obtained value 0.75. The agreement is as good as the accuracy of the experiments would lead us to expect.

## VI. Summary

The absence of a constricting effect in a *uniform* longitudinal magnetic field has been confirmed. The primary electrons from the cathode formed a collimated beam which bore the imprint of cathode structure and which disappeared rather abruptly into a general glow. "Abnormal" electron distributions were found to be transmitted with decreasing amplitude along the arc in the direction of the anode.

A magnetic field of 70 oersteds distorted probe characteristics at $5.4\times 10^{-3}$ mm pressure so badly as to make them uninterpretable. Therefore attention was confined to a 50-oersted field. The stable cross-sectional distribution of electron density was the same with this field as without field and a distribution which differed from the "normal" tended to become normal in the direction of the anode.

The abnormal distribution was found to obey a modified Boltzmann distribution in accordance with the theory of an arc in a longitudinal field.

Unfortunately the magnetic field distortion of the probe characteristics began in the range where the effects on the arc were just becoming large enough to exhibit significant differences from the zero field condition. As a consequence the experimental results did not exhibit the magnetic effects as vividly as could be wished and the test of the theory was a rather mild one.

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# Transmission of an Electron Density Disturbance Along a Positive Column in a Longitudinal Field

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The distribution of ions and electrons in the cross section of a uniform positive column is maintained by the radial motions of these particles. This distribution is designated as "normal." A disturbance of this distribution at some point in the column is followed on the anode side by an asymptotic approach to the normal. In the absence of a longitudinal magnetic field the recovery of a normal distribution occurs within a very short distance, but a longitudinal magnetic field slows down the readjustment by decreasing the radial mobility of the electrons. By making certain simplifying assumptions a theory for the approach of a disturbed column back to normal when the disturbance is cylindrically symmetrical has been worked out. The distribution is developed in a series of zero order Bessel functions, and it is found that the first term, which corresponds to the normal distribution, approaches a constant amplitude along the column whereas successive terms have successively greater space decrements. The decrement of the $i$th term approaches the constant value

$$\frac{(x_i^2-x_1^2)\alpha(T_e+T_p)}{(1-\alpha)T_e a^2(-\epsilon)},$$

where $x_i$ is the $i$th root of $J_0(x)=0$, $\alpha$ is the factor giving the reduction in transverse electron mobility, $T_e$ and $T_p$ are the electron and positive ion temperatures, $a$ is the radius of the column and $\epsilon$ is $(e/kT_e)(\partial V/\partial z)$. When compared with the experimental results of Cummings and Tonks, the theory calls for a reduction of the second term of the series expansion to 50 percent in 12.5 cm of arc length whereas experiment gave 33 percent.

## 1. Introduction

In the uniform positive column of a low pressure discharge there is a *normal* distribution of electron (and ion) density in the cross section which maintains a balance, in each element of volume, between the new ions and electrons formed there and the net rate at which these particles escape radially under the influence of the electric fields and concentration gradients. The net longitudinal outflow is zero because of the axial uniformity of the arc. Such a normal distribution exists whether the column be in a finite uniform magnetic field or none at all but the distribution will of course be different depending on the orientation and strength of the magnetic field. Experiment[1] indicates in agreement with theory[2] that the normal distributions for no field and for uniform longitudinal field are identical.

With no field the distribution reaches the normal in very short distances, evidence of which is found in the uniformity of the positive column even close to the cathode or to the point where a change in column diameter occurs. This does not mean that deviations from the normal electron *temperature* may not extend along the column for some distance, but that the charge reaches its normal distribution quite promptly.

The presence of a longitudinal magnetic field, however, reduces the transverse mobility of the electrons and thus decreases the ease with which an abnormal distribution disappears. The electrons, moving longitudinally with ease and transversely with difficulty, propagate the maldistribution along the column. Such effects have been definitely found in experiments on a mercury arc column reported by C. S. Cummings and L. Tonks.[1] It therefore becomes of interest to analyze the problem theoretically to gain, if possible, a quantitative understanding of the phenomenon. The problem is that of dealing with a non-uniform column in which longitudinal transfer of ions and electrons do play a part.

## 2. *A Priori* Limitations of the Theory

The present application of the theory will be only to the case for which the plasma extends to within a short distance of the tube wall. The wall sheath is supposed to be thin, so that the positive ions originating in the plasma flow freely to the walls. Any case in which the column has been pulled away from the wall, as by a sharply increasing magnetic field, a constriction in the arc tube, or by a cathode of limited area brings in factors which will not be considered here. The present treatment is concerned rather with what goes on beyond the point at which such a constricted column first expands to fill the whole cross section.

The theory to be developed will suffer from one of the same errors as the present theory of the positive column, namely that arising from the assumption that the ions drift with a velocity proportional to the electric field. We know that in the range of pressures and fields which exist in the plasma their motion is more nearly proportional to the square root of the field.[3] But such a law would result in nonlinear equations whose solution would be practically impossible. Only disturbances which have cylindrical symmetry will be treated because it has not been found possible to solve the equations which include azimuthal variation.

Other limitations of the theory, which are consequences of mathematical necessity, will appear as the theory is developed.

## 3. The Fundamental Equations

For definiteness let the positive $z$ direction be toward the anode so that it is the direction of electron drift.

As usual, equality of electron and positive ion densities $n_e$ and $n_p$, respectively, is assumed:

$$n_e = n_p, \tag{1}$$

so that where necessary these quantities can be used interchangeably and the subscripts can be dropped.

There are six other equations which form the system which will be used to describe the behavior of the plasma. The first four are the drift equations for the transverse ($u$) and longitudinal ($w$) drift velocities of the electrons (subscript $e$) and ions (subscript $p$):

$$u_e = -\alpha D_e\left(\frac{\partial n_e}{n_e \partial r} + \frac{e}{kT_e}\frac{\partial V}{\partial r}\right), \tag{2}$$

$$u_p = -D_p\left(\frac{\partial n_p}{n_p \partial r} - \frac{e}{kT_p}\frac{\partial V}{\partial r}\right), \tag{3}$$

$$w_e = -D_e\left(\frac{\partial n_e}{n_e \partial z} + \frac{e}{kT_e}\frac{\partial V}{\partial z}\right), \tag{4}$$

$$w_p = -D_p\left(\frac{\partial n_p}{n_p \partial z} - \frac{e}{kT_p}\frac{\partial V}{\partial z}\right). \tag{5}$$

Here $D_e$ and $D_p$ are electron and ion diffusion coefficients in the absence of magnetic field and $\alpha$ is the fraction giving the decrease of transverse electron mobility due to the magnetic field.[2]

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[1] C. S. Cummings and L. Tonks, preceding paper.
[2] L. Tonks, Phys. Rev. **56**, 360 (1939).
[3] K. H. Kingdon and E. J. Lawton, Phys. Rev. **56**, 215 (1939).

The other two equations are those of continuity for electrons and ions, respectively:

$$\frac{\partial(n_e u_e r)}{r\partial r}+\frac{\partial(n_e w_e)}{\partial z}=\lambda n_e, \quad (6)$$

$$\frac{\partial(n_p u_p r)}{r\partial r}+\frac{\partial(n_p w_p)}{\partial z}=\lambda n_e, \quad (7)$$

$\lambda$ being the rate of production of ion-electron pairs per electron per second.

Even on the simplest theoretical grounds another relation giving the relation between $\lambda$ and the arc gradient should be included. Our knowledge of this relationship is meager, however. In addition, its introduction would create great mathematical difficulties, and there is a sufficiently satisfactory detour around this complication, which will be introduced later.

Finally, besides the usual boundary condition that $n$ be zero at the tube wall, we must seek a solution in which (1) $\partial V/\partial z$ becomes constant for large enough $z$ and (2) the wall current is small so that the total arc current does not change rapidly along the arc length.

## 4. Necessary Approximations

In order to solve the six equations in convenient form it is necessary to make certain simplifications. First, we shall assume that the "partial" drift arising from longitudinal diffusion is small compared to that arising from the electric field so that the quantities $\partial n/n\partial r$ may be dropped from Eqs. (4) and (5). If the diffusion terms are left in these equations, a $\partial^2 n/\partial z^2$ term appears later in Eq. (11) which makes the solution under the assumed conditions far more difficult.

Second, we assume that the arc gradient is essentially constant, an assumption that will be connected with the question of the relation between $\lambda$ and $\partial V/\partial z$.

The various simplifications result in the following two equations as a consequence of substituting Eqs. (2) to (5) into Eqs. (6) and (7):

$$-\alpha D_e \nabla_r^2 n-\alpha D_e\frac{\partial}{r\partial r}\left(rn\frac{\partial\eta}{\partial r}\right)-D_e\epsilon\frac{\partial n}{\partial z}=\lambda n, \quad (8)$$

$$-D_p\nabla_r^2 n+\frac{D_pT_e}{T_P}\frac{\partial}{r\partial r}\left(rn\frac{\partial\eta}{\partial r}\right)+\frac{D_pT_e}{T_P}\epsilon\frac{\partial n}{\partial z}=\lambda n, \quad (9)$$

where $\nabla_r^2$ is the Laplacian with respect to $r$ only, and

$$\eta=eV/kT_e, \quad \epsilon=\partial\eta/\partial z. \quad (10)$$

Note that $\epsilon$ is intrinsically negative for electron flow toward the anode.

These equations contain the two dependent variables $n$ and $\eta$. Multiplying the first by $D_pT_e/T_p$, the second by $\alpha D_e$ and adding gives

$$\nabla_r^2 n+P\partial n/\partial z+Q\lambda n=0 \quad (11)$$

with

$$P=\frac{(1-\alpha)T_e\epsilon}{\alpha(T_e+T_p)}, \quad (12)$$

$$Q=\frac{\alpha D_eT_p+D_pT_e}{\alpha D_eD_p(T_e+T_p)}. \quad (13)$$

The form of Eq. (11) suggests zero-order Bessel functions as appropriate solutions. It is already known that the function whose first root lies at the wall, namely $J_0(2.40\ r/a)$, is the solution when $\partial n/\partial z$ is zero. This normal solution, as has already been stated, just balances the net radial outflow of ions and electrons from each elementary volume against the creation of new particles. As a consequence, $N$, the number of electrons per cm length of arc, which is given by

$$N=2\pi\int_0^a nrdr \quad (14)$$

is constant.

Now it is obvious that any distribution other than the normal will fail to maintain such a balance. For example, a distribution in which the relative concentration at the axis is larger would result in a slower escape of electrons and hence an increase in $N$ toward the anode. The constancy of the arc current then would require that the gradient decrease, and this in turn would cause a decrease in $T_e$ and $\lambda$, thus bringing compensating factors into play. The faster $\lambda$ changes with $T_e$, the smaller would be the actual changes in $T_e$ and $\epsilon$ which would accomplish the compensation. Actually $\lambda$ varies rapidly with $T_e$, and for present purposes it will be assumed that this variation is so great as to bring about complete compensation with only infinitesimal, and hence negligible, changes in $T_e$ and $\epsilon$. Accordingly $\lambda$ is to be treated as a function of $z$ in Eq. (11), and is to be determined by the condition that $N$ remain constant, if there is no net current to the wall, or by the actual variation in $N$ which is consistent with a wall current.

## 5. Solution of the Equations*

We suppose that the actual charge distribution is analyzed by a Bessel-Fourier analysis into the sum of a set of Bessel function distributions, each with its unique dependence on $z$, represented by $Z_i$:

$$n=\sum_{i=1}^{i=\infty} n_i J_0\left(\frac{x_i r}{a}\right) Z_i. \tag{15}$$

Here $x_i$ is the $i$th root of $J_0(x)=0$ so that the wall-boundary condition, $n=0$, is automatically fulfilled. If each term in the series of Eq. (15) satisfies Eq. (11), substitution reveals that

$$n=\sum n_i J_0\left(\frac{x_i r}{a}\right) \exp\left[P^{-1}\int_0^z \left(\frac{x_i^2}{a^2}-Q\lambda\right)dz\right], \tag{16}$$

where, as has been remarked, $\lambda$ is a function of $z$.

With insulating walls $N$ is constant, so that

$$0=\frac{\partial N}{\partial z}=\frac{2\pi\partial}{\partial z}\int_0^a nrdr$$

$$=2\pi a^2 \exp\left[\int_0^z \frac{-Q\lambda}{P}dz\right]\sum \frac{n_i}{x_i}\left(\frac{x_1^2}{a^2}-Q\lambda\right) \times J_1(x_i)\exp\left(\frac{x_i^2 z}{a^2 P}\right)$$

and solving for $\lambda$:

$$a^2 Q\lambda=\frac{\sum n_i x_i J_1(x_i)\exp(x_i^2 z/a^2P)}{\sum n_i x_i^{-1} J_1(x_i)\exp(x_i^2 z/a^2P)}. \tag{17}$$

Turning now to the transverse gradient in the column, Eq. (16) is to be substituted into either Eq. (8) or (9) giving

$$\frac{D_eD_p(T_p+\alpha T_e)x_i^2-a^2\lambda(D_eT_p+D_pT_e)}{aD_eD_pT_e(1-\alpha)} \times\sum\frac{n_i'}{x_i}J_1\left(\frac{x_i r}{a}\right)+n\frac{\partial\eta}{\partial r}=0, \tag{18}$$

where

$$n_i'=n_i\exp\left[P^{-1}\int_0^z\left(\frac{x_i^2}{a^2}-Q\lambda\right)dz\right], \tag{19}$$

is the amplitude of the $i$th partial distribution at the axial distance $z$.

## 6. Interpretation

The reasonable expectation is that the amplitude of each of the partial distributions, except the normal, will decrease with distance, and that the normal will approach a constant value. It will now appear that the solution describes just such behavior. It is necessary that $\partial n/\partial r$ shall be negative or zero at $r=a$. At $z=0$, then,

$$0\geqq \partial n/\partial r|_a=-a^{-1}\sum n_i x_i J_1(x_i),$$

which assures a positive or zero numerator in the right member of Eq. (17). The denominator, being proportional to $N$, is positive. It follows that $\lambda$ is initially positive or zero. Now each successive term in the two sums decreases more rapidly than the last because of the increase in $x_i$ from term to term and the fact that $P$ is negative. This assures that even if $\lambda$ was initially zero, it becomes and remains positive as $z$ increases, and approaches the limiting value

$$\lambda\rightarrow x_1^2/a^2Q. \tag{20}$$

Examination of Eq. (16) then shows that the normal ($i=1$) term of the sum approaches constancy, whereas every subsequent term decreases with a variable "absorption coefficient" $\gamma_i$ which approaches the value $-(x_i^2-x_1^2)/a^2P$.

Using Eq. (12) we have

$$\gamma_i\rightarrow(x_i^2-x_1^2)\frac{\alpha(T_e+T_p)}{(1-\alpha)T_e a^2(-\epsilon)}. \tag{21}$$

From Bessel function tables it is found that the coefficients for successive $\gamma$'s have the values:

| $i$: | 2 | 3 | 4 |
|---|---|---|---|
| $(x_i^2-x_1^2)$: | 24.7 | 69.0 | 133.3 |

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* I am indebted to Dr. A. V. Hershey for valuable help in solving the differential equations.

## 7. Limitations of the Solution

For zero magnetic field $\alpha$ is unity and Eq. (21) becomes meaningless. As this condition is approached by decreasing the magnetic field the limiting values of the $\gamma$'s become so large that the original neglect of longitudinal diffusion is seen to be no longer justified. This approximation therefore limits the validity of the present analysis to cases where the magnetic field is not too small. Since the longitudinal diffusion is relatively more important for the higher order components of the disturbance, the analysis is also limited in that direction and ceases to apply for values of $i$ that are too great.

The more crucial of the two places where diffusion has been neglected is in Eq. (4). There the assumption was that

$$1 \ll \frac{\epsilon}{n_i^{-1}\partial n_i/\partial z} = \frac{(1-\alpha)T_e\epsilon^2}{\alpha(T_e+T_p)(x_i^2/a^2-Q\lambda)}.$$

As this condition is chiefly of interest for $i$ greater than unity the particular value of $\lambda$ used makes little difference. We therefore choose its limiting value from Eq. (20). Since $T_e+T_p \backsimeq T_e$, the condition becomes

$$\frac{(1-\alpha)\epsilon^2a^2}{\alpha(x_i^2-x_1^2)} \gg 1. \tag{22}$$

## 8. Theory for Non-Insulating Walls

The case to which attention has been confined so far is that in which the arc current remained constant along the length of the column, that is, the case of insulating walls which impose the condition that $u_e = u_p$ at the wall. Equation (11) was derived, however, without any such assumption, so that it is valid even when a net wall current flows. Only $\lambda$ depended on that assumption, so that a change in $\lambda$ should take care of its abandonment.

To apply the theory to this case we assume that the cross section distribution is normal so that $n_2$, $n_3$, etc. are zero. Any other assumption presents great complications. Also, for simplicity, we suppose that the current variation is exponential, that is, in terms of $N$,

$$N = N_0\epsilon^{\beta z}. \tag{23}$$

Comparison with Eq. (16) shows that

$$x_1^2/a^2 - Q\lambda = P\beta. \tag{24}$$

If now this is used to eliminate $\lambda$ from Eq. (18) we find that

$$\frac{\partial\eta}{\partial r} = \tau_a \frac{x_1 J_1(x_1 r/a)}{a J_0(x_1 r/a)}, \tag{25}$$

where

$$\tau_a = \frac{\alpha D_e - D_p + a^2 x_1^{-2}\beta(-\epsilon)(D_e + D_p T_e/T_p)}{\alpha D_e + D_p T_e/T_p}. \tag{26}$$

Integration of Eq. (25) gives

$$J_0(x_1 r/a) = \epsilon^{-\eta/\tau a}, \tag{27}$$

a result of the same form as has already been derived[2] without considering longitudinal effects.

These results, of course, are consistent with the continuity of current in the arc and to the walls. A calculation to demonstrate this simply serves as a check on the mathematics and tells us nothing new.

We can, however, confirm a previous assumption[2] that the ratio of $u_e$ to $u_p$ is a constant over the cross section of the column under these conditions. From Eqs. (2) and (3):

$$u_e = -\alpha D_e x_1(\tau_a - 1)J_1(x_1 r/a)/aJ_0(x_1 r/a),$$
$$u_p = -D_p x_1(1+\tau_a T_e/T_p)J_1(x_1 r/a)/aJ_0(x_1 r/a)$$

with the previous definition of $\mu$,[2]

$$\mu = \frac{u_e}{u_p} = \frac{\alpha D_e(1-\tau_a)}{D_p(1+\tau_a T_e/T_p)}, \text{ a constant,} \tag{27}$$

and the previously given value for $\tau_a$ ($\tau$ in the reference) is confirmed:

$$\tau_a = \frac{\alpha D_e - \mu D_p}{\alpha D_e + \mu D_p T_e/T_p}. \tag{28}$$

## 9. Comparison with Theory

The only experimental results which are available for comparison with this theory are those of Cummings and Tonks. Their Fig. 8 shows the approach to a normal distribution with increasing $z$ of a distribution which was initially peaked at the axis. We assume that of the abnormal partial distributions, only $n_2$ is appreciable and that in the range shown $\gamma_2$ is a constant.

It would be simplest if the 47-ma normal distribution contained the same total number of electrons as the abnormal distribution. Then it could be shown,[4] in terms of the values in the figure, that

$$\epsilon^{-12.5\gamma_2}=(51-47)/(59-47)$$

whence

$$\gamma_2=0.088.$$

We know, however, that $N$ for the 59-ma plasma is 10 percent greater than for the 47-ma plasma.[1] But since this difference is itself proportional to the magnitude of $n_2$, it will be proportionately less with increasing $z$ so that the above result is not affected.

To calculate $\gamma_2$ from Eq. (21) we take the following values from Cummings and Tonks: $T_e=18{,}500°K$, $T_p=350°K$, $\alpha=1.92\times10^{-3}$, $a=2.3$ cm, $-\epsilon=0.28\times11{,}600/18{,}500$. It follows that

$$\gamma_2=0.052.$$

The agreement, though not close, is reasonable in view of theoretical approximations and experimental uncertainties.

Finally, it is necessary to find to what degree the condition of (22) is fulfilled. It appears that the left member has the value 3.4 for the $n_2$ distribution and would be only 1.2 for the $n_3$. Therefore the neglect of the diffusion term from Eq. (4) introduces considerable inexactness in the present calculation, and the analysis becomes exact only when larger magnetic fields giving considerably smaller values of $\alpha$ are used.

[4] See Appendix.

## Appendix

Calculation of absorption coefficient of second partial distribution from axial concentration measurements.

Let $n_f$ = axial concentration at first point, with field.
$n_s$ = ditto second point.
$n_0$ = axial concentration with no field but for same total charge in cross section.
$n_1$ = axial concentration of first partial distribution at first point.
$n_2$ = ditto second partial at first point.
$n_1'$ = ditto first partial at second point.
$n_2'$ = ditto second partial at second point.
$f=\epsilon^{-\gamma_2 z}$ = fractional decrease in second partial between points.

Equating total charges in the cross section gives,

$$n_1J_1(x_1)/x_1-n_2J_1(x_2)/x_2=n_0J_1(x_1)/x_1,$$
$$n_1'J_1(x_1)/x_1-n_2'J_1(x_2)/x_2=n_0J_1(x_1)/x_1.$$

The axial concentrations at first and second points are:

$$n_1+n_2=n_f;\quad n_1'+n_2'=n_s.$$

By definition, $n_2'=fn_2$. By using the last equation to eliminate $n_2'$ and substituting $\lambda$ for $x_1J_1(x_2)/x_2J_1(x_1)$:

$$n_1-\lambda n_2=n_1'-\lambda fn_2=n_0,$$
$$n_1+n_2=n_f,\quad n_1'+fn_2=n_s.$$

From these

$$n_s-n_0=f(1+\lambda)n_2;\quad n_f-n_0=(1+\lambda)n_2,$$

whence

$$f=(n_s-n_0)/(n_f-n_0)=\epsilon^{-\gamma_2 Z}.$$