{1}{2}}$ | $h'$ |
| $k$ | $\gamma(2x-1)^{\frac{1}{2}}$ | $k'$ |
| $\varphi(y)$ | $yJ_0(y)/J_1(y)$ | — |

With these exceptions we adhere to Love's usage. Clearly $x$ is intrinsically positive. We shall also assume that $0<\beta<1$, a restriction which implies $0<\sigma<\frac{1}{2}$. If materials should be found with $\sigma<0$, our analysis would have to be re-examined, and important modifications would presumably be forthcoming.

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[1] S. K. Shear and A. B. Focke, Phys. Rev. **57**, 532 (1940).
[2] L. Pochhammer, J. f. Math. (Crelle) **81**, 324 (1876).
[3] A. E. H. Love, *Mathematical Theory of Elasticity* (Cambridge University Press, 1927), fourth edition, p. 287.
[4] G. S. Field, Can. J. Research **11**, 254 (1934), etc.
[5] R. Ruedy, Can. J. Research **5**, 149 (1931).
[6] Reference 3, p. 289.

The frequency equation, derived from Love's Eq. (54), p. 289, takes the form

$$\begin{vmatrix} 2\mu\dfrac{\partial^2 J_0(ha)}{\partial a^2}-\dfrac{p^2\rho\lambda}{\lambda+2\mu}J_0(ha) & 2\mu\gamma\dfrac{\partial J_1(ka)}{\partial a} \\ 2\gamma\dfrac{\partial J_0(ha)}{\partial a} & \left(2\gamma^2-\dfrac{p^2\rho}{\mu}\right)J_1(ka) \end{vmatrix}=0. \quad (1)$$

By means of suitable substitutions, and a somewhat space-consuming manipulation, (1) reduces to

$$(x-1)^2\varphi(ha)-(\beta x-1)[x-\varphi(ka)]=0. \quad (2)$$

Referring to the definitions, we see that the latter equation is of the form $F(x, \beta, \gamma a)=0$, and that therefore the transformation to a function of three variables has been accomplished.

In discussing the characteristics of Eq. (2), the following properties of the function, $\varphi(y)$, are important:

$$\varphi(y)=\varphi(-y); \quad \varphi(0)=2;$$
$$\lim_{y\to\infty}\varphi(iy)=y.$$

Zeros and poles of $\varphi$ correspond to zeros of $J_0$ and $J_1$, respectively.

Let us now suppose Eq. (2) solved explicitly for $x$, so that

$$x=x(\beta, \gamma a). \quad (3)$$

The surface described by Eq. (3) consists of sheets over the $(\beta, \gamma a)$ plane. These sheets will be designated as the zeroth, first, etc., according to their heights above the plane. The case of the zeroth sheet is trivial, for Eq. (2) is satisfied identically by $x=0$, independent of $\beta$ and $\gamma a$. The first sheet is the one corresponding to the ordinary propagation of longitudinal waves, and is doubtless the most important, but the higher sheets may be of some interest.

The first sheet is completely described by Table I, over a reasonably wide range of the independent variables $\beta$ and $\gamma a$ and the results have been plotted in Fig. 1. The parameters of Eq. (2) are not particularly convenient except for purposes of computation; accordingly more conventional variables have been used in the preparation of the final table. The dependent variable is $v/v_0$, while the independent variables are $\sigma$ and $d/L$. Numerical uncertainty should not exceed 2 units in the last place, and interpolation in the table should be accurate to four significant figures up to $d/L=1$.

Of particular interest are the values of $x$ corresponding to $\gamma a=0$ and $\gamma a=\infty$. When $\gamma a=0$, Eq. (2) yields at once

$$x(\beta, 0)=(3-2\beta)/(2-\beta)=1+\sigma,$$

but some manipulation is required to obtain $x(\beta, \infty)$. We must suppose that for some large value of $\gamma a$, the arguments of the $\varphi$ functions become and remain pure imaginaries as we move out along the sheet, and also that the arguments, $ha$ and $ka$, increase without limit. Under these conditions we have $\varphi(ha)\doteq -iha$ and $\varphi(ka)\doteq -ika$, and Eq. (2) reduces to

$$x^3-4x^2+(6-2\beta)x+(\beta-2)=0. \quad (4)$$

If we make suitable substitutions to eliminate $x$ and to express this result in terms of Poisson's

TABLE I. $v/v_0$ *as a function of* $d/L$ *and* $\sigma$.

| | $\sigma$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $d/L$ | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 |
| 0.00 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.05 | 0.99994 | 0.99986 | 0.99975 | 0.99961 | 0.99944 | 0.99924 | 0.99901 |
| 0.10 | 0.99975 | 0.99943 | 0.99899 | 0.99843 | 0.99774 | 0.99694 | 0.99602 |
| 0.15 | 0.99941 | 0.99868 | 0.99766 | 0.99638 | 0.99482 | 0.99302 | 0.99097 |
| 0.20 | 0.99890 | 0.99754 | 0.99568 | 0.99333 | 0.99054 | 0.98732 | 0.98373 |
| 0.25 | 0.99816 | 0.99591 | 0.99287 | 0.98909 | 0.98466 | 0.97967 | 0.97418 |
| 0.30 | 0.99710 | 0.99362 | 0.98899 | 0.98337 | 0.97691 | 0.96979 | 0.96214 |
| 0.35 | 0.99556 | 0.99038 | 0.98366 | 0.97572 | 0.96688 | 0.95739 | 0.94747 |
| 0.40 | 0.99323 | 0.98569 | 0.97627 | 0.96559 | 0.95410 | 0.94218 | 0.93007 |
| 0.45 | 0.98951 | 0.97866 | 0.96592 | 0.95220 | 0.93810 | 0.92397 | 0.91001 |
| 0.50 | 0.98296 | 0.96771 | 0.95133 | 0.93479 | 0.91854 | 0.90277 | 0.88758 |
| 0.55 | 0.97014 | 0.95037 | 0.93119 | 0.91288 | 0.89549 | 0.87899 | 0.86333 |
| 0.60 | 0.94487 | 0.92436 | 0.90502 | 0.88681 | 0.86964 | 0.85341 | 0.83806 |
| 0.65 | 0.90658 | 0.89086 | 0.87432 | 0.85800 | 0.84222 | 0.82709 | 0.81265 |
| 0.70 | 0.86493 | 0.85471 | 0.84201 | 0.82841 | 0.81466 | 0.80110 | 0.78792 |
| 0.75 | 0.82653 | 0.82009 | 0.81074 | 0.79982 | 0.78818 | 0.77632 | 0.76452 |
| 0.80 | 0.79306 | 0.78893 | 0.78202 | 0.77332 | 0.76357 | 0.75330 | 0.74284 |
| 0.85 | 0.76445 | 0.76167 | 0.75644 | 0.74943 | 0.74125 | 0.73236 | 0.72310 |
| 0.90 | 0.74013 | 0.73812 | 0.73402 | 0.72826 | 0.72130 | 0.71355 | 0.70532 |
| 0.95 | 0.71949 | 0.71791 | 0.71454 | 0.70967 | 0.70365 | 0.69682 | 0.68946 |
| 1.00 | 0.70196 | 0.70058 | 0.69768 | 0.69344 | 0.68814 | 0.68203 | 0.67537 |
| 1.20 | 0.65419 | 0.65266 | 0.65030 | 0.64712 | 0.64321 | 0.63869 | 0.63368 |
| 1.40 | 0.62836 | 0.62623 | 0.62361 | 0.62048 | 0.61687 | 0.61284 | 0.60844 |
| 1.60 | 0.61393 | 0.61118 | 0.60815 | 0.60479 | 0.60111 | 0.59713 | 0.59289 |
| 1.80 | 0.60565 | 0.60236 | 0.59892 | 0.59526 | 0.59139 | 0.58731 | 0.58304 |
| 2.00 | 0.60080 | 0.59707 | 0.59326 | 0.58932 | 0.58524 | 0.58101 | 0.57664 |
| $\infty$ | 0.60213 | 0.59491 | 0.58804 | 0.58148 | 0.57516 | 0.56903 | 0.56307 |

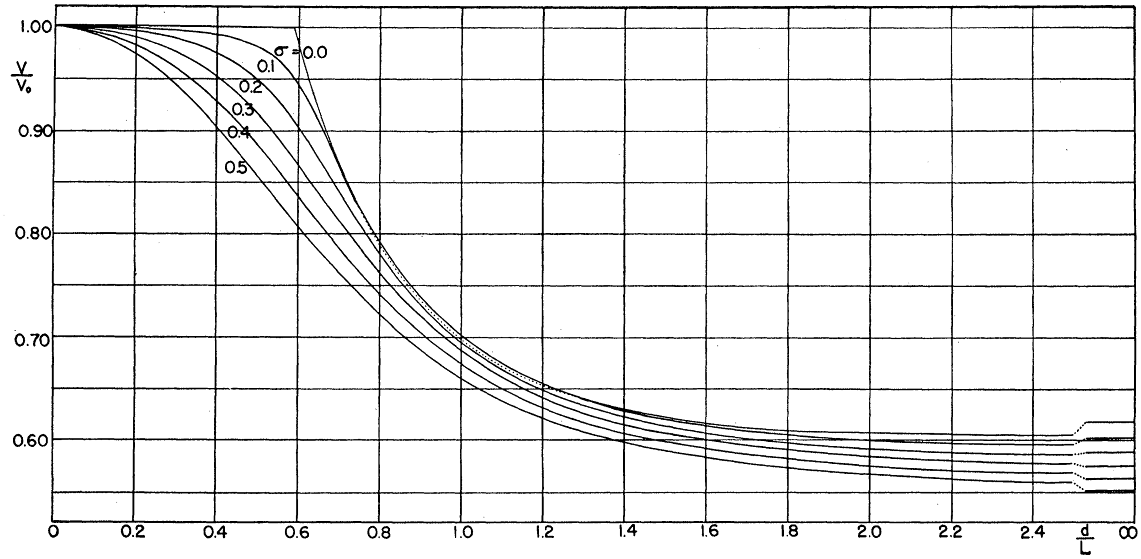


FIG. 1. $v/v_0$ as a function of $d/L$ for various values of $\sigma$.

ratio and the velocity, we find that this cubic for $x$ expresses a result identical with that deduced by Rayleigh for the velocity of surface waves (Rayleigh waves) upon a semi-infinite isotropic solid. The essential properties of Rayleigh's equation are well known; for the possible values of $\sigma$ (or $\beta$) there is always exactly one real root, which is positive. Computation of the roots of (4) for various values of $\beta$ reveals at once that the initial suppositions made in deriving it are both satisfied. The results of the computation appear in Table I, opposite the entry $d/L=\infty$.

It is interesting to note in Fig. 1 that the curves for $v/v_0$ drop below the asymptotic values deduced from Eq. (4) and approach the asymptotes from beneath. We were led at one point to doubt that the curves actually approached the proper values, but computations carried out for $d/L=20$ indicated clearly that beyond a certain point, the values of $v/v_0$ increased as predicted. The possibility of an oscillatory behavior for $v/v_0$ at large values of $d/L$ is precluded by the imaginary character of the arguments of the $\varphi$ functions in Eq. (2), and by the monotonic nature of the $\varphi$ function of an imaginary argument. The minimum value thus implied for the velocity of longitudinal waves is a somewhat unexpected result, and one which would perhaps be difficult to verify experimentally. We have not computed the values of $d/L$ for which the minimum occurs, but a rough estimate indicates that for most substances, the wave-length would have to be about $\frac{1}{5}$ the diameter. The determination of the position of the minimum by the methods of the calculus is rendered laborious by the transcendental nature of Eq. (2).

Further examination of the curves of Fig. 1 reveals the possibility of determining Poisson's ratio from measurements of the dispersion of longitudinal waves. If we plot $v/v_0$ as a function of $\sigma$ for various values of $d/L$, it will be observed that $\sigma$ is best determined by measurements of the velocity when $d/L=\frac{1}{2}$, or thereabouts, a condition which may readily be obtained experimentally.

The statement is occasionally made[7] that at very high frequencies, the velocity of propagation of longitudinal waves in a bar should approach the velocity of compressional waves in an infinite medium. In the case of the infinite medium, the wave involves no motion in the plane of the wave front, and the displacement is uniform. In the case of a long bar, motion in the plane of the wave front is inevitable, and the displacement is far from uniform, as will appear below. It is hard to see how a valid analogy can be drawn between the two cases.

Whether or not the Rayleigh velocity can be observed experimentally at very short wave-

[7] G. S. Field, Can. J. Research 5, 624 (1931).

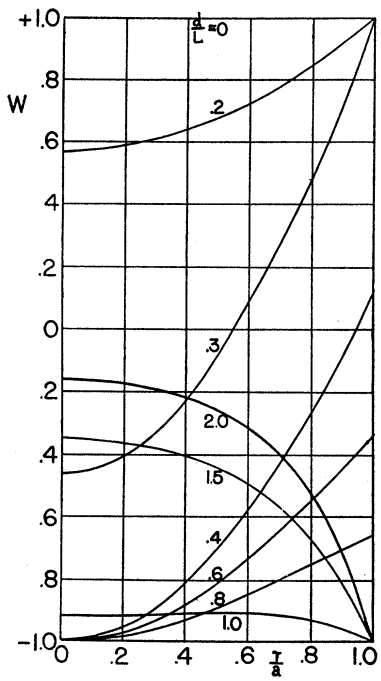


FIG. 2. Axial component of the displacement as a function of $r/a$ for various values of $d/L$, $\sigma=\frac{1}{4}$.

lengths depends on the stability of the mode of vibration with which it is associated. It is possible that beyond some limiting value of the frequency, velocities corresponding to one of the higher sheets of Eq. (3) will prevail. At the present time these sheets appear to be of purely academic interest. It may be shown that they all approach asymptotically the plane $x=\frac{1}{2}$ when $\gamma a$ becomes infinite, and that they do so from above. For extremely small values of $\gamma a$, the values of $x$ become infinite. Thus each of the higher sheets contains a curve corresponding to the velocity of compressional waves in an infinite medium (as a function of $\beta$). For any particular value of $\beta$, this curve determines a discrete value of $\gamma a$, a circumstance which has been pointed out by Field.[7] The solutions for the displacements corresponding to velocities given by the higher sheets always involve one or more nodes as a function of the radius; the problem of verifying the existence of such nodes of vibration appears somewhat difficult.

It seems worth while to give brief attention to the displacements corresponding to the first sheet. The axial component, $u_z$, is described by the equation

$$u_z = We^{i(\gamma z+pt)},$$

where $W$ is a function of the radius alone; *viz.*,

$$W = Ai\gamma J_0(hr) + \frac{Ci}{r}\frac{\partial}{\partial r}\{rJ_1(kr)\}. \qquad (5)$$

The ratio $A/C$ in (5) was determined in the course of obtaining Eq. (1). Utilizing this fact, and converting into the notation which we have introduced for the purposes of computation, we find

$$W = B\left\{\frac{1-x}{(\beta x-1)^{\frac{1}{2}}}\frac{J_0(hr)}{J_1(ha)} + (2x-1)^{\frac{1}{2}}\frac{J_0(kr)}{J_1(ka)}\right\}, \qquad (6)$$

where $B$ is arbitrary, and is proportional to the amplitude of vibration. It will be noted that the signs in (6) are independent of the choice of sign in the extraction of the square root, for the radicals occur only in conjunction with $J_1$'s having the same radicals for their arguments. Similar considerations exclude the possibility of a complex solution. The latter point is of interest in that it implies that the displacement is in phase at all points of the bar.

We have plotted $W$ as a function of $r/a$ for various values of the wave-length in Fig. 2, using values of the velocity taken from Table I, with $\sigma=\frac{1}{4}$. Each curve for $W(r/a)$ has been adjusted to make the maximum amplitude $\pm 1$. When $d/L=0$, we have of course a uniform displacement. But as the wave-length is decreased, the displacement at the center of the bar rapidly decreases, and vanishes when $d/L$ is about $\frac{1}{4}$. Then for a small range of wave-length the displacements at the center of the bar are of opposite sign to those on the surface, and at some point along the radius, we encounter a node. This situation is altered as $d/L$ becomes slightly greater than 0.4, when the displacement at the surface of the bar vanishes. As the wave-length is still further decreased, the displacement reappears at the surface, and when $d/L=1$, it is nearly uniform once more. After this, the displacement at the center of the bar gradually decreases, until finally, when the wave-length becomes infinitesimal, the motion is confined strictly to the outside surface. The Rayleigh wave velocity obtained for the limiting case of Eq. (2) corresponds to these surface waves.

The reduction of the frequency equation to a function of three variables can be carried out as above for the cases of both torsional and flexural waves. In both cases, the parameters and functions required are the same as for the longitudinal waves.

For a torsional disturbance, we find at once

$$\varphi(ka)-2=0 \tag{7}$$

which yields the velocities not only for the familiar case where the displacement is proportional to the radius, but for the more complicated cases as well. For these cases we find

$$v/v_0=\left(\frac{C(L/d)^2+\frac{1}{2}}{\sigma+1}\right)^{\frac{1}{2}}, \tag{8}$$

where $v_0$ is again $(E/\rho)^{\frac{1}{2}}$, and $C$ is a constant obtained by solving Eq. (7).

For the case of flexural waves, even the simplified frequency equation is probably too complicated to be of much use, but it seems worth while recording it; *viz.*,

$$\begin{vmatrix} x-1 & 2x-1 & 2x-1 \\ \varphi(ha)-2 & \varphi(ka)-2 & -2[\varphi(ka)-2]-(\gamma a)^2(2x-1) \\ \varphi(ha)-1 & -(x-1)[\varphi(ka)-1] & 1 \end{vmatrix}=0. \tag{9}$$

A partial check on the accuracy of (9) was obtained by substituting the first few terms of the series expansion for the $\varphi$ functions, to obtain the simple formula for the approximate velocity of flexural waves. Construction of a table of roots of (9) similar to Table I would involve a considerable expenditure of time. If limiting values of the $\varphi$ functions of an imaginary argument are substituted in (9), we again obtain Rayleigh's equation. The form of the displacements for this limiting case has not been attempted, but it seems likely that here also the displacement becomes confined to the surface.

In applying the solutions of Eqs. (2), (7), and (9) to bars of finite length, only torsional waves yield a solution which permits rigorous satisfaction of the boundary conditions on plane surfaces at the ends of the bar. For the torsional case, then, we may expect that the solution applies regardless of the length of the bar, even when it is so short as to become a disk. For the other two cases, it is possible to determine the wave-length so as to make the normal traction across the end surfaces vanish, though a small shearing traction remains. Experimentally, we know that this shearing traction is zero when resonance conditions obtain. If we consider a slender bar, vibrating so that it contains many wave-lengths, we expect Pochhammer's solutions to be exact, except in the neighborhood of the ends. Here we may suppose that the wave-length becomes slightly altered, and if we attempt to compute it by dividing the length of the bar by the number of waves it contains at resonance, a small error will be introduced. This discrepancy has led to considerable confusion as to the precision which may be expected of Pochhammer's theory in any specific case. It is certain that the failure makes it impossible to develop a solution for a thin disk from the solution for a cylindrical bar, for in the case of the thin disk, the boundary conditions on the flat surfaces are of paramount importance, while those on the curved surface are trivial—quite the reverse of the situation for the bar. On the other hand, experimental evidence indicates that when the ratio of diameter to length is less than 0.4, the error involved in neglecting these stresses is less than 0.05 percent. Table II gives velocities of longitudinal waves in a set of specimens cut from a single length of $\frac{3}{8}$-inch drill rod. The measure-

TABLE II. *Observed velocity of longitudinal waves in steel bars of different lengths.*

| LENGTH (CM) | WAVE-LENGTH (CM) | RESONANT FREQUENCY | VELOCITY (KM/SEC.) |
|---|---|---|---|
| 15.235 | 10.157 | 50473 | 5.126 |
| 5.083 | 10.166 | 50426 | 5.126 |
| 15.235 | 7.518 | 67184 | 5.118 |
| 3.812 | 7.624 | 67162 | 5.120 |
| 15.235 | 5.078 | 100390 | 5.098 |
| 2.540 | 5.081 | 100364 | 5.099 |

ments were made by the method of Bancroft and Jacobs.[8] It will be seen that, within the experimental error, the observed velocity as computed from the assumed wave-length is independent of the length of the specimen, and is a function of the wave-length alone.

Except for the effects of the above-mentioned discrepancy, which we have shown experimentally to be slight in the case of longitudinal waves, it is felt that the velocities determined from Pochhammer's solutions may be used with perfect confidence.

In conclusion, it appears that the work of Shear and Focke[1] is explained on a quantitative basis. Table I reproduces the behavior of the longitudinal vibrations which they observed with remarkable fidelity. The flexural vibrations behave in much the way one would expect in the light of Eq. (9), and it is noteworthy that the experimental data suggest that a common asymptote for the flexural and longitudinal velocities at high frequency is not improbable. The observed torsional vibrations may possibly belong to one of the sheets of Eq. (7). It also seems likely that the unexplained points at high frequency lie upon one of the higher sheets of Eqs. (2), (7), or (9), for in a qualitative way they lie in the region associated with these more complicated vibrations.

It is a pleasure to acknowledge our indebtedness to Professor Francis Birch for his encouragement and help, particularly in checking most of the rather tedious algebraic work.

[8] D. Bancroft and R. B. Jacobs, Rev. Sci. Inst. 9, 279 (1938).

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# Non-Uniform Particle Density in Nuclear Structure

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The Coulomb repulsion between protons must give rise to a tendency for the proton density to vary within a nucleus from a minimum value at the center to a maximum near the boundary. A non-uniform proton density may be expected to create forces which distort the neutron distribution and tend to make the two distributions vary in the same manner. If surface effects are neglected, it is possible to calculate the energy correction associated with the non-uniform densities without making special assumptions about the nuclear forces. The neglect of surface effects permits the assumption that the variations in density are small departures from essentially constant distributions. It is found that the departure from uniform density is appreciable in heavy nuclei, but the energy correction is negligible.

## 1. Introduction

The Coulomb repulsion between protons must give rise to a tendency for the proton density to vary within a nucleus from a minimum value at the center to a maximum near the boundary. A non-uniform proton density may be expected to create forces which distort the neutron distribution and tend to make the two particle densities vary in the same manner. If surface effects are neglected, it is possible to calculate the energy correction associated with the non-uniform densities as well as the densities themselves in a comparatively rigorous and simple manner. The neglect of surface effects permits the assumption that the variations in density are small departures from essentially constant distributions. A suitable model for the systematic neglect of surface effects is provided by supposing the nuclear system enclosed in a box of radius $R$. At the boundary the radial derivative of the wave function with respect to any radial coordinate must vanish:

$$\frac{\partial}{\partial r_l}\psi(x_1y_1z_1\cdots x_Ay_Az_A)=0,\ r_l=R. \qquad (1)$$