On the Magnitude of Electronic Charges

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A charged point particle is characterized first by its classical rest energy $\epsilon_0 = mc^2$, second by the characteristic wave period $t_0 = 4\pi e^2/3mc^3$ that occurs in the classical formula $\phi_\tau/\phi_\infty = [1+(t_0/\tau)^2]^{-1}$ for the ratio of the scattering cross section for light of period τ compared with the Thomson cross section ϕ_∞ for light of period ∞. Quantum theory asks that the product $\epsilon_0 t_0$ be as small as $h/210$ as the smallest eigenvalue of a proper value problem. The classical values of ϵ_0 and t_0 together with their quantum product yield the value $\alpha \approx 1/140$ for Sommerfeld's fine-structure constant. The discrepancy may be due to our *classical* treatment of the interaction between light and charged matter. The Lorentz invariance of the energy transmission coefficient $R_\tau = \phi_\tau/\phi_\infty$ makes R_τ apt to serve as a reduction factor in the theory of radiation for avoiding the usual infinities. As a first example, the energy reduction R_τ leads to a modified Coulomb energy together with a finite electrostatic self-energy corresponding to an electrostatic mass $m_{\text{stat}} = m$.

THE velocity of a particle can be found in two ways; first, by measuring the energy ϵ and the momentum p so that

$$v/c = pc/\epsilon \quad \text{whereby} \quad \epsilon^2 - (pc)^2 = \epsilon_0^2, \tag{1}$$

if ϵ_0 stands for the rest energy; and second, by measuring the path r during a time interval t so that

$$v/c = r/ct \quad \text{whereby} \quad t^2 - (r/c)^2 = t_0^2, \tag{2}$$

if t_0 stands for the rest value of the interval t.

The quantities p and ϵ can be measured optically if the particle is charged. The Compton scattering effect yields p and ϵ within certain margins of accuracy since p and ϵ change during the observation. The universal rest energy ϵ_0 which also is the critical photonic energy for the pair production, has a definite value $\epsilon_0 = mc^2$ without uncertainty.

The quantities r and t can also be measured optically by means of light waves scattered by the particle. The Lorentz rest system in which the scattering process takes place is defined by the system of interference fringes of the incident and reflected light and matter waves that move through the distance r during t. The position of the maxima cannot be measured exactly at any time. Therefore r and t are determined within a certain margin only, although the rest period t_0 has a definite universal value (see (7)).

The two uncertainties (corpuscular p and ϵ, wave r and t) are reciprocal in the sense of quantum theory. The probability amplitudes $\psi(r, t)$ and $\chi(p, \epsilon)$ comply with the quantum rule (for free particles) that they are Fourier expansions of one another. At the same time ϵ is determined by p and ϵ_0, and t is determined by r and t_0 by virtue of (1) and (2). The Fourier integrals[1] read

$$\begin{aligned} \chi(p) &= h^{-\frac{3}{2}} \int \psi(r) \\ &\quad \times \exp\left[i/\hbar(p\cdot r - \epsilon t)\right] dx dy dz (t_0/t)^{\frac{1}{2}}, \\ \psi(r) &= h^{-\frac{3}{2}} \int \chi(p) \\ &\quad \times \exp\left[-i/\hbar(p\cdot r - \epsilon t)\right] dp_x dp_y dp_z (\epsilon_0/\epsilon)^{\frac{1}{2}}. \end{aligned} \tag{3}$$

The square roots on the right provide for invariant volume elements. The factors $h^{-\frac{3}{2}}$ are chosen so that the two integral equations are mere inversions of one another in the nonrelativistic limit. The integrals are carried over positive and negative values of the square roots; they are soluble only for certain proper values of the quantity

$$\epsilon_0 t_0/\hbar = \mu. \tag{4}$$

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[1] The integrals are modifications of Born's integrals. Born modified the writer's original wrong integrals. See A. Landé, J. Frank. Inst. **228**, 459 (1939). M. Born, Proc. Roy. Soc. Edinburgh **59**, 219 (1939). The proper value theory of Born and Fuchs, Proc. Roy. Soc. Edinburgh **60**, 100, 141 (1940) starts from a different background and arrives at different results.

The smallest proper value of μ turns out to be[2]

$$\mu = 0.02985037\cdots \tag{4'}$$

as the smallest root of the transcendental equation $2\pi\mu[Y_0(\mu)]^2 = 1$.

The smallness of the product $\epsilon_0 t_0$ in terms of h, namely $\epsilon_0 t_0/h \approx 1/210$ must be considered as the chief reason for the smallness of the Sommerfeld fine-structure constant α. Indeed, the latter is

$$\alpha = \frac{e^2}{c\hbar} = \frac{e^2}{ct_0\epsilon_0}\cdot\frac{t_0\epsilon_0}{\hbar} = \frac{e^2}{ct_0mc^2}\cdot\mu = \frac{\mu}{\gamma}, \tag{5}$$

if γ is the numerical factor expressing t_0 in terms of e^2/mc^2 by the formula

$$ct_0 = \gamma e^2/mc^2. \tag{6}$$

A tentative value of γ is obtained by identifying t_0 with the characteristic period t_0 that occurs in the scattering cross section ϕ_τ for light of period τ, as compared with the Thomson cross section ϕ_∞ for light of period ∞. The classical theory of the scattering of infinitely weak light waves yields the formula

$$\phi_\tau = \phi_\infty\cdot[1+(\tau/t_0)^2]^{-1} \text{ where } t_0 = 4\pi e^2/3mc^3. \tag{7}$$

From identifying t_0 of (2) with t_0 of (7) we learn that γ is $4\pi/3$, hence

$$\alpha = \mu/\gamma = 0.0298\cdot(3/4\pi) \approx 1/140 \tag{8}$$

instead of the experimental value $\approx 1/137$. The discrepancy may be due partly to the value of ϕ_τ which was obtained from the *classical* scattering of infinitely weak light waves, as against quantized waves of zero-point energy. A uniform theory is wanted in order to replace the two incoherent sections of computing $\alpha = \mu/\gamma$ from a quantum calculation of μ and a classical calculation of γ. The not quite convincing classical reasons for choosing $\gamma = 4\pi/3$ will be discussed in a paper, Part III, in the *Journal of the Franklin Institute* (1941) to which we also refer for the following remarks.

[2] A. Landé, J. Frank. Inst. **229**, 767 (1940). Part I.

The ratio $R_\tau = \phi_\tau/\phi_\infty$ is apt to serve as a reduction factor to rid the radiation theory of infinities. *R is Lorentz-invariant* since ϕ_∞ is a universal constant, and ϕ_τ depends on the scattered period τ and the speed of the scattering particle in the following way. The light may have the period τ for an observer who is at rest together with the particle. If the observer moves with velocity v' relative to the former rest system he will observe a different period τ'. But the scattering cross section, that is, the ratio of the scattered to the incident intensity per unit area will be the same as in the rest system namely R_τ rather than $R_{\tau'}$. Indeed, the scattering cross section can be thought of as the cross section of a (missing) column of light cut out of the incident parallel light rays, the walls of the column being light rays.

A first application of this reduction factor for the energy transmission is offered by the Dirac-Fermi wave theory of the electrostatic interaction between particles and light. This theory leads to the Coulomb energy e_je_k/r_{jk} and to an infinite self-energy. If one assumes, however, that the energy contribution of the waves of period τ is reduced by the invariant factor $R_\tau = \phi_\tau/\phi_\infty$ of (7), then one obtains a modified Coulomb energy

$$(e_je_k/r_{jk})\cdot[1-\exp(-r_{jk}3mc^2/2e^2)] \tag{9}$$

and a finite electrostatic self-energy of value $E_{\text{stat}} = \frac{3}{4}\,mc^2$. This corresponds to an electrostatic mass

$$m_{\text{stat}} = m, \tag{9'}$$

since, according to Abraham, any spherical electric field of energy E_{stat} has an inertia $m_{\text{stat}} = E_{\text{stat}}\cdot(4/3c^2)$. The reduction of the infinite self-energy through radiation damping is preferable to an arbitrary cutting-off process, and perhaps also to those theories (Born-Infeld and Born's reciprocity) that yield the correct finite self-energy only with a certain selection of an adjustable parameter r_0 called "electronic radius" (determination of the factor γ *a posteriori*).