Interaction of Alpha-Particles

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(Received November 8, 1940)

The properties of the forces between alpha-particles are discussed on the basis of a Heitler-London analysis augmented by a "van der Waals" calculation. The results are in some respects unfavorable to the alpha-particle model of the nucleus inasmuch as they contradict the assumptions often made when this model is applied. Thus, for instance, it is pointed out that there are no attractive van der Waals forces beyond the range of the exchange forces, the range of the second-order forces being no greater than that of the first-order forces. The forces between alpha-particles are not additive. The range of the forces is approximately equal to that of the forces between elementary nuclear particles. The Heitler-London forces are repulsive if the usual choice of a symmetric Hamiltonian is made. The details of the forces are first worked out in connection with a simple fictitious example (§1) in terms of which these various features are most easily discussed. Calculations for the alpha-particle, for which a ψ-function of the Gauss type is assumed, are made in §§2 and 3. Finally (§4) the results are used in an examination of the problem of the scattering of alpha-particles in helium. It appears that the scattering of the S waves can be explained only partially by means of the present interaction, its fault being the failure to produce resonance at the correct energy.

THE attempts to understand the properties of nuclei have led to the invention of a number of models which are thought to represent, in an approximate fashion, the significant details of nuclear structure. The earliest of these, the independent particle or Hartree model, has been discussed so frequently that its merits, and more particularly its faults, are now commonly recognized. Of more recent interest has been the alpha-particle model[1] which springs from the consideration that the alpha-particle is a tightly bound group, and likely, as such, to have a fairly permanent identity within a nucleus. In agreement with this picture it is sometimes supposed that alpha-particles interact in a manner typical of the constituent parts of a molecule; indeed the machinery of molecular theory is then brought into action and a number of qualitative results are obtained. But the detailed manner of interaction of two alpha-particles, about which apparently plausible but totally *ad hoc* assumptions are made in such calculations, has not been investigated with due thoroughness. Even the simple question as to the attractive or repulsive nature of these supposed building blocks of nuclei does not seem to have been satisfactorily answered, nor is there any evidence that the scattering of alpha-particles in helium can or cannot be understood in terms of basic postulates about forces between primary nuclear constituents. The present paper deals with these questions.

A common argument relative to the alpha-particle interaction relies on the analogy with the molecular problem: At close distances of approach, it is pointed out, there must be strong exchange forces, and these are repulsive because the interaction occurs between closed shells; at larger distances of separation, attractive van der Waals forces come into play, and finally the Coulomb force alone remains active. In addition to this, one often assumes these forces to be additive.

A little reflection shows, however, that these conclusions may not be taken for granted, for there are, after all, essential differences between the molecular and the nuclear case. For example, the exchange repulsion between two H atoms which occurs at small distances when they approach each other in the triplet state, depends quite definitely upon the character of the Coulomb force; it might well turn into an attraction if the force between electrons were itself of an exchange nature. The force between nuclear elementary particles is of this type; hence a special investigation is needed to ascertain the direction of the force acting between alpha-particles. Secondly, to speak of a van der Waals force is in this instance almost meaningless; for this terminology implies that there exists a

[1] Cf., for instance, L. R. Hafstad and E. Teller, Phys. Rev. 54, 681 (1938). For a general discussion of the various aspects of particle groups in nuclei see J. A. Wheeler, Phys. Rev. 52, 1083 (1937). Here the limitations of the simple alpha-particle picture are recognized.

region in which second-order forces are large compared to the first-order forces. In molecular interactions this is true because, as one may easily see, the Coulomb force has a range which is much larger than the diameter of a molecule—an alpha-particle, however, has dimensions of the same order of magnitude as the range of the elementary nuclear forces. It will also be seen that the assumption of the additivity of inter-alpha-forces is quite untenable.

A simplified quantitative calculation of the interaction in question has been made by Heisenberg.[2] He assumes that the centers of mass of the alpha-particles are not exactly localized but distributed in space with Gaussian probability. This artifice has the advantage of obviating an explicit calculation of the kinetic energy of the particles, and furthermore respects the uncertainty principle in more adequate measure than the formalism of fixed alpha-centers would. However, it makes the results obtained more difficult to interpret.[3] Heisenberg's calculation differs from the one presented in this paper in several other respects: It neglects forces between like particles and assumes only Majorana forces between unlike particles, restrictions which are no longer permissible; finally, in the calculation of the exchange integrals, terms which arise from double transpositions of coordinates in the ψ-function are omitted. The latter approximation is a valid one when the overlap integral is small, but this condition is strongly violated in the present case over the entire region of the interaction.

The method to be employed in the present paper and its results may be discussed most simply in connection with a fictitious two-particle problem which bears a certain resemblance to the interaction of two deuterons. The ψ-functions used, however, are very poor approximations to the deuteron functions, so that the results have only qualitative significance. But the essential differences between the nuclear and the molecular problem can be seen very clearly.

[2] W. Heisenberg, Zeits. f. Physik **96**, 473 (1935).

[3] Cf. H. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936). The result stated in footnote 67 of this paper is not valid since Heisenberg's Eq. (14) holds only for small η.

1. Two-Particle Problem. Comparison Between Exchange and van der Waals Forces

Let us assume a particle to be bound to a fictitious center by simple harmonic forces. Its state function corresponding to the lowest energy will then be

$$u=(q/\pi)^{3/4}\exp\,[-\tfrac{1}{2}qr^2], \tag{1}$$

q being a parameter which measures the strength of binding. If an identical particle is situated with its center of force a distance $\mathbf{R}$ from that of the first particle, the ψ-function for this system may be written in the Heitler-London approximation:

$$\psi(1,2)=(2-2\delta^2)^{-\frac{1}{2}}\times[u(1)v(2)-v(1)u(2)]\alpha(1)\alpha(2). \tag{2}$$

The spins are here taken to be parallel; $v(r)$ stands for $u(\mathbf{r}-\mathbf{R})$, and

$$\delta=\int uv\,d\tau=\exp\,[-\tfrac{1}{4}qR^2].$$

Suppose that particles 1 and 2 interact, their classical potential energy being V. The first-order energy between the two is then given as the sum of two terms:

$$\Delta_1 E=\bar{V}+\Delta E_{\text{kin}},$$

where $\bar{V}=\int\psi^* V\psi\,d\tau$ and ΔE_{kin} is the difference between the mean of $-(\hbar^2/2M)(\nabla_1^2+\nabla_2^2)$ taken with the function ψ, and the kinetic energy of the two oscillators at infinite separation. Turning our attention first to the computation of ΔE_{kin} we find

$$\int\psi\nabla_1^2\psi\,d\tau=\int\psi\nabla_2^2\psi\,d\tau$$

$$=(1-\delta^2)^{-1}\left[\int u\nabla^2 u\,d\tau-\delta\int u\nabla^2 v\,d\tau\right]$$

$$=-3q/2-\tfrac{1}{4}q^2R^2\delta^2/(1-\delta^2).$$

The first term of this expression, when multiplied by $-\hbar^2/2M$, is just the kinetic energy of one particle not interacting with the other, that is, $-(\hbar/2M)\int u\nabla^2 u\,d\tau$. Hence

$$\Delta E_{\text{kin}}=(\hbar^2/4M)q^2R^2\delta^2/(1-\delta^2). \tag{3}$$

It is perhaps noteworthy that in the limit $R\to 0$, ΔE_{kin} becomes $\hbar^2 q/2M=\tfrac{1}{2}h\nu$, ν being the classical

frequency of the oscillator. Now the kinetic energy of an oscillator in its ground state is $\frac{3}{4}h\nu$, one-half its total energy; in the next excited state, the kinetic energy is $(\frac{3}{4}+\frac{1}{2})h\nu$. Thus our value of ΔE_{kin} indicates a promotion of one of the oscillators to an excited state as $R\to 0$. The same result may be read from the function (2) which becomes

$$\psi_{R\to 0}=q^{\frac{1}{2}}(\mathbf{r}_2-\mathbf{r}_1)\cdot\mathbf{R}u(1)u(2)/|R|$$

and this represents the $(1s)(2p)$-configuration of two (isotropic) oscillators.

As to V, two different suppositions will be made. First we take it to be an ordinary potential energy: $V=-Ae^{-r^2/a^2}$. Evaluation of the integrals (cf. formulas 15) leads to

$$\bar{V}=-A\tau^{\frac{3}{2}}(1-\delta^2)^{-1}\times(\exp[-\tau R^2/a^2]-\exp[-\tfrac{1}{2}\sigma R^2/a^2]). \quad (4)$$

Here σ and τ are two parameters which occur throughout this work:

$$\sigma\equiv qa^2;\quad \tau\equiv\sigma/(\sigma+2). \quad (5)$$

Second, we may choose for V a Majorana potential: $V=-Ae^{-r^2/a^2}P_{12}$. In that case $\bar{V}$ is given by (4) but with its sign reversed. Collecting these results we obtain for the first-order energy (using the notation 5)

$$\Delta_1E=\frac{\hbar^2}{4Ma^2}\sigma^2\frac{R^2}{a^2}\frac{\delta^2}{1-\delta^2}\pm\bar{V}. \quad (6)$$

For $R\to 0$, this expression takes the form $\hbar^2\sigma/2Ma^2\pm A\tau^{5/2}\sigma$; as R increases Δ_1E changes in monotone fashion. Clearly, if V is a Majorana (or Heisenberg) potential, Δ_1E represents a repulsion at all distances, while an ordinary potential may, for suitable values of A and a, produce attraction.

Next we consider the second approximation, Δ_2E, which is given by $-\langle V^2\rangle_{\text{Av}}/E_m$. The quantity E_m is some mean of all the excited energies of the system, the exact value of which can be ascertained by performing the summation appearing in the expression for the second-order perturbation energy. For the present purposes no exact attempt to evaluate it will be made. In molecular problems, it is in general sufficient to calculate $\langle V^2\rangle_{\text{Av}}$ with the function $u(1)v(2)$, a procedure which neglects exchange. To test the validity of this simplification let us first compute $\langle V^2\rangle_{\text{Av}}$ with exchange, that is with the use of the function $\psi(1,2)$ given by (2). The result is easily seen to be

$$\langle V^2\rangle_{\text{Av}}=A^2\left(\frac{\sigma}{\sigma+4}\right)^{\frac{3}{2}}(1-\delta^2)^{-1}\times\left(\exp\left[-\frac{2\sigma}{\sigma+4}\frac{R^2}{a^2}\right]-\exp\left[-\frac{\sigma}{2}\frac{R^2}{a^2}\right]\right). \quad (7)$$

If $\langle V^2\rangle_{\text{Av}}$ had been calculated with the use of $u(1)v(2)$ alone, the bracket appearing here would have contained only the first item, and the factor $(1-\delta^2)$ would be replaced by 1. Thus we identify the first term in the bracket as the "direct" one, the second as the exchange term.

The range of the nuclear force is a, the "radius" of the ψ-functions of each particle is $q^{-\frac{1}{2}}$. Hence $q^{\frac{1}{2}}a\equiv\sigma^{\frac{1}{2}}$ is the ratio of these two lengths. Now it is seen from (7) that, if σ is large, the exchange term decreases very rapidly in comparison with the direct term as R increases. This provides the justification for neglecting the exchange term in ordinary molecular problems. In the nuclear problem, however, σ is of order unity, and the exchange term must be retained.

It is equally instructive to compare the range of the first-order forces with that of the second-order forces. If σ is large, only the constituent $\bar{V}$ of Δ_1E remains ($\delta^2=\exp[-\frac{1}{2}\sigma R^2/a^2]$), and the exchange term in $\bar{V}$ (second term in the bracket of (6)) becomes small in comparison with the other; hence the dominating part of Δ_1E is $\exp[-\tau R^2/a^2]$ and this becomes e^{-R^2/a^2}. The leading term of $\langle V^2\rangle_{\text{Av}}$ in this case is

$$\exp\left[-\frac{2\sigma}{\sigma+4}\frac{R^2}{a^2}\right]\to\exp\left[-2\frac{R^2}{a^2}\right].$$

Thus the first-order forces have the *longer* range even for large σ. This is indeed characteristic of the nuclear situation. The fact that in the molecular case the reverse is true is conditioned by the nature of the Coulomb force and by the circumstance that each molecule has as many attracting as repulsive constituents. In the interaction of two H atoms, for example, the direct term (corresponding to $\exp[-\tau R^2/a^2]$ in (6)) becomes zero as soon as the atoms cease to overlap, whereas the direct term of (7) continues

to be appreciable and furnishes the van der Waals attraction.

These considerations illuminate the marked differences which exist between nuclear interactions and those between molecules. They prepare for an understanding of the results which will now be deduced for the interaction between alpha-particles.

2. Alpha-Particles. First-Order Forces

The energy of an alpha-particle is given variationally with good approximation by a product of Gauss functions of the type (1),[4] which yields more than 90 percent of the true energy if the range of the nuclear forces is 2.25×10^{-13} cm.[5] The value of q will depend on this range; indeed the lowest value of the energy is obtained by choosing

$$qa^2=\sigma=2.60.$$

Let the particles composing the first alpha-particle be labeled 1256, those composing the second 3478; 1, 2, 3, and 4 are protons. At large separation the first alpha-particle will have the state function

$$\tfrac{1}{2}\cdot\begin{vmatrix} u(1)\alpha(1) & u(1)\beta(1)\\ u(2)\alpha(2) & u(2)\beta(2)\end{vmatrix}\cdot\begin{vmatrix} u(5)\alpha(5) & u(5)\beta(5)\\ u(6)\alpha(6) & u(6)\beta(6)\end{vmatrix}, \quad (8)$$

the second a similar one with u replaced by v. α and β are spin functions. The system of two will be represented by the product of these, that is, by a product of four 2-rowed determinants. Such a function neglects the exchange of particles between the alpha-groups. The function which includes this exchange, and which will serve as the basis for the present calculation, is the following:

$$\psi=N_pU(1234)\cdot N_nU(5678) \quad (9)$$

with $N_p=N_n=(4!)^{-\frac{1}{2}}(1-\delta^2)^{-1}$, and

$$U(1234)=\begin{vmatrix} u_+(1) & u_-(1) & v_+(1) & v_-(1)\\ u_+(2) & u_-(2) & v_+(2) & v_-(2)\\ u_+(3) & u_-(3) & v_+(3) & v_-(3)\\ u_+(4) & u_-(4) & v_+(4) & v_-(4)\end{vmatrix} \quad (10)$$

$u_+(1)$ means $u(1)\alpha(1)$ etc., δ is again $\exp[-\frac{1}{4}qR^2]$, and $v(i)=v(\mathbf{r}_i)=u(|\mathbf{r}_i-\mathbf{R}|)$. $\mathbf{R}$ is the vector from the center of the first to that of the second alpha-particle. This coordinate will here be treated as a fixed parameter in conformity with molecular methods. The justification for such a procedure is not nearly so good in nuclear problems as it is in the molecular one, where the individual moving particles (electrons) are very much lighter than the nuclei. Our results will therefore be in error on this account; but this error will hardly affect the conclusions we wish to draw. To avoid it would require the use of more complicated methods such as that of Wheeler.[1]

The calculation of the first-order forces proceeds as follows: From the function ψ [Eq. (9)], $\bar{V}$ is computed, V being the total potential energy of the particles 1 to 8. From this must be subtracted the internal potential energy, V_0, of two alpha-particles which are not interacting, and finally ΔE_{kin}, the kinetic energy of promotion of 2 neutrons and 2 protons and the mutual Coulomb energy, E_c, are to be added. V is the sum over all 28 particle pairs of the elementary interaction V_{ij}, and this will here be taken as the sum of 3 exchange potentials and one ordinary potential with coefficients to be discussed later. Thus

$$V_{ij}=-AJ_{ij}(w+mP_{ij}+bQ_{ij}+hP_{ij}Q_{ij}) \quad (11)$$

with $J_{ij}=\exp[-r_{ij}^2/a^2]$. The operator P permutes space coordinates, Q permutes the spins. This form of V_{ij} no longer appears consistent in detail with recent developments in nuclear physics, meson potentials being now generally favored. The latter differ from (11) chiefly in two respects: They depend on r_{ij} in a slightly different way and they vary with the angle between $\mathbf{r}_i$ and $\mathbf{r}_j$. This last feature is not likely to be important in the present problem because the alpha-particles, being closed shells, will not exhibit polar forces even if the elementary particles do. As to the dependence on r_{ij}, the meson potential introduces a singularity at the origin which must be meaningless and is avoided by some cutting-off process. It is not yet known whether the radial function of the meson potential is more satisfactory than that of (11).

[4] H. Margenau and W. A. Tyrrell, Phys. Rev. **54**, 422 (1938).

[5] According to Breit, Thaxton and Eisenbud, Phys. Rev. **55**, 1018 (1939) the range is smaller. The degree of approximation by Gauss functions is then poorer.

V consists of 12 like-particle terms and 16 unlike-particle terms. Typical of the first class is V_{34}, of the second, V_{48}. We first establish the fact that all like-particle $\bar{V}_{ij}$ are equal, and similarly that all unlike-particle $\bar{V}_{ij}$ are equal, so that

$$\bar{V}=12\bar{V}_{34}+16\bar{V}_{48}. \tag{12}$$

The first follows immediately from the indistinguishability of like particles in the determinants U which compose the function ψ. Equality of unlike-particle interactions may be proved as follows: Since ψ is antisymmetrical with respect to (1234) and also to (5678), we have, for instance,

$$\int \psi V_{17}\psi d\tau=\int (14)(78)\psi(14)(78)\,V_{17}(14)(78)\psi d\tau=\int \psi V_{48}\psi d\tau.$$

(The numbers in parentheses indicate transpositions of coordinates to be performed in the function, ψ or V, which follows; and integrals include summations over spins.) Thus $\bar{V}_{17}=\bar{V}_{48}$ etc., and this establishes (12). In calculating the like-particle interaction we note that

$$\int \psi J_{34}P_{34}Q_{34}\psi d\tau=-\int \psi J_{34}\psi d\tau \quad \text{and} \quad \int \psi J_{34}P_{34}\psi d\tau=-\int \psi J_{34}Q_{34}\psi d\tau.$$

Therefore

$$\bar{V}_{34}=-A\left\{(w-h)\int \psi J_{34}\psi d\tau+(b-m)\int \psi J_{34}Q_{34}\psi d\tau\right\}. \tag{13}$$

The calculation of the integrals which appear here involves first an integration over particles (5678). There will then be left in ψ only one determinant, and this may be decomposed in a suitable way to form the products ψ^2 and $\psi\cdot Q_{34}\psi$. The final integrations are all elementary and can be performed with the use of the single formula

$$\int\int_{-\infty}^{\infty} \exp\,(-ax_1^2-bx_2^2+cx_1+dx_2+ex_1x_2)dx_1dx_2=\pi\left(ab-\frac{e^2}{4}\right)^{-\frac{1}{2}}\exp\left(\frac{ad^2+bc^2+cde}{4ab-e^2}\right). \tag{14}$$

The following basic integrals occur:

$$\left.\begin{aligned}
&\lambda_1\equiv\int u^2(1)u^2(2)J_{12}d\tau_1d\tau_2 &&=\int v^2(1)v^2(2)J_{12}d\tau_1d\tau_2 &&=\tau^{\frac{3}{2}}\\
&\lambda_2\equiv\int u^2(1)v^2(2)J_{12}d\tau_1d\tau_2 &&&&=\tau^{\frac{3}{2}}\exp\left[-\tau\rho^2\right]\\
&\lambda_3\equiv\int u(1)v(1)u(2)v(2)J_{12}d\tau_1d\tau_2 &&&&=\tau^{\frac{3}{2}}\exp\left[-\tfrac{1}{2}\sigma\rho^2\right]\\
&\lambda_4\equiv\int u(1)v(1)u^2(2)J_{12}d\tau_1d\tau_2 &&=\int u(1)v(1)v^2(2)J_{12}d\tau_1d\tau_2 &&=\tau^{\frac{3}{2}}\exp\left[-\tfrac{1}{4}(\sigma+3)\tau\rho^2\right]\\
&\delta\equiv\exp\left[-\tfrac{1}{4}\sigma\rho^2\right]
\end{aligned}\right\} \tag{15}$$

with $\sigma=qa^2$, $\tau=\sigma/(\sigma+2)$, $\rho=R/a$. One then obtains

$$\left.\begin{aligned}
&\int \psi J_{34}\psi d\tau=\tfrac{1}{3}(1-\delta^2)^{-2}\{\lambda_1+2\lambda_2-\lambda_3-4\delta\lambda_4+\delta^2(3\lambda_3-\lambda_2)\},\\
&\int \psi J_{34}Q_{34}\psi d\tau=\tfrac{1}{3}(1-\delta^2)^{-2}\{-\lambda_1+\lambda_2-2\lambda_3+4\delta\lambda_4-2\delta^2\lambda_2\}.
\end{aligned}\right\} \tag{16}$$

Turning now to the unlike-particle interaction, $\bar{V}_{48}$, we note first that

$$\int \psi J_{48} Q_{48} \psi d\tau = \tfrac{1}{2} \int \psi J_{48} \psi d\tau \quad \text{and} \quad \int \psi J_{48} P_{48} Q_{48} \psi d\tau = \tfrac{1}{2} \int \psi J_{48} P_{48} \psi d\tau.$$

The factor $\frac{1}{2}$ on the right side of these equations results because half the terms in the expanded determinants drop out on account of spin orthogonality after the Q permutations have been carried out. Thus

$$\bar{V}_{48} = -A\left\{(w+\tfrac{1}{2}b)\int \psi J_{48}\psi d\tau + (m+\tfrac{1}{2}h)\int \psi J_{48} P_{48}\psi d\tau\right\}, \tag{17}$$

and the integrals are

$$\begin{aligned} \int \psi J_{48}\psi d\tau &= \tfrac{1}{2}(1-\delta^2)^{-2}\{\lambda_1+\lambda_2-4\delta\lambda_4+2\delta^2\lambda_3\}, \\ \int \psi J_{48} P_{48}\psi d\tau &= \tfrac{1}{2}(1-\delta^2)^{-2}\{\lambda_1+\lambda_3-4\delta\lambda_4+\delta^2(\lambda_2+\lambda_3)\}. \end{aligned} \tag{18}$$

From (13), (16), (17), and (18) $\bar{V}$ can be compounded in accordance with Eq. (12). The internal potential energy of two alpha-particles, V_0, is easily seen to be

$$V_0 = -12A(w+m)\lambda_1.$$

Thus

$$\begin{aligned} \bar{V}-V_0 = -4A(1-\delta^2)^{-2}\{&w[(4-\delta^2)\lambda_2+(5+4\delta^2)\lambda_3-12\delta\lambda_4]+m[(4\delta^2-1)\lambda_2+(10-\delta^2)\lambda_3-12\delta\lambda_4] \\ &+2(b-h)(1-\delta^2)(\lambda_2-\lambda_3)\}. \end{aligned} \tag{19}$$

The remaining λ's are all exponential functions of $-R^2$, but have different ranges. The expression is finite at $R=0$ and tends to the limit

$$-4A\tau^{\frac{3}{2}}\sigma^{-2}\{w[4(\sigma-\tau)^2+\tfrac{1}{2}\tau^2]+m[11\tau^2/2-(\sigma-\tau)^2]+2(b-h)\sigma^2\tau\}. \tag{20}$$

The coefficients of w, m, b, and h in (19) are plotted in Fig. 1. It is to be noted that ordinary forces produce strong attraction, spin exchange forces half as strong attraction, while Majorana and Heisenberg forces produce repulsion between alpha-particles, results which are seen to be closely related to the saturation properties of these forces.

To evaluate Eq. (19) it is necessary to fix the nuclear constants w, m, b, and h. Fortunately, the interaction is almost independent of the particular choice as long as it satisfies the relations[6]

$$\left.\begin{aligned} w+m+b+h &= 1 \\ m+h &\geqq 2w+2b, \end{aligned}\right\} \quad \text{(a)}$$

$$m+2h \geqq 4w+2b. \quad \text{(b)}$$

In addition to these, let us impose: $b+h=0.2$, a value which seems to be in agreement with most facts known about light nuclei. One can then plot h as a function of w, first in accordance with relation (a) taken as an equality and disregarding (b), then in accordance with equality (b) disregarding (a). To eliminate all ambiguity we now select that value of h, for any given w, which makes the repulsion smallest; that is, the smaller of the two permitted values of h. The locus of these is very nearly

$$h = 1.25w - 0.1.$$

[6] N. Kemmer, Nature **140**, 192 (1937); G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936).

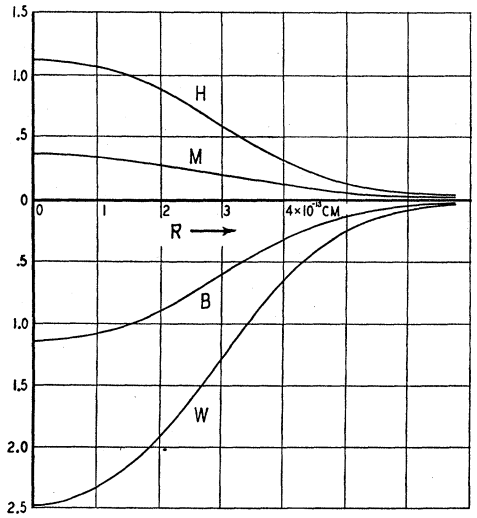

FIG. 1. Contributions to the potential energy of two alpha-particles, as functions of separation between their centers, from the different constituents of V. Heisenberg and Majorana forces (H, M) produce repulsion, the others attraction.

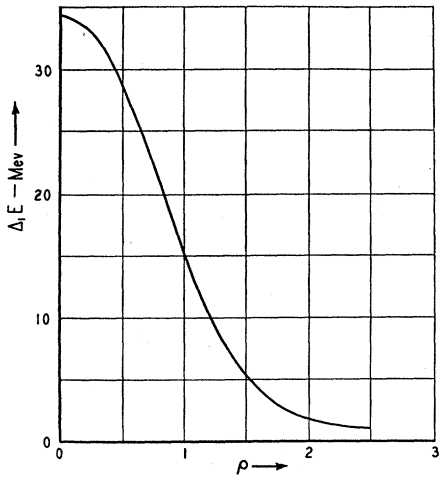

FIG. 2. Total first-order interaction between alpha-particles as function of ρ, the distance of separation in units a, the range of the nuclear force [cf. Eq. (11)].

This allows the parameters appearing in $\bar{V}-V_0$ to be expressed in terms of w: $m=0.80-w$, $b-h=0.40-2.50w$. Now, as mentioned before, minimization of the alpha-particle energy requires σ to be 2.60, hence $\tau=0.565$. It is seen from Fig. 1 that the ratio of the coefficients of w, m, b, h is approximately constant at all values of R, so that the value of $\bar{V}-V_0$ at any particular R, and most conveniently at $R=0$, is a measure of this interaction. Let us, therefore, evaluate $\bar{V}-V_0$ at $R=0$ as a function of w. Equation (20) gives

$$-60.5 \text{ Mev } [2.47w-0.35m+1.13(b-h)],$$

if A is taken to be 35.6 Mev in conformity with the theory of light nuclei. When the above values are inserted here, it turns out that w cancels almost exactly and the result is -10.4 Mev. We conclude, therefore, that the first-order interaction between alpha-particles is almost independent of the choice of force parameters. The potential energy is negative, but it will be seen that $\Delta_1 E$, which includes also the kinetic energy of promotion and the Coulomb energy E_C, is everywhere positive.

The total kinetic energy is

$$-\frac{\hbar^2}{2M}\langle\sum_i \nabla_i^2\rangle_{\text{Av}}=-\frac{4\hbar^2}{M}(1-\delta^2)^{-1}\int[u\nabla^2 u-\delta u\nabla^2 v]d\tau=\frac{4\hbar^2}{M}(1-\delta^2)^{-1}\{\tfrac{3}{2}q-(\tfrac{3}{2}q-\tfrac{1}{4}q^2R^2)\delta^2\}.$$

From this must be subtracted $(8\hbar^2/2M)(3q/2)$, the kinetic energy at infinite separation. Thus

$$\Delta E_{\text{kin}}=\frac{\hbar^2}{Ma^2}\sigma^2\rho^2\frac{\delta^2}{1-\delta^2}.$$

The Coulomb energy is most conveniently calculated in the "two center" system of coordinates familiar from the theory of the hydrogen molecule. If exchange is neglected, the result is very simple:

$$E_C=(4e^2/R)\text{ erf }[(\tfrac{1}{2}q)^{\frac{1}{2}}R].$$

Inclusion of exchange makes the expression more complicated for small values of R, but the modifications are hardly of interest because in this region E_C is much smaller than the other constituents of

the interaction. It takes the value 3.27 Mev at $R=0$ if we take for a the value 2.25×10^{-13} cm. When ΔE_{kin} and E_C are added to (19) and the λ-integrals are introduced, the result is

$$\Delta_1 E=\frac{\exp\left[-\frac{1}{2}\sigma\rho^2\right]}{1-\exp\left[-\frac{1}{2}\sigma\rho^2\right]}\left\{\frac{\hbar^2}{Ma^2}\sigma^2\rho^2-\frac{4A\tau^{\frac{3}{2}}}{1-\exp\left[-\frac{1}{2}\sigma\rho^2\right]}\left[w(5+4\exp\left[-\frac{1}{2}\sigma\rho^2\right]+4\exp\left[\frac{1}{2}\sigma\tau\rho^2\right]\right.\right.$$
$$-\exp\left[-\tau\rho^2\right]-12\exp\left[-\tfrac{1}{4}\tau\rho^2\right])+m(10-\exp\left[-\tfrac{1}{2}\sigma\rho^2\right]+4\exp\left[-\tau\rho^2\right]-\exp\left[\tfrac{1}{2}\sigma\tau\rho^2\right]$$
$$\left.\left.-12\exp\left[-\tfrac{1}{4}\tau\rho^2\right])+2(b-h)(1-\exp\left[-\tfrac{1}{2}\sigma\rho^2\right])(\exp\left[\tfrac{1}{2}\sigma\tau\rho^2\right]-1)\right]\right\}+(4e^2/R)\operatorname{erf}\left[(\tfrac{1}{2}\sigma)^{\frac{1}{2}}\rho\right]. \quad (21)$$

This expression is plotted in Fig. 2 for $b+h=0.2$, a choice which, as noted, makes $\Delta_1 E$ practically independent of w. The largest component of (21) is ΔE_{kin} which represents the energy of promotion of 4 particles to p states. It is to be remarked, however, that the function ψ of (9), while approaching the correct configuration $(1s)^4(2p)^4$ as $R\to0$, does not acquire the proper symmetry of a ^1S-state but contains an admixture of ^1D and ^1G as well. This implies increasing poorness of the approximation to the energy as $R\to0$, a defect which did not arise in the two-particle problem discussed in the foregoing section. The range of the first-order forces is seen to be practically the same as that of the nuclear forces themselves.

3. Second-Order Forces

An exact calculation of the second-order forces would be extremely tedious. We shall use here the method outlined in §1 and compute $-\langle V^2\rangle_{\text{Av}}$, leaving aside the constant factor $1/E_m$, since our interest concerns chiefly the range of the second-order forces. Furthermore, $\langle V^2\rangle_{\text{Av}}$ will be calculated *without* exchange. The effect of this neglect may be understood by examining Eq. (7) again. With the value of σ for the alpha-particle, the exchange term in that expression is $e^{-1.30\rho^2}$, the direct term is $e^{-0.79\rho^2}$. The latter persists at larger distances. Our procedure will therefore yield the constituent of the second-order forces which has the longer range. Finally, we shall avoid the complications arising from the fact that V is a mixture of different exchange terms and calculate $\langle V^2\rangle_{\text{Av}}$ for ordinary, pure Majorana, etc., interactions. After making all these simplifications, no quantitative inferences may be drawn from the results of this section, but several characteristic features become evident. The calculations need not be given in detail. Suffice it to say that, when

$$V=\sum_{i\neq j}V_{ij}\,^7$$

is squared and the integrations are performed, there result three groups of terms: 2-particle integrals like $\langle J_{15}^2\rangle_{\text{Av}}$, 16 in number; 3-particle integrals like $\langle J_{15}J_{16}\rangle_{\text{Av}}$, 96 in number; 4-particle integrals, 144 in number. The integrals are in general different for the different types of interactions. For spin-dependent interactions, the numbers of integrals will be effectively reduced by orthogonality of spin functions which fail to match after permutation. This explains the different coefficients in the expressions which follow. We find

$V_{ij}=J_{ij}$:

$$\langle V^2\rangle_{\text{Av}}=16\left(\frac{\sigma}{\sigma+4}\right)^{\frac{3}{2}}\exp\left[-\frac{2\sigma}{\sigma+4}\rho^2\right]+96\left[\frac{\sigma^2}{(\sigma+1)(\sigma+3)}\right]^{\frac{3}{2}}\exp\left[-\frac{2\sigma}{\sigma+3}\rho^2\right]$$
$$+144\left(\frac{\sigma}{\sigma+2}\right)^3\exp\left[-\frac{2\sigma}{\sigma+2}\rho^2\right],$$

$V_{ij}=J_{ij}P_{ij}$:

$$\langle V^2\rangle_{\text{Av}}=16\left(\frac{\sigma}{\sigma+4}\right)^{\frac{3}{2}}\exp\left[-\frac{2\sigma}{\sigma+4}\rho^2\right]+96\left[\frac{\sigma^2}{(\sigma+1)(\sigma+3)}\right]^{\frac{3}{2}}\exp\left[-\frac{\sigma+4}{\sigma+3}\cdot\frac{\sigma}{2}\rho^2\right]$$
$$+144\left(\frac{\sigma}{\sigma+2}\right)^3\exp\left[-\sigma\rho^2\right],$$

[7] The summation here is to be extended only over pairs not belonging to the same alpha-particles.

$V_{ij}=J_{ij}Q_{ij}$:

$$\langle V^2\rangle_{Av}=16\left(\frac{\sigma}{\sigma+4}\right)^{\frac{3}{2}}\exp\left[-\frac{2\sigma}{\sigma+4}\rho^2\right]+48\left(\frac{\sigma}{\sigma+2}\right)^{3}\exp\left[-\frac{2\sigma}{\sigma+2}\rho^2\right],$$

$V_{ij}=J_{ij}P_{ij}Q_{ij}$:

$$\langle V^2\rangle_{Av}=16\left(\frac{\sigma}{\sigma+4}\right)^{\frac{3}{2}}\exp\left[-\frac{2\sigma}{\sigma+4}\rho^2\right]+48\left(\frac{\sigma}{\sigma+2}\right)^{3}\exp\left[-\sigma\rho^2\right].$$

$\rho\equiv R/a$.

As an example, $\langle V^2\rangle_{Av}$ and $\bar{V}$ have been plotted for the case of pure Majorana forces in Fig. 3, the ordinates having been so adjusted as to make the two equal at $R=0$. $\langle V^2\rangle_{Av}$, which corresponds to the van der Waals force in molecular interactions, falls off much more rapidly than the exchange force $\bar{V}_{exch}$. To suppose that there exists a "van der Waals minimum" beyond the range of the repulsive first-order forces would be fallacious.

This result has also an important bearing on the additivity of the interaction between alpha-particles. In molecular problems, and in nuclear problems as well, additivity holds for second-order forces, but not for first-order exchange forces. Inasmuch as there exists for molecules a range in R in which only second-order forces are present, additivity holds within this range. The absence of such a range in nuclear interactions destroys additivity completely. It is clear, therefore, that attempts to calculate energies of complex nuclei based on the assumption of unique and additive inter-alpha forces must fail.[8] On the other hand, the validity of considerations respecting symmetries, multiplicities and the general topology of levels of complex nuclei is not impaired by the results here obtained, even though these considerations involve the concept of alpha-particles as constituent and relatively stable groups.

Apart from this general conclusion, there is reason to wonder if an interaction composed of the first- and second-order forces deduced above is applicable to the two-body problem, in which additivity is of no concern.

4. The Scattering of Alpha-Particles in Helium

The question thus arises as to whether and to what extent the interactions derived in the foregoing sections will explain the scattering of alpha-particles in helium. A very careful analysis of the experimental scattering data has been made by Wheeler,[9] who has also deduced phase shifts which account for the experimental results. These will here be used as a basis for comparison with expectations to be derived from the present model.

To be sure, this model is very indefinite because the second-order forces have not been calculated completely. It is certain, however, that whatever modifications these latter would introduce in the calculated Δ_1E, they are confined to the region $R<4.5\times10^{-13}$ cm. Somewhere in this region there must be a minimum which produces the level corresponding to the ground state of Be^8. To fix one parameter, the fact will be used that the energy of this level is very nearly zero[10] with respect to the energy of two alpha-particles at infinite separation. In the calculation of phase shifts, we shall, therefore, use a scattering potential indicated by the solid line in Fig. 4. For $R>R_0=4.5\times10^{-13}$ cm, the potential will be appreciably Coulombian. The potential trough occurring at smaller distances will be considered to have a fixed outer radius R_0, and the internal radius R' will be adjusted so that there will

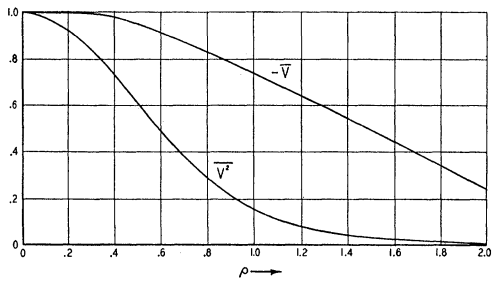

FIG. 3. Comparison of $\bar{V}$ and $\langle V^2\rangle_{Av}$. Ordinates are arbitrarily adjusted to make the two equal at $\rho=0$.

[8] Cf. in this connection also the conclusions of B. O. Grönblom and R. E. Marshak, Phys. Rev. **55**, 229 (1939).

[9] J. A. Wheeler, Phys. Rev. **59**, 16 (1941).

[10] S. K. Allison, L. S. Skaggs and N. M. Smith, Jr., Phys. Rev. **57**, 550 (1940).

exist but one level, and this at zero energy. The depth of the trough, D, will then be a function of R'. (Cf. Fig. 5.)

This model is highly idealized. The results for a smooth potential with sloping walls, however, differ insignificantly from those of this simple model. Even a potential having the form of the dashed curve in Fig. 4, which is not excluded by the previous analysis, yields results for the phase shifts which are qualitatively the same as

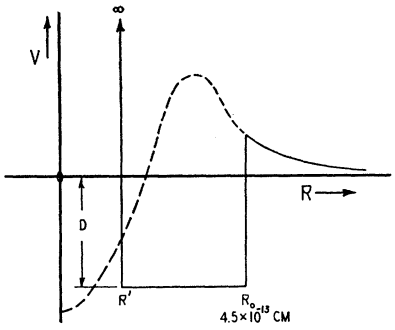

FIG. 4. Potential assumed in scattering calculations.

the ones here discussed. Details would change the local curvature of the phase shifts plotted in Fig. 6 but do not alter appreciably the region of the diagram to which the values of K_0 are confined.

The phase shift K_0, corresponding to the alpha-wave of zero angular momentum, is taken to be zero for scattering in a pure Coulomb field. For the potential of Fig. 4, it is computed by fitting the wave function inside the hole, $\sin k'(r-R')$ with $k'^2=(2\mu/\hbar^2)(D+E)$, to a proper function in the Coulomb field at $r=R_0$. This work is greatly abbreviated if use is made of the tables published by Wheeler.[11] In Fig. 6, K_0 is plotted against the energy of the incident alpha-particles for different values of R'. The lowest of these graphs corresponds to an infinitely deep trough of zero width at $R=R_0$, the highest to the absence of all repulsion near the origin. It may be noticed that all curves converge to the value $K_0=\pi$ at $V=0$. This feature is, of course, independent of our assumptions about the potential; it is a consequence solely of the existence of a stable ^1S level at zero energy.

[11] J. A. Wheeler, Phys. Rev. **52**, 1123 (1937).

The K_0 curve which Wheeler, in an adjoining paper, has deduced from scattering data and found compatible with the facts known about Be^8, is also plotted in Fig. 6. At lower energies it is seen to lie in a region permitted by the simple alpha-particle model under discussion, but it rapidly moves out of this region as the energy increases. The reason for this behavior is the resonance occurring in the S wave at an energy of about 3 Mev according to Wheeler's analysis. The potential deduced in this paper will not yield a resonance level at so low an energy.

The existence of an excited level in the neighborhood of 3 Mev above the lowest state of Be^8 and of width ≈ 0.8 Mev is quite definitely indicated in the observations by Dee and Gilbert[12] on the disintegration of B^{11} by protons, and in the scattering experiments of Devons.[13] Previously it had been identified as a $^1D_2$ state, but Wheeler[9] shows it to be a ^1S level. One can easily see that the simple alpha-particle model is incompetent to account for these facts inasmuch as it gives too small a range for the non-Coulombian forces. The difficulty is with the spacing of the levels, which comes out too large. Let us adjust the parameter R' so that the

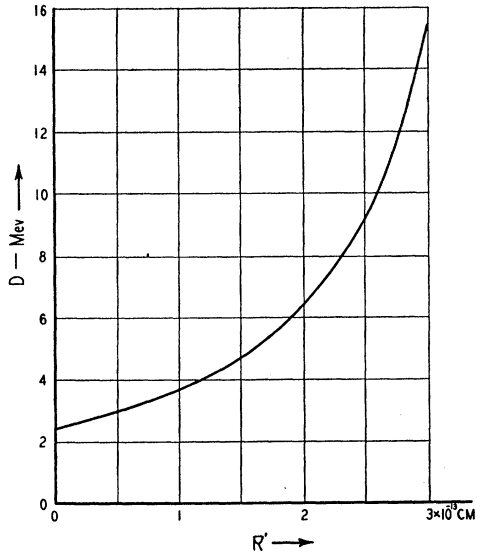

FIG. 5. Relation between D (cf. Fig. 4) and R' which will produce the lowest level in Be^8 at zero energy.

[12] P. I. Dee and C. W. Gilbert, Proc. Roy. Soc. **A154**, 279 (1936).
[13] S. Devons, Proc. Roy. Soc. **A172**, 559 (1939).

energy levels have the closest possible spacing, that is, let $R'=0$. One can then compute the position of the first ^1S resonance in the continuum[14] and it is found to lie about $6D$ above zero. Its width is of the order of the level spacing, that is, around $3D$. Now to obtain the lower ^1S level at zero energy, D must be 2.4 Mev. Hence the lowest position obtainable for the ^1S resonance would be 14 Mev, and the width would be of that order of magnitude, all of which disagrees with experience. The state in question can obviously not be understood on the basis of the simple alpha-particle model; it must involve a considerable rearrangement of elementary particles and seems to be much better described by the Hartree model.[15]

To produce the resonance at the correct place one must assume a force range far greater than is compatible with our former conclusions, and one would then encounter difficulties in explaining the initial trend of Wheeler's K_0 curve at low energies. The inconsistency attaching to the width of the excited level has already been pointed out by Rosenthal[16] who, using a Morse curve for the potential, finds the resonance to be too diffuse for consideration.

Conclusion

The application to the interaction between alpha-particles of methods similar to those which describe successfully the forces between atoms, leads to results which leave no basis for the supposition that the forces are describable as a superposition of repulsive first-order and attractive second-order effects. There is indeed no indication that the various orders of the perturbation scheme converge, and the forces are not additive. It is possible, however, to fix their range with reasonable certainty if detailed assumptions are made regarding the interaction between elementary particles. This range is approximately equal to that of the elementary forces.

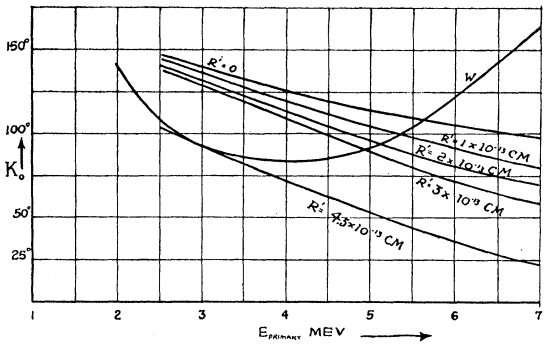

FIG. 6. *Solid lines:* Phase shift of S wave, K_0, as calculated on the basis of the model of Fig. 4, for different values of R'. *Curve marked W:* K_0 values derived from scattering data by Wheeler.

When the positive results of this study are incorporated in a simple model, the scattering of alpha-particles in He can be discussed. The computed phase shifts for the wave of zero angular momentum are in accord with observations at low energies of the incident particles, but fail to explain the resonance observed at higher energy.

The results definitely discredit the early alpha-particle model which endowed these groups with fairly permanent existence in higher nuclei and sought to explain nuclear structure by assuming unique and sometimes additive forces between them. They indicate in addition that even the interaction of two alpha-particles cannot be treated successfully by methods familiar from the theory of molecular forces. On the other hand they do not dispute the legitimate rôle which these alpha-groups may play in determining symmetries and multiplicities of nuclear states.

The author records with gratitude the privilege of using Professor Wheeler's results before their publication. His thanks are likewise extended to the Institute for Advanced Study which permitted him a most stimulating stay in Princeton.

[14] H. Margenau, Phys. Rev. **46**, 613 (1934).
[15] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937).
[16] J. E. Rosenthal, Phys. Rev. **54**, 315 (1938).