lated terms for V II $3d^4$ from Ostrofsky's formulas are given in Table III. The parameters were found from ^3H, ^3G, ^3D, ^1I, and ^1F as these terms were considered the most reliable. Finally Table IV gives the formulas for the term values of d^5s and the observed and calculated values from Mn II.

Although the agreement of the observed and calculated term values is rough in terms of the accuracy with which spectra may be observed, some idea of the relative positions of the terms is a help in analysis as Meggers and Moore[1] have pointed out. As a further aid in the analysis of Mn II, it would be advisable to separate the different terms of the same kind in d^5 by the methods[6] used for d^3. I am grateful to Professor C. W. Ufford for suggesting this problem and for his assistance in solving it.

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Magnetization Near Saturation in Polycrystalline Ferromagnets

T. HOLSTEIN, *New York University, New York, New York*

AND

H. PRIMAKOFF, *Queens College, Flushing, New York*

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An important mechanism involved in the variation of the magnetization of ferromagnets near saturation is the rotation of the magnetization vector under the combined influence of the magnetic field and crystalline anisotropy torques. The effect of this mechanism in polycrystalline specimens has been calculated by Akulov and Gans; their result is $M_{\text{Av}}=M_0[1-(c/M_0^2)H^{-2}]$, where H is the sum of the external and demagnetizing fields, M_0 the saturation magnetization, and c a constant proportional to the square of the crystalline anisotropy constant. The Akulov-Gans derivation, however, is subject to a serious error; namely, neglect of the internal magnetic field arising from the magnetization itself. In the present paper, this internal field is taken into account; then, with the dual assumption, of randomness of orientation of crystallographic axes, and irregularity of shapes of the individual crystal grains, one obtains the formula, $M_{\text{Av}}=M_0[1-(c'/M_0^2)H^{-2}]$. Here, c' is a slowly varying function of H; for $H\gg 4\pi M_0$, $c'=c$; $H\ll 4\pi M_0$, $c'=\frac{1}{2}c$. Applications of the last formula to the analysis of the experimental data are discussed.

I. Introduction

ACCORDING to present day theories[1] of the magnetization curve of a polycrystalline ferromagnetic in high fields, an important contribution to the deviation of the observed magnetization, $|\mathbf{M}|_{\text{Av}}$, from its saturation value $|\mathbf{M}_0|$, arises from the crystalline anisotropy torques. These torques, acting on the magnetization vector of each crystal grain, tend to rotate this vector away from the direction of the external magnetic field, and toward the direction of the nearest easy axis of magnetization within the grain in question. Thus, in the equilibrium position, the direction of the magnetization vector in any crystal grain is determined by the balance of the crystalline and magnetic torques acting on it, *viz.*:

$$\mathbf{T}=\mathbf{M}_0\times(\mathbf{H}+\mathbf{h}), \tag{1}$$

where $\mathbf{M}_0$ is the magnetization of the crystal grain under consideration; $\mathbf{H}$ the external magnetic field (the demagnetizing field due to the magnetization on the surface of the polycrystalline specimen is incorporated into $\mathbf{H}$); $\mathbf{h}$ the internal magnetic field, i.e., the magnetic field due to the magnetization vectors of the totality of crystal grains; $\mathbf{T}$ the torque due to the crystalline anisotropy forces within the crystal grain.[1]

$\mathbf{T}$ in any grain, depends on the direction of $\mathbf{M}_0$ with respect to the crystallographic axes of that grain, and is independent of position within any given single grain.[1] The magnitude of $\mathbf{T}$ is proportional to the "anisotropy constant" K.[1]

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[1] N. S. Akulov, Zeits. f. Physik **69**, 822 (1931); R. Gans, Ann. d. Physik **15**, 28 (1932). Cf. also, E. C. Stoner, *Magnetism and Matter* (Methuen, London, 1934), p. 405; F. Bitter, *Introduction to Ferromagnetism* (McGraw-Hill, New York, 1937), p. 222; R. Becker and W. Döring, *Ferromagnetismus* (Springer, Berlin, 1939), p. 167.

It is obvious that $\mathbf{h}$ and $\mathbf{M}_0$ cannot be assumed constant even within a single crystal grain and must hence be considered as functions of position within the polycrystalline specimen. However, only the direction of $\mathbf{M}_0$ varies with position, since the magnitude of $\mathbf{M}_0$ is the intrinsic domain magnetization and is therefore constant.

The problem of calculating the deviation of $|\mathbf{M}|_{\text{Av}}$ from $|\mathbf{M}_0|$ as a function of $\mathbf{H}$ was first considered by Akulov and by Gans[1] on the basis of the general mechanism outlined above. These authors obtained the result:

$$|\mathbf{M}|_{\text{Av}}=|\mathbf{M}_0|\left(1-\left(\frac{\beta K^2}{|\mathbf{M}_0|^2}\right)H^{-2}\right), \tag{2}$$

with $\beta=0.0872$ for Fe; 0.0762 for Ni.

However, Akulov and Gans, in their derivation of (2), neglected completely the internal magnetic field $\mathbf{h}$, and thus the torque term $\mathbf{M}_0\times\mathbf{h}$ in Eq. (1). The neglect of this torque term cannot be considered satisfactory since it is of the same order of magnitude as the term $\mathbf{M}_0\times\mathbf{H}$, for the range of external fields $\mathbf{H}$ used in the experiments.[2] It is not possible to dismiss the term $\mathbf{M}_0\times\mathbf{h}$ with the observation that $|\mathbf{h}|\ll|\mathbf{H}|$, since $\mathbf{H}$ and $\mathbf{M}_0$ are almost parallel, while $\mathbf{h}$ has a direction which in general, is quite different from that of $\mathbf{M}_0$ and $\mathbf{H}$.

In this paper, the deviation of $|\mathbf{M}|_{\text{Av}}$ from $|\mathbf{M}_0|$, considered as a function of $\mathbf{H}$, will be obtained taking the internal magnetic field $\mathbf{h}$ into account.

II. DERIVATION OF EXPRESSION FOR THE DEVIATION OF $|\mathbf{M}|_{\text{Av}}$ FROM $|\mathbf{M}_0|$

It is convenient to resolve $\mathbf{M}_0$ into components $\mathbf{M}$ and $\mathbf{m}$, parallel and perpendicular, respectively, to $\mathbf{H}$. Thus,

$$\mathbf{M}_0=\mathbf{M}+\mathbf{m} \tag{3}$$

and Eq. (1) becomes

$$\mathbf{T}=\mathbf{m}\times\mathbf{H}+\mathbf{M}\times\mathbf{h}+\mathbf{m}\times\mathbf{h}, \tag{4}$$

while the observed magnetization $|\mathbf{M}|_{\text{Av}}$ is given by the integral of $|\mathbf{M}|$ over the volume of the polycrystalline specimen, i.e.,

$$|\mathbf{M}|_{\text{Av}}\equiv\frac{1}{V}\int_V|\mathbf{M}|d\mathbf{r}=\frac{|\mathbf{M}_0|}{V}\int_V\left(1-\frac{|\mathbf{m}|^2}{|\mathbf{M}_0|^2}\right)^{\frac{1}{2}}d\mathbf{r}. \tag{5}$$

Since only the magnetization observed at high external fields[3] is under consideration, the deviation of $|\mathbf{M}|_{\text{Av}}$ from $|\mathbf{M}_0|$ is small, and the following approximations may be considered as holding at each point within the specimen:

$$|\mathbf{m}|/|\mathbf{M}_0|\ll 1, \tag{6}$$

$$\frac{|\mathbf{M}|-|\mathbf{M}_0|}{|\mathbf{M}_0|}=\frac{|\mathbf{M}_0-\mathbf{m}|-|\mathbf{M}_0|}{|\mathbf{M}_0|}\cong\frac{|\mathbf{m}|^2}{|\mathbf{M}_0|^2}. \tag{7}$$

From (6), (7), it follows that with neglect of terms $\cong|\mathbf{m}|^2/|\mathbf{M}_0|^2$ or higher, $\mathbf{M}$ is constant in *magnitude*, as well as in direction.

Further,

$$\mathbf{m}\times\mathbf{h}\cong 0+O(|\mathbf{m}|^2). \tag{8}$$

since

$$\mathbf{h}=-\text{grad}\int\frac{-\text{div}\,\mathbf{M}_0}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\cong-\text{grad}\int\frac{-\text{div}\,\mathbf{m}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}',$$

and hence $\mathbf{h}\cong\mathbf{m}$.

The approximations (6)–(8) applied to Eqs. (5), (4) yield:

$$|\mathbf{M}|_{\text{Av}}=|\mathbf{M}_0|\left(1-\frac{1}{2|\mathbf{M}_0|^2V}\int_V|\mathbf{m}|^2d\mathbf{r}\right), \tag{9}$$

$$\mathbf{T}=\mathbf{m}\times\mathbf{H}+\mathbf{M}\times\mathbf{h}. \tag{10}$$

Further, in view of (6)–(8), $\mathbf{T}$ in any crystal grain can be taken as dependent only on the direction of the vector $\mathbf{M}$ or $\mathbf{H}$ with respect to the crystallographic axes of that grain, since $\mathbf{M}_0$ may be taken parallel to $\mathbf{M}$ to the first order in $|\mathbf{m}|/|\mathbf{M}_0|$. $\mathbf{T}$ is hence independent of $\mathbf{m}$.

Thus to find $|\mathbf{M}|_{\text{Av}}$, it remains to calculate from Eq. (10) $\mathbf{m}$ as a function of position within the specimen, and then to evaluate the integral in (9). This procedure will now be carried out.

First, taking the vector product of both sides

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[2] These experiments are discussed in Section IV; the magnitude of the fields used is ~2000—4000 gauss. Cf. reference 5.

[3] These fields are such that $|\mathbf{M}|_{\text{Av}}$ is well above the "knee" of the magnetization curve; e.g., $H>1000$ gauss for Fe.

of Eq. (10) with $\mathbf{H}$, and solving for $\mathbf{m}$, one obtains,

$$\mathbf{m}=\mathbf{F}+(4\pi)^{-1}\alpha[\mathbf{h}-(\mathbf{h}\cdot\mathbf{H})(\mathbf{H}/|\mathbf{H}|^2)] \quad (11)$$

with the abbreviations

$$\mathbf{F}\equiv(\mathbf{H}\times\mathbf{T})/|\mathbf{H}|^2, \quad (12)$$

$$\alpha\equiv 4\pi|\mathbf{M}|/|\mathbf{H}|\cong 4\pi|\mathbf{M}_0|/|\mathbf{H}|. \quad (13)$$

One also has the general field equations relating $\mathbf{h}$ and $\mathbf{m}$,

$$\operatorname{div}\mathbf{h}=-4\pi\operatorname{div}\mathbf{M}_0\cong-4\pi\operatorname{div}\mathbf{m}, \quad (14)$$

$$\operatorname{curl}\mathbf{h}=0. \quad (15)$$

Introducing the Fourier resolution[4] of $\mathbf{m}$,

$$\mathbf{m}=\int\mathbf{m}_{\mathbf{k}}\exp(2\pi i\mathbf{k}\cdot\mathbf{r})d\mathbf{k}, \quad (16)$$

one obtains from (9)

$$|\mathbf{M}|_{\mathrm{Av}}=|\mathbf{M}_0|\left(1-\frac{1}{2|\mathbf{M}_0|^2V}\int|\mathbf{m}_{\mathbf{k}}|^2d\mathbf{k}\right), \quad (17)$$

while from (11), (14), (15) the x, y, z components of $\mathbf{m}_{\mathbf{k}}$ satisfy the relations:

$$\mathrm{m}_{\mathbf{k}x}=F_{\mathbf{k}x}+(4\pi)^{-1}\alpha \mathrm{h}_{\mathbf{k}x}, \quad (18a)$$

$$\mathrm{m}_{\mathbf{k}y}=F_{\mathbf{k}y}+(4\pi)^{-1}\alpha \mathrm{h}_{\mathbf{k}y}, \quad (18b)$$

$$\mathrm{m}_{\mathbf{k}z}=0, \quad (18c)$$

$$\mathbf{k}\cdot\mathbf{h}_{\mathbf{k}}=-4\pi\mathbf{k}\cdot\mathbf{m}_{\mathbf{k}}, \quad (19)$$

$$\mathbf{k}\times\mathbf{h}_{\mathbf{k}}=0. \quad (20)$$

In Eqs. (18), (19), (20), $\mathbf{F}_{\mathbf{k}}$, $\mathbf{h}_{\mathbf{k}}$ are the vector Fourier components of $\mathbf{F}$ and $\mathbf{h}$, similar to $\mathbf{m}_{\mathbf{k}}$ being the vector Fourier component of $\mathbf{m}$ in (16); $F_{\mathbf{k}x}$, $\mathrm{h}_{\mathbf{k}x}$ are the x components of $\mathbf{F}_{\mathbf{k}}$, $\mathbf{h}_{\mathbf{k}}$; the z axis has been chosen parallel to $\mathbf{H}$.

To proceed further one eliminates $\mathbf{h}_{\mathbf{k}}$ from (18), (19), (20), finding $\mathbf{m}_{\mathbf{k}}$ as a function of $\mathbf{k}$. One has as the solution of (19), (20) for $\mathbf{h}_{\mathbf{k}}$,

$$\mathbf{h}_{\mathbf{k}}=-4\pi\mathbf{n}\mathbf{n}\cdot\mathbf{m}_{\mathbf{k}}, \quad (21)$$

with $\mathbf{n}\equiv\mathbf{k}/|\mathbf{k}|$,

whence, substituting (21) into (18), and solving for $\mathrm{m}_{\mathbf{k}x}$, $\mathrm{m}_{\mathbf{k}y}$, one obtains

$$\mathrm{m}_{\mathbf{k}x}=\frac{F_{\mathbf{k}x}(1+\alpha n_y^2)+F_{\mathbf{k}y}\alpha n_xn_y}{1+\alpha(n_x^2+n_y^2)}, \quad (22)$$

$$\mathrm{m}_{\mathbf{k}y}=\frac{F_{\mathbf{k}x}\alpha n_xn_y+F_{\mathbf{k}y}(1+\alpha n_x^2)}{1+\alpha(n_x^2+n_y^2)},$$

$$\mathrm{m}_{\mathbf{k}z}=0.$$

Finally, substituting (22) into (17), one obtains

$$|\mathbf{M}|_{\mathrm{Av}}=|\mathbf{M}_0|\left(1-\frac{1}{2|\mathbf{M}_0|^2V}\left[\int\frac{(1+\alpha n_y^2)^2+\alpha^2n_x^2n_y^2}{(1+\alpha(n_x^2+n_y^2))^2}|F_{\mathbf{k}x}|^2d\mathbf{k}\right.\right.$$
$$\left.\left.+\int\frac{(1+\alpha n_x^2)^2+\alpha^2n_x^2n_y^2}{(1+\alpha(n_x^2+n_y^2))^2}|F_{\mathbf{k}y}|^2d\mathbf{k}+2\int\frac{(2+\alpha n_x^2+\alpha n_y^2)\alpha n_xn_y}{(1+\alpha(n_x^2+n_y^2))^2}|F_{\mathbf{k}x}F_{\mathbf{k}y}|d\mathbf{k}\right]\right). \quad (23)$$

Now it will be shown in Section III below, that as far as the integrations over $\mathbf{k}$ in Eq. (23) are concerned, one may effectively take

$$|F_{\mathbf{k}x}|^2=|F_{\mathbf{k}y}|^2=\tfrac{1}{2}|F_{\mathbf{k}}|^2, \quad (24a)$$

$$|F_{\mathbf{k}x}F_{\mathbf{k}y}|=0, \quad (24b)$$

and take $|F_{\mathbf{k}}|^2$ to depend only on the magnitude of $\mathbf{k}$, i.e.,

$$|F_{\mathbf{k}}|^2=\Phi(|\mathbf{k}|). \quad (24c)$$

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[4] The Fourier analysis method has been used by W. F. Brown, Jr., in a related problem, cf. Phys. Rev. **58**, 736 (1940).

Thus,

$$|\mathbf{M}|_{\mathrm{Av}}=|\mathbf{M}_0|\left(1-\frac{(4\pi)^{-1}}{4|\mathbf{M}_0|^2V}\int\Phi(|\mathbf{k}|)|\mathbf{k}|^2d|\mathbf{k}|4\pi\int\int\left(1+\frac{1}{(1+\alpha(n_x{}^2+n_y{}^2))^2}\right)\sin\theta_{\mathrm{k}}d\theta_{\mathrm{k}}d\phi_{\mathrm{k}}\right)$$

$$=|\mathbf{M}_0|\left(1-\frac{1}{8|\mathbf{M}_0|^2V}\int_V|\mathbf{F}|^2d\mathbf{r}\int_0^{\pi}\left(1+\frac{1}{(1+\alpha\sin^2\theta_{\mathrm{k}})^2}\right)\sin\theta_{\mathrm{k}}d\theta_{\mathrm{k}}\right)$$

$$=|\mathbf{M}_0|\left(1-\frac{\{|\mathbf{T}\sin(\mathbf{T},\mathbf{H})|^2\}_{\mathrm{Av}}}{4|\mathbf{M}_0|^2}H^{-2}\left[1+\tfrac{1}{2}(1+\alpha)^{-1}+\tfrac{1}{4}(1+\alpha)^{-\frac{3}{2}}\alpha^{-\frac{1}{2}}\log\left(\frac{(1+\alpha)^{\frac{1}{2}}+\alpha^{\frac{1}{2}}}{(1+\alpha)^{\frac{1}{2}}-\alpha^{\frac{1}{2}}}\right)\right]\right).\quad(25)$$

Equation (25) is the desired expression for the functional dependence of $|\mathbf{M}|_{\mathrm{Av}}-|\mathbf{M}_0|$ on **H**. In the limiting cases of $\alpha\ll 1$ and $\gg 1$, Eq. (25) reduces to the following simple forms:

$$|\mathbf{M}|_{\mathrm{Av}}=|\mathbf{M}_0|\left(1-\frac{\{|\mathbf{T}\sin(\mathbf{T},\mathbf{H})|^2\}_{\mathrm{Av}}}{2|\mathbf{M}_0|^2}H^{-2}\right),\quad\text{when }\alpha\ll 1,\text{ i.e., }4\pi M_0\ll H,\quad(26)$$

$$|\mathbf{M}|_{\mathrm{Av}}=|\mathbf{M}_0|\left(1-\frac{\{|\mathbf{T}\sin(\mathbf{T},\mathbf{H})|^2\}_{\mathrm{Av}}}{4|\mathbf{M}_0|^2}H^{-2}\right),\quad\text{when }\alpha\gg 1,\text{ i.e., }4\pi M_0\gg H.\quad(27)$$

Equation (26) may be derived by neglecting the term proportional to α in Eq. (11), or equivalently, by neglecting $\mathbf{M}\times\mathbf{h}$ in Eq. (10), or $\mathbf{M}_0\times\mathbf{h}$ in Eq. (1). Equation (26) is thus identical with Eq. (2) derived by Akulov and Gans. Thus it is seen that the Eq. (2) or (26), of Akulov and Gans is only applicable, if $H\gg 4\pi M_0=21{,}800$ gauss in Fe, 6400 gauss in Ni. It should further be emphasized that only in the limiting cases $4\pi M_0\gg H$, $4\pi M_0\ll H$, does $|\mathbf{M}|_{\mathrm{Av}}-|\mathbf{M}_0|$ vary, as const. H^{-2} (with different constants for the two limits). For intermediate field strengths, the dependence of $|\mathbf{M}|_{\mathrm{Av}}$ on **H** is much more involved, being given by Eq. (25).

III. Justification of Equations (24)

It now remains to justify Eqs. (24). One must consider the evaluation of the integrals in Eq. (23). These integrals are of the type:

$$\int\Psi_x(\mathbf{n})F_{\mathrm{k}x}^2d\mathbf{k},\quad\int\Psi_y(\mathbf{n})F_{\mathrm{k}y}^2d\mathbf{k},$$
$$\int\Psi_{xy}(\mathbf{n})F_{\mathrm{k}x}F_{\mathrm{k}y}d\mathbf{k}.\quad(28)$$

Now

$$\mathbf{F}_{\mathrm{k}}=\int_V\mathbf{F}(\mathbf{r})\exp(2\pi i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}$$
$$=\sum_j\int_{V_j}\mathbf{F}(\mathbf{r})\exp(2\pi i\mathbf{k}\cdot\mathbf{r})d\mathbf{r},\quad(29)$$

where the volume integrals in the sum over j extend over V_j, the volume of the jth crystal grain. In V_j, **T** and hence **F**, are constant both in magnitude and direction; one writes the constant value of **F** in V_j as $\mathbf{F}_j$. Thus,

$$\mathbf{F}_{\mathrm{k}}=\sum_j\mathbf{F}_j\int_{V_j}\cdot\exp(2\pi i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}$$
$$=\sum_j\mathbf{F}_j\exp(2\pi i\mathbf{k}\cdot\mathbf{R}_j)\int_{V_j}\exp[2\pi i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_j)]d\mathbf{r}$$
$$\equiv\sum_j\mathbf{F}_j\exp(2\pi i\mathbf{k}\cdot\mathbf{R}_j)V_jW_j(\mathbf{k}),\quad(30)$$

where $\mathbf{R}_j$ is a vector from the center of the polycrystalline specimen to the center of the jth crystal grain, and $W_j(\mathbf{k})$ is the spatial average of $\exp[2\pi i\mathbf{k}\cdot(\mathbf{r}-\mathbf{R}_j)]$ over V_j.

The perfect randomness of the distribution of the orientations of the crystallographic axes of the various crystal grains within the polycrystalline specimen leads to the following "randomness properties" of the vectors $\mathbf{F}_j$:

(A) There is no correlation between the magnitudes and directions of the vectors $\mathbf{F}_j$ and

$\mathbf{F}_l$. ($l \mp j$), i.e.,

$$\Big|\sum_j \mathbf{F}_j\Big| \sim \Big(\sum_j |\mathbf{F}_j|^2\Big)^{\frac{1}{2}} \equiv \big[(V/\langle V_j\rangle_{\mathrm{Av}})\langle |\mathbf{F}_j|^2\rangle_{\mathrm{Av}}\big]^{\frac{1}{2}}, \quad (31)$$

where $V/\langle V\rangle_{\mathrm{Av}}$ is the number of terms in the sum over j.

(B) There is no correlation between the magnitude and direction of $\mathbf{F}_j$ and the position of the jth crystal grain within the specimen, i.e.,

$$\sum_j f(\mathbf{F}_j)g(\mathbf{R}_j) \sim \langle |g(\mathbf{R}_j)|\rangle_{\mathrm{Av}} \sum_j f(\mathbf{F}_j), \quad (32)$$

where $f(\mathbf{F}_j)$ is any function of $\mathbf{F}_j$, $g(\mathbf{R}_j)$ is any function of $\mathbf{R}_j$, and

$$\langle |g(\mathbf{R}_j)|\rangle_{\mathrm{Av}} = (V/\langle V_j\rangle_{\mathrm{Av}})^{-1} \sum_j |g(\mathbf{R}_j)|.$$

(C) There is no correlation between the direction of $\mathbf{F}_j$ and any fixed direction within the specimen, i.e.,

$$\sum_j F_{jx}^2 = \sum_j F_{jy}^2 = \tfrac{1}{2}|\mathbf{F}_j|^2;$$

or

$$\langle F_{jx}^2\rangle_{\mathrm{Av}} = \langle F_{jy}^2\rangle_{\mathrm{Av}} = \langle \tfrac{1}{2}|\mathbf{F}_j|^2\rangle_{\mathrm{Av}}. \quad (33)$$

From Eq. (30) one sees that $\mathbf{F}_{\mathbf{k}}$ oscillates rapidly when $\mathbf{k}$ varies over a region V^{-1}, since $\mathbf{R}_j$ in $\exp(2\pi i\mathbf{k}\cdot\mathbf{R}_j)$ is $\sim V^{\frac{1}{3}}$. On the other hand, from Eqs. (30), (32), (31),

$$\begin{aligned}|\mathbf{F}_{\mathbf{k}}| &\sim \langle |\exp(2\pi i\mathbf{k}\cdot\mathbf{R}_j)\, V_j W_j(\mathbf{k})|\rangle_{\mathrm{Av}} \\ &\qquad \times\big[(V/\langle V_j\rangle_{\mathrm{Av}})\langle |\mathbf{F}_j|^2\rangle_{\mathrm{Av}}\big]^{\frac{1}{2}} \\ &\sim (V^{\frac{1}{2}}/\langle V_j\rangle_{\mathrm{Av}}^{\frac{1}{2}})\langle |V_j W_j(\mathbf{k})|\rangle_{\mathrm{Av}}\langle |\mathbf{F}_j|^2\rangle_{\mathrm{Av}}^{\frac{1}{2}} \quad (34)\end{aligned}$$

and thus the variation with $\mathbf{k}$ of the order of magnitude of $\mathbf{F}_{\mathbf{k}}$ is independent of the rapid oscillations due to $\exp(2\pi i\mathbf{k}\cdot\mathbf{R}_j)$ and is determined only by the $W_j(\mathbf{k})$. Also, from their definition, the $W_j(\mathbf{k})$ vary appreciably with $\mathbf{k}$ only when $\mathbf{k}$ ranges over a region $\sim V_j^{-1}$; in particular, $W_j(\mathbf{k}) \cong 1$ if $|\mathbf{k}| \lesssim V_j^{-\frac{1}{3}}$, and $W_j(\mathbf{k})$ decreases if $|\mathbf{k}|$ exceeds $V_j^{-\frac{1}{3}}$.

Thus, turning back to the integrals of Eq. (28), one sees that the effective region of integration in $\mathbf{k}$ space is defined by the condition $|\mathbf{k}| \lesssim V_j^{-\frac{1}{3}}$. Now Ψ depends only on $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$; one calculates the variation in $\mathbf{n}$, $\Delta\mathbf{n}$, corresponding to a variation in $\mathbf{k}$, $\Delta\mathbf{k}$, and finds

$$\Delta\mathbf{n} \cong [\mathbf{k}\times(\Delta\mathbf{k}\times\mathbf{k})]/|\mathbf{k}|^3$$

over most of the effective region of integration. Therefore, Ψ varies appreciably only when $\mathbf{k}$ varies over a region V_j^{-1}.

Thus, it is possible to divide the $\mathbf{k}$ space into cells of size Ω, which, if $V^{-1} \ll \Omega \ll V_j^{-1}$ possess the following properties: (1) Many oscillations of $\mathbf{F}_{\mathbf{k}}$, $|\mathbf{F}_{\mathbf{k}}|^2$, etc., take place in Ω. (2) The order of magnitude of $|\mathbf{F}_{\mathbf{k}}|^2$ and of the value of Ψ, remain effectively constant over Ω.

It therefore follows that the integrals (28) can be written as

$$\begin{aligned}\int \Psi_x F_{\mathbf{k}x}^2 d\mathbf{k} &\cong \sum_\rho \Psi_x(\mathbf{k}_\rho/|\mathbf{k}_\rho|)\int_{\Omega_\rho} F_{\mathbf{k}x}^2 d\mathbf{k} \\ &\cong \int \Psi_x(\mathbf{k}/|\mathbf{k}|)\Big\{\Omega^{-1}\int_\Omega F_{\mathbf{k}'x}^2 d\mathbf{k}'\Big\} d\mathbf{k} \quad (35)\end{aligned}$$

and hence the $F_{\mathbf{k}x}^2$ in (28) are effectively replaced by their "smoothed over" values:

$$\Omega^{-1}\int_\Omega F_{\mathbf{k}'x}^2 d\mathbf{k}'.$$

Now the "smoothed over" values of $F_{\mathbf{k}x}^2$, $F_{\mathbf{k}y}^2$, $F_{\mathbf{k}x}F_{\mathbf{k}y}$ satisfy, as is shown immediately below the following relations:

$$\Omega^{-1}\int_\Omega F_{\mathbf{k}x}^2 d\mathbf{k} = \Omega^{-1}\int_\Omega F_{\mathbf{k}y}^2 d\mathbf{k} = \tfrac{1}{2}\Omega^{-1}\int_\Omega |\mathbf{F}_{\mathbf{k}}|^2 d\mathbf{k}, \quad (36a)$$

$$\Omega^{-1}\int_\Omega F_{\mathbf{k}x}F_{\mathbf{k}y} d\mathbf{k} = 0, \quad (36b)$$

and thus, by use of Eqs. (35), (36a), (36b), the validity of Eqs. (24a), (24b), is established.

To prove Eq. (36a), for example, one has:

$$\begin{aligned}\Omega^{-1}\int_\Omega F_{\mathbf{k}x}^2 d\mathbf{k} &= \Omega^{-1}\sum_j F_{jx}^2 V_j^2 \int_\Omega [W_j(\mathbf{k})]^2 d\mathbf{k} + \Omega^{-1}\sum_{j\neq l} F_{jx}F_{lx}V_jV_l \int_\Omega \exp(2\pi i\mathbf{k}\cdot\mathbf{R}_{jl})\, W_j(\mathbf{k})\, W_l(\mathbf{k})\, d\mathbf{k} \\ &\cong \sum_j F_{jx}^2 V_j^2 [W_j(\mathbf{k}_0)]^2 + \sum_{j\neq l} F_{jx}F_{lx}V_jV_l W_j(\mathbf{k}_0)\, W_l(\mathbf{k}_0) \exp(2\pi i\mathbf{k}_0\cdot\mathbf{R}_{jl}) \\ &\qquad \times(\Omega \mathrm{X}_{jl}\mathrm{Y}_{jl}\mathrm{Z}_{jl})^{-1}\sin(\pi\Delta\mathrm{k}_x\mathrm{X}_{jl})\sin(\pi\Delta\mathrm{k}_y\mathrm{Y}_{jl})\sin(\pi\Delta\mathrm{k}_z\mathrm{Z}_{jl}) \quad (37)\end{aligned}$$

where $\mathbf{k}_0$ is a vector from the origin of $\mathbf{k}$ space to the center of Ω and $\Delta k_x \Delta k_y \Delta k_z = \Omega$. The last (approximate) equality in Eq. (37) is valid, since in Ω each $W_j(\mathbf{k})$ is approximately constant.

From Eq. (32), the order of magnitude of the single sum is:

$$\sim \langle V_j^2 [W_j(\mathbf{k}_0)]^2 \rangle_{\mathrm{Av}} \sum_j F_{jx}^2$$
$$\equiv \langle V_j^2 W_j^2 \rangle_{\mathrm{Av}} (V/\langle V_j \rangle_{\mathrm{Av}}) \langle F_{jx}^2 \rangle_{\mathrm{Av}}, \qquad (38)$$

while the order of magnitude of the double sum,

$$\sim (\Omega V)^{-1} \langle | V_j V_l W_j(\mathbf{k}_0) W_l(\mathbf{k}_0) \times \exp(2\pi i \mathbf{k}_0 \cdot \mathbf{R}_{jl}) | \rangle_{\mathrm{Av}} \sum_{j \neq l} F_{jx} F_{lx},$$
$$\sim (\Omega V)^{-1} \langle | V_j V_l W_j(\mathbf{k}_0) W_l(\mathbf{k}_0) \times \exp(2\pi i \mathbf{k}_0 \cdot \mathbf{R}_{jl}) | \rangle_{\mathrm{Av}} (\sum_j F_{jx})^2,$$
$$\sim (\Omega V)^{-1} \langle | V_j^2 W_j^2 | \rangle_{\mathrm{Av}} (V/\langle V_j \rangle_{\mathrm{Av}}) \langle F_{jx}^2 \rangle_{\mathrm{Av}}, \qquad (39)$$

the last (order of magnitude) equality being a consequence of Eq. (31).

Thus, from (38) and (39) the ratio of the orders of magnitude of the double sum and the single sum is $(V\Omega)^{-1} \ll 1$. Hence,

$$\Omega^{-1} \int_\Omega F_{\mathrm{k}x}^2 = \Omega^{-1} \sum_j F_{jx}^2 V_j^2 [W_j(\mathbf{k}_0)]^2, \qquad (40)$$

and similarly

$$\Omega^{-1} \int_\Omega F_{\mathrm{k}y}^2 d\mathbf{k} = \Omega^{-1} \sum_j F_{jy}^2 V_j^2 [W_j(\mathbf{k}_0)]^2. \qquad (41)$$

"Randomness properties" (B) and (C) now insure the equality of the right-hand sides of Eqs. (40) and (41), and Eq. (36a) follows immediately.

To prove Eq. (36b), one observes that $\Omega^{-1} \int_\Omega F_{\mathrm{k}x} F_{\mathrm{k}y} d\mathbf{k}$ can be expressed as $\sum_j + \sum_{j \neq l}$, and that from Eq. (32),

$$\sum_j \sim \langle V_j^2 W_j^2 \rangle_{\mathrm{Av}} \sum_j F_{jx} F_{jy}$$
$$\sim \langle V_j^2 W_j^2 \rangle_{\mathrm{Av}} [(V/\langle V_j \rangle_{\mathrm{Av}}) \langle F_{jx}^2 \rangle_{\mathrm{Av}} \langle F_{jy}^2 \rangle_{\mathrm{Av}}]^{\frac{1}{2}}, \qquad (42)$$

the last (order of magnitude) equality being a consequence of "randomness properties" (A) and (C).

Thus the ratio of the orders of magnitude of the single sum in $\Omega^{-1} \int_\Omega F_{\mathrm{k}x} F_{\mathrm{k}y} d\mathbf{k}$, and of the single sum in $\Omega^{-1} \int_\Omega F_{\mathrm{k}x}^2 d\mathbf{k}$, is (cf. Eqs. (38) and (42)), $\sim \langle V_j \rangle_{\mathrm{Av}}^{\frac{1}{2}} / V^{\frac{1}{2}} \ll 1$. Further, by an argument similar to that given in connection with Eq. (39), it can be shown that the double sum in $\Omega^{-1} \int_\Omega F_{\mathrm{k}x} F_{\mathrm{k}y} d\mathbf{k}$ is negligible compared with the single sum. Therefore, in comparison to the integral $\Omega^{-1} \int_\Omega F_{\mathrm{k}x}^2 d\mathbf{k}$ the integral $\Omega^{-1} \int_\Omega F_{\mathrm{k}x} F_{\mathrm{k}y} d\mathbf{k}$ can be taken to be equal to zero (to within terms $\sim \langle V_j \rangle_{\mathrm{Av}}^{\frac{1}{2}} / V^{\frac{1}{2}} \ll 1$, and $(V\Omega)^{-1} \ll 1$).

It now remains to justify Eq. (24c). For this purpose it is necessary to make use of another "randomness property"; *viz.:* The shapes of the individual crystal grains are not correlated in any way with a fixed set of directions in the crystal. Then, apart from the rapid oscillations of $|\mathbf{F}_\mathrm{k}|^2$ with $\mathbf{k}$ which are smoothed out by the integrations over Ω in Eq. (35) or (36) or (23),

$$|\mathbf{F}_\mathrm{k}|^2 = \left| \sum_j \mathbf{F}_j \int_{V_j} \exp(2\pi i \mathbf{k} \cdot \mathbf{r}) d\mathbf{r} \right|^2,$$

can only depend on $|\mathbf{k}|$. Hence the validity of Eq. (24c).

IV. Discussion of Experiments

The experimentally observed variation[5] of the magnetization of polycrystalline ferromagnetics with external magnetic field at high fields is customarily described by an empirical relation of the form:[5]

$$\frac{d|\mathbf{M}|_{\mathrm{obs}}}{dH} = AH^{-2} + BH^{-3} + CH, \qquad (43)$$

where A, B, C, are assumed to be independent of H.

It has further been observed that C is independent of the metallurgical history of the specimen;[5] it therefore must arise from the variation of the intrinsic domain magnetization with the field. The magnitude, temperature dependence, and actual slow variation with field,

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[5] A. R. Kaufmann, Phys. Rev. **55**, 1142 (1939); **57**, 1089 (1939). H. Polley, Ann. d. Physik **36**, 625 (1939). Only Ni and Fe but not Co, were investigated.

of C, have been treated in a recent paper by the present authors.[6]

A and B depend on the metallurgical history of the polycrystalline specimen;[5] A is conditioned by the plastic deformation of the material,[5] (small for well-annealed specimens) and is temperature independent,[5] whereas, B is determined by the crystalline properties and the elastic state,[5] and is furthermore strongly temperature dependent.[5] The theoretical interpretation of the term AH^{-2} is still in doubt; however, a recent paper by W. F. Brown, Jr.,[4] ascribes this term to the existence of anisotropy torques which are *concentrated* along lines of dislocation produced by plastic deformation.

Finally, the term BH^{-3} arises from the mechanism discussed in this paper, i.e., from the rotation of the magnetization vector away from the direction of H and toward the various directions of easy magnetization (in the various individual crystal grains) by crystalline anisotropy torques constant within any one single grain. Comparing the identical Eqs. (2) and (26), one sees that

$$K=[\{|\mathbf{T}\sin(\mathbf{T},\mathbf{H})|^2\}_{\mathrm{Av}}/2\beta]^{\frac{1}{2}}, \tag{44}$$

whence from Eqs. (26), (27), $d|\mathbf{M}|_{\mathrm{Av}}/dH$ may be written in the form BH^{-3} in the two limiting cases, i.e.,

$$d|\mathbf{M}|_{\mathrm{Av}}/dH=(2\beta K^2/|\mathbf{M}_0|^2)H^{-3}, \quad \text{when } 4\pi M_0 \ll H; \tag{45}$$

$$d|\mathbf{M}|_{\mathrm{Av}}/dH=(\beta K^2/|\mathbf{M}_0|^2)H^{-3}, \quad \text{when } 4\pi M_0 \gg H. \tag{46}$$

For intermediate values of H, $d|\mathbf{M}|_{\mathrm{Av}}/dH$ is obtained by a differentiation in Eq. (25), and can be written in the form BH^{-3} only if B itself is considered as a (slowly varying) function of H.

Now in all the experiments so far performed, H was considerably less than $4\pi M_0$, so that, in evaluating K from the experiments, one should use the relation:[7]

$$K=[\beta^{-1}|\mathbf{M}_0|^2 \times(\text{observed coefficient of } H)]^{\frac{1}{2}}. \tag{47}$$

On the other hand, these experiments have previously been interpreted in terms of the Akulov-Gans equation (2) or (26) or (45); thus the values of K, previously deduced from the existing experimental data, are smaller than they should be by a factor equal to $2^{\frac{1}{2}}$. If these previously deduced values of K be all multiplied by $2^{\frac{1}{2}}$, they agree somewhat better than before with the values of K obtained directly from the magnetization curves of single crystal grains; cf. especially the experimental data given in Polley.[8]

The analysis of the present paper can be tested by experiments on the magnetization curves of polycrystalline specimens, in which, both the limiting cases, $4\pi M_0 \gg H$, $4\pi M_0 \ll H$, and also the case of intermediate H (described by the complete Eq. (25)) are studied. Now, both for iron and nickel, K is so small, that unless H is considerably less than $4\pi M_0$, the whole term BH^{-3} is negligible. For cobalt, however, the crystalline anisotropy energy and hence K, are so large, that the portion of $d|\mathbf{M}|_{\mathrm{obs}}/dH$ arising from the mechanism under discussion, could easily be observed as a function of H for values of H less than, of the order of, and considerably greater than, $4\pi M_0$.[9] In this way, a complete experimental test of the Eqs. (25), (26), (27) can be obtained.

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[6] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[7] This formula, as well as the rest of the analysis in the present paper and in the derivation of the Akulov-Gans equation (2), is based on the assumption that the crystalline anisotropy torques are just those present in a *strain-free* material. The existence of strains creates additional anisotropy torques, which, if constant over regions comparable in size to a single crystal grain (if not, cf. W. F. Brown, Jr., reference 4) can be easily incorporated into the present treatment. The only resulting difference in the final formulae is that K in Eq. (44) and hence in Eqs. (45), (46), (47) has to be amplified to include the "strain anisotropy" constant, and that the numerical value of β must be changed somewhat.

[8] Reference 5, p. 643, Fig. 12.

[9] Since cobalt has a hexagonal lattice, the numerical value of β for Co will differ from the values quoted in Eq. (2), which are only valid for the cubic lattices of Ni and Fe.