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Radiative Transition Probabilities in Heavy Nuclei

Excitation of Nuclei by X-Rays

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From experimental absorption cross sections for excitation of nuclei by x-rays, radiation widths, i.e., emission probabilities are deduced. For the lowest activation levels of In^{115} and Pb widths of the order of a few millivolts are obtained. These widths are of the same order as those of the low levels of the natural radioactive nuclei. The quadrupole radiation widths obtained are within the upper limits set by a sum formula. Formulae for the transition probabilities are given depending both upon energy and spin. A numerical constant entering into the formulae, which is usually calculated from assumed models, is obtained from experiment. A simple expression is given, valid in the neighborhood of the short wave limit, for the intensity of x-rays produced by fast electrons. At the short wave limit the intensity is finite and changes but little both with energy and frequency in the neighborhood of the limit. Therefrom an excitation curve with many flat plateaus follows, each plateau indicating another nuclear level. These flat plateaus reveal directly the shape of isochromats, thus providing a qualitative check on the theory of production of x-rays by fast electrons.

I. INTRODUCTION

UNTIL recently, direct information about the absolute values of nuclear transition probabilities and the level systems of the nuclei involved was obtained only for a few elements. Three groups of experiments have been used for this purpose. (1) The transition probabilities between low levels of natural radioactive nuclei were estimated by comparing the intensity of the long range α-particle groups from the excited levels of these nuclei with that of the γ-rays from the same level. Γ-widths of about a few millivolts were obtained. (2) The transition probabilities between high levels of heavy nuclei were estimated from experiments on the capture of slow neutrons. Γ-widths of about a few tenths of a volt were obtained. (3) The transition probability between a continuous and a discrete state of light nuclei was estimated from experiments on the capture of protons. Γ-widths from a tenth to about a few volts were obtained.

Recently, however, Waldman and Collins excited low metastable states of In^{115} and Pb by x-rays and electrons.[1] This method is capable of extension to higher states and probably to other elements. The present note shows how the experimental excitation curves may be understood quantitatively and how the radiation widths, i.e., the *emission* probability, may be calculated from the experimental *absorption* cross sections. The nuclear levels are given directly by the distances between discontinuities

[1] These data were furnished me prior to their publication by Drs. Waldman and Collins. Cf. also B. Waldman, G. B. Collins, E. M. Stubblefield and M. Goldhaber, Phys. Rev. **55**, 1129 (1939); G. B. Collins and B. Waldman, Phys. Rev. **59**, 109A (1940); B. Waldman and G. B. Collins, Phys. Rev. **57**, 338 (1940).

in the excitation curves. The radiation widths obtained for In^{115} and Pb are of the order of a few millivolts, i.e., of the same order as those of the low states of the natural radioactive nuclei. Incidentally, the excitation curves revealing the shape of the isochromats provide a qualitative check of the theory of the production of x-rays by fast electrons, particularly in the neighborhood of the short wave limit.

For the interpretation of the experimental data, theoretical formulae had to be introduced for: (a) the intensity of the x-rays produced by electrons near the short wave limit; (b) the dependency of the multipole matrix elements involved upon both energy and spin.

The importance of such semi-empirical information about heavy nuclei is borne out by the well-known fact that there are no reliable models for these nuclei. Calculations of the transition probabilities for an α-particle model or the liquid droplet model may and were, in fact, carried out. But the results are not trustworthy.

The excitation of a nucleus by photons may be (1) a line absorption, (2) a Raman effect, and (3) a photo-effect. We consider here the first process only.[2] For In^{115} it may be concluded directly that a line absorption must take place. In the experiments of Goldhaber, Hill, and Szilard, γ-rays from $\frac{1}{2}$ g of Ra irradiated indium without producing any activity.[3] It is now known that the energy of these γ-rays is sufficient to excite In^{115} and the absence of activity is attributed to the fact that none of the known γ-ray energies are equal to the energy of the activation level. The probability of excitation is therefore small. Thus, only irradiation by a continuous x-ray spectrum is likely to lead to a line absorption.

II. Connection Between Absorption Cross Section and Emission Transition Probability

A general formula may be derived, connecting the cross section σ_a for the excitation in line absorption of a level with the emission transition probability.

Let us define $\sigma(h\nu)$ as the absorption cross section for excitation by one photon. Then the dispersion formula holds:

$$\sigma(h\nu)=\pi\lambda\!\!\!^{-2}\cdot\frac{2j_a+1}{2(2j_g+1)}\cdot\frac{\Gamma\Gamma_r}{h^2(\nu-\nu_0)^2+\Gamma^2/4}. \quad (1)$$

Γ is the total gamma-width of a level a with the energy $h\nu_a$ above the ground level g. Γ_r is the partial width for the resonance transition. At the peak of the line, $\sigma(h\nu)$ reduces to the expression

$$4\pi^2\lambda\!\!\!^{-2}\cdot\frac{2j_a+1}{2(2j_g+1)}\cdot\frac{\Gamma_r}{\Gamma};\quad \lambda\!\!\!^{-}=\frac{\lambda}{2\pi};$$

j_a and j_g denote the spins of the states a and g, respectively. Formula (1) is valid not only for dipole, but also for electric and magnetic 2^l- and 2^{l-1}-pole radiation, if $\Gamma=\Gamma^{(l)}$ and $\Gamma_r=\Gamma_r^{(l)}$ are taken then as widths for the corresponding multipole radiation.

Let $n(h\nu)d(h\nu)$ designate the number of x-ray quanta produced by one incident electron of the energy E with frequencies between ν and $\nu+d\nu$. $h\nu\cdot n(h\nu, E)d(h\nu)$ is then the intensity. Then the absorption cross section σ_a for production of nuclei in the excited state with energy $h\nu_a$ above the ground state is given by

$$\sigma_a^{(l)}=\int_{\text{over the line}} n(h\nu_a, E)\sigma(h\nu_a)d(h\nu_a)$$

$$=n(h\nu_a, E)\frac{2j_a+1}{2(2j_g+1)}\cdot 2\pi^2\lambda\!\!\!^{-}_a{}^2\Gamma_r^{(l)} \quad (2a)$$

or conversely

$$\Gamma_r^{(l)}=\frac{2(2j_g+1)}{2j_a+1}\cdot\frac{\sigma_a^{(l)}}{2\pi^2\lambda\!\!\!^{-}_a{}^2 n(h\nu_a, E)}. \quad (2b)$$

III. Connection of the Cross Section σ_a with the Experimental Cross Section σ_{exp}

In the actual experiments, the cross section σ_a as defined above cannot be observed. Rather, the internally converted electrons are observed directly because of the decay of a metastable state which lies between the ground state g and the

[2] Estimates show that the cross section for Raman processes on nuclei is rather small and probably not observable. An estimate of widths from photo-effect and photo-fission was carried out by V. F. Weisskopf, to be published soon.

[3] M. Goldhaber, R. D. Hill, and L. Szilard, Phys. Rev. 55, 57 (1939). Also, no effect was observed for the γ-rays of ThC, according to a personal communication of Dr. Goldhaber.

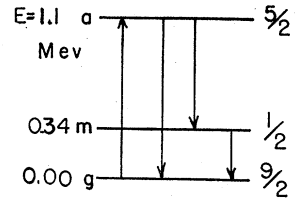

FIG. 1.

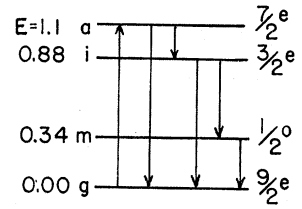

FIG. 2.

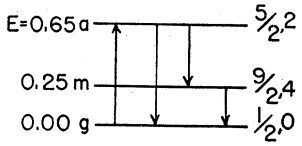

FIG. 3.

activation state a, giving an experimental cross section σ_{exp}. The cross section σ_a refers to the production of this activation level. From σ_{exp} the cross section for production of stable nuclei in the metastable state σ_m may be obtained by division by the coefficient of internal conversion α:

$$\sigma_m = \sigma_{\text{exp}} \cdot \alpha^{-1}. \tag{3}$$

The next step is to connect σ_m with σ_a. In order to do that, we must know the relative transition probabilities, activation→ground and activation→metastable. The ratio, f, of the transition probabilities

$$\frac{\text{activation}\rightarrow\text{ground}}{\text{activation}\rightarrow\text{metastable}}$$

consists of the product of two factors: $f = f_S \cdot f_E$. The factor f_S (statistical factor) is simply equal to the ratio of the statistical weights of the ground and metastable levels. The factor f_E (energy factor) arises from the dependency of the transition probabilities upon the energy.

Knowing this ratio, f, one has only to multiply it by σ_m, the cross section for production of nuclei in the metastable state in order to obtain the excitation cross section σ_a:

$$\sigma_a = f \cdot \sigma_m; \quad f = f_S \cdot f_E. \tag{4}$$

This holds if the two transitions to be compared have the same multipole order as is the case for the level scheme of Fig. 1. However, if the transitions have different multipole orders, as is the case for the level scheme of Fig. 2, f will contain a third factor f_l where l denotes the difference in spins of the levels considered. f_l may be taken from formula (10) of Appendix 1. One then has:

$$f = f_S \cdot f_E \cdot f_l. \tag{4a}$$

IV. Dependency of Multipole Transition Probabilities upon the Energy

The factor f_E consists of the ratio of two multipole transition probabilities which depend upon the energy E.

For an electric 2^l-pole or a magnetic 2^{l-1}-pole we have, according to definition

$$\begin{Bmatrix} \Gamma_{\text{el}}^{(l)} \\ \Gamma_{\text{magn}}^{(l-1)} \end{Bmatrix} \sim \frac{1}{\lambda^{2l+1}} \begin{Bmatrix} |M_{\text{el}}^{(l)}|^2 \\ |M_{\text{magn}}^{(l-1)}|^2 \end{Bmatrix}, \tag{5}$$

where $|M_{\text{el}}^{(l)}|^2$ and $|M_{\text{magn}}^{(l-1)}|^2$ designate the square of the absolute values of the coordinate electric 2^l- and magnetic 2^{l-1}-pole matrix elements, respectively. There are good arguments which make it very plausible that the coordinate matrix elements $|M|^2$ are to a good approximation, independent of the energy. Rough estimates and more accurate simple model calculations indicate this, and also a more general argument put forward by Weisskopf and Ewing[4] yields the same conclusion.

We have then:

$$\begin{Bmatrix} \Gamma_{\text{el}}^{(l)} \\ \Gamma_{\text{magn}}^{(l-1)} \end{Bmatrix} = \begin{Bmatrix} C \\ C' \end{Bmatrix} \cdot E^{2l+1}, \tag{6}$$

where the C and C' depend upon the actual values of the matrix elements involved and are best taken, whenever possible, from experiment. Model calculations are unreliable in this respect and generally give too high values for the constants. For a more complete formula which, however, is based on more special assumptions, cf. Appendix 1.

[4] V. F. Weisskopf and D. H. Ewing, Phys. Rev. **57**, 472 (1940); cf. also Weisskopf, reference 2.

V. Assumptions About the Level Scheme Needed for the Evaluation of the Factors f

The values of the factors f_S and f_E, introduced in formula (4), depend upon the level scheme assumed, namely, upon the values of the spin assigned to the metastable and the activation level and also whether another level between the metastable and the activation level is assumed to exist or not.

For In^{115} the simplest plausible scheme is shown in Fig. 1. On the right side of the scheme we inserted the spins attributed to the levels. All three levels may have the same parity. The height of the activation level corresponds to the threshold of the x-ray excitation and the height of the metastable level is the energy of the electrons arising from internal conversion of the γ-rays emitted by the decay of the metastable state. Both have been observed directly.

This system does not contain the level at approximately 800 kev postulated by Lawson and Cork,[5] since the evidence for it is meager. Contrary to the usual conclusion that the relatively large lifetime of In^{115} necessitates the assumption of a change in parity between the ground and metastable states in addition to a change of the spin by two, no change in parity is postulated.[6] The need for introducing a change in parity may be overcome either by the introduction of a second level between the metastable level and the activation level, or by assuming that the matrix elements involved are much smaller than is generally thought.[7]

It should be pointed out that the transition from the ground to the activation level may turn out eventually to be a magnetic or even an electric dipole transition. In either case, it would be necessary to assume the existence of an intermediate level between the activation level and the metastable level. A probable level scheme is shown in Fig. 2 (cf. ref. 5). The indices e and o designate even and odd. According to well-known selection rules, Fig. 2 means that the transition $g \rightarrow a$ is assumed to be a magnetic dipole transition. For the height of the intermediate level i the value given by Lawson and Cork may be used, better evidence being lacking.

From data on excitation by x-rays alone it is hardly possible to arrive at an unambiguous decision about the multipole character of the activation transition. However, additional data about excitation of nuclear levels by electrons and heavy charged particles may help in this decision, as will be shown at another place (cf. also end of Appendix 1).

In the case of lead, it is not known to which of its four known isotopes, namely, Pb^{204}, Pb^{206}, Pb^{207}, and Pb^{208}, the observed activity should be attributed. For the even isotopes with zero spins a transition having a 0–0 spin change between two states of different parity is also possible,[8] in addition to transitions with changes of the spin. For a spin change 0–0, the internally converted electrons will not be monochromatic. Furthermore, the conversion coefficient will be much smaller than for a transition with change of the spin. Neither of these conclusions has as yet been experimentally tested, because of the short life of the metastable lead level.

Another possibility is a 0–0 transition between states of the same parity by direct interaction between the nucleus the atomic electrons, which penetrate, in this case, into the nucleus. Such a nonradiative transition takes place, according to Fowler,[9] from the metastable level of 1.41 Mev of RaC′ to its ground level. One obtains, however, on all plausible assumptions much too short lifetimes ($\tau < 10^{-8}$ sec.) to account for the observed periods of lead or any other known isomers.[7]

For simplicity, it is assumed here that in the actual transition, a change in the spin takes place just as for In^{115}. A possible level scheme is then shown in Fig. 3. The first column on the

[5] J. L. Lawson and J. M. Cork, Phys. Rev. **57**, 982 (1940), Fig. 12, p. 993.

[6] My attention was called by Drs. Serber and Dancoff to the interesting fact that the assumption of a change of parity between m and g fits the lifetime of m; K/L conversion ratio and K conversion. However, in the estimate of the lifetime model calculations were used. Furthermore, the conversion coefficients for the magnetic 2^4-pole radiation were taken from nonrelativistic calculations of M. H. Hebb and E. Nelson, Phys. Rev. **58**, 486 (1940) which do not hold well for In ($Z=49$) and conversion electrons of 0.34-Mev energy.

[7] Cf. M. H. Hebb and G. E. Uhlenbeck, Physica **5**, 605 (1938).

[8] R. G. Sachs, Phys. Rev. **57**, 194 (1940).

[9] R. H. Fowler, Proc. Roy. Soc. **A129**, 1 (1930). It may be, however, that the observed transition is connected with a spin change of two or more instead of being a 0–0 transition.

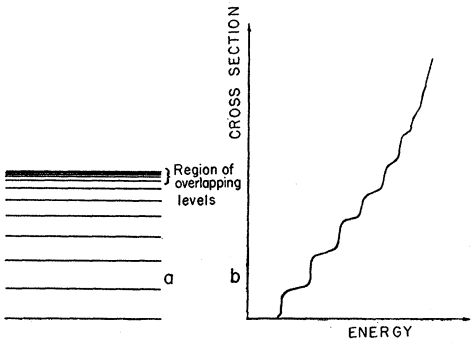

FIG. 4(a). Schematic nuclear level system and (b) the corresponding excitation curve. The theoretically discontinuous corners of the plateaus are rounded off.

right-hand side of the scheme refers to the even active isotope Pb^{204} or Pb^{206} or Pb^{208}, the second to Pb^{207}.

VI. Number of X-Ray Quanta Produced by Fast Electrons Near the Short Wave Limit

In the experiments, excitation curves were taken as a function of the energy of the electrons incident on a gold or lead target. The x-rays produced this way were used to excite In^{115} or Pb. We need, therefore, $n(h\nu, E)$, i.e., the number of quanta produced by one electron at a fixed $h\nu$, namely, $h\nu = h\nu_a$, the value of the threshold (energy of activation level) as a function of energy. $I(h\nu_a \cdot E) = h\nu_a \cdot n(h\nu_a \cdot E)$, as a function of E at fixed ν_a is a so-called isochromat. In checking the theory of the continuous x-ray radiation, it is, from the experimental point of view, best to take isochromats. *It is interesting to point out that the shape of an excitation curve in passing an activation threshold reveals exactly the shape of the corresponding isochromat.* For in formula (2), $n(h\nu_a, E)$ is the only quantity varying with E.

According to the Born approximation the number of quanta at a fixed ν produced by one electron would increase with increasing energy. In fact, the Bethe-Heitler formula gives $n(h\nu_a, h\nu_a) = 0$, i.e., zero number of quanta at the short wave limit. However, Born's approximation does not hold near the short wave limit. The exact theory is known only in the nonrelativistic region. Both Sommerfeld's original approximation and the exact formulae of Sommerfeld and Maue[10] show that the number of quanta is *finite* at the short wave limit and changes but little both with ν and E in the immediate neighborhood of the threshold. As a matter of fact, the isochromats may even decrease with increasing energy. This is true both for the number of quanta integrated over all angles or in a certain specified direction. Quantitatively, one may prove that the exact formulae of Sommerfeld and Maue may be obtained from the Born approximation (which holds rigorously at the long wave end) by multiplication with a factor $f(Z; v, v')$ where v and v' are the velocities of the electron before and after collision and Z is the atomic number.[11]

$$f(Z; v, v') = \frac{1 - e^{-2\pi\alpha Zc/v}}{1 - e^{-2\pi\alpha Zc/v'}} \cdot \frac{v'}{v}. \tag{7a}$$

It is now a plausible assumption that the same factor will convert the Bethe-Heitler formula into a formula valid with good approximation at and around the threshold; i.e.,

$$n(h\nu, E) = n(h\nu, E)_{B-H} \cdot f(Z; v, v'). \tag{7b}$$

Of course, the exact $n(h\nu, E)$ may contain additive terms which go to zero in the nonrelativistic limit. Calculations in which approximate wave functions of Sommerfeld and Maue are used point in this direction. However, comparison of $n(h\nu, E)$, according to formula (7b), and of exact numerical calculations of Jaeger[12] shows that such additive terms are not too important. In fact, formula (7b) gives for $h\nu_a \sim 1$ Mev and $E \sim 1$ Mev an $n(h\nu, E)$ differing by about 15 percent from Jaeger's value. In the immediate neighborhood of the threshold, (7b) may be written in the form:

$$n(h\nu, E) = A \cdot \frac{Z^2 r_0^2}{137} \cdot \frac{1}{h\nu} \cdot N \cdot dx;$$
$$A \leqq 3.44 \text{ for } E \leqq 1.1 \text{ Mev}, \tag{7b'}$$
$$A \leqq 4.89 \text{ for } E \leqq 0.65 \text{ Mev},$$

[10] A. Sommerfeld and A. W. Maue, Ann. d. Physik **23**, 589 (1935).

[11] The factor $\frac{2\pi\alpha Zc/v}{e^{2\pi\alpha Zc/v} - 1} \cdot \frac{2\pi\alpha Zc/v'}{1 - e^{-2\pi\alpha Zc/v'}}$ converts the Born approximation into the old Sommerfeld formula (Sommerfeld, Ann. d. Physik **9**, 257 (1931)) which, however, is itself only an approximation.

[12] J. C. Jaeger, Nature **140**, 108 (1937).

where N is the number of atoms per cm^3 and dx is the thickness of the (thin) target. Further away from the threshold, however, the full formula (7b) has to be used. (7b′) does not depend upon E at all.

This behavior of $n(h\nu, E)$ shows that an excitation curve with many flat plateaus is to be expected, each plateau indicating another level; cf. Fig. 4(a) and (b). In fact, for In^{115} and Pb, the first plateau was observed and for In^{115} indication of a second level was also observed. The "flatness" of the first plateau is illustrated by the fact that, according to (7b), $n(h\nu, E)$ increases by less than 2 percent if E is raised from 1.1 to 1.4 Mev. The experimental curves, of course, do not show sharp discontinuities at each level. This is just as for the isochromats, because of the unavoidable slight inhomogeneity of the electron beam.

VII. Calculation of the Widths

The experimental cross sections of Waldman and Collins[13] are:

$$\sigma_{\text{exp}} = 1\times10^{-34} \text{ for } In^{115}$$

and

$$\sigma_{\text{exp}} = 2.5\times10^{-34} \text{ for Pb.}$$

The internal conversion coefficients are[14]

$$\alpha = 0.5 \text{ for } In^{115}, \quad \alpha \sim 1 \text{ for Pb.}$$

Therefore, one has according to formula (2)

$$\sigma_m = 2\times10^{-34} \text{ cm}^2 \text{ for } In^{115},$$
$$\sigma_m = 2.5\times10^{-34} \text{ cm}^2 \text{ for Pb.}$$

For the simple schemes of Figs. 1 and 3, the statistical factors are:

$$f_S = (2j_g+1)/(2j_m+1) = 5 \text{ for } In^{115}$$

and

$$f_S = \tfrac{1}{5} \text{ for Pb (odd isotope).}$$

The energy factors are, according to the energy values indicated in these figures: $f_E = 6.4$ for In^{115} and $f_E = 11.4$ for Pb. Inserting these factors into formula (4), one has

$$\sigma_a^{(2)} = 6.4\times10^{-33} \text{ for } In^{115},$$
$$\sigma_a^{(2)} = 5.7\times10^{-34} \text{ for Pb.}$$

Finally, from formula (2b), using formula (7b′) for $n(h\nu, E)$, one obtains

$$\Gamma_r^{(2)} = 4.09 \text{ millivolts for } In^{115},$$
$$\Gamma_r^{(2)} = 0.28 \text{ millivolt for Pb.}$$

These values for the widths are smaller than an upper limit set by a sum formula for quadrupole transitions. (Cf. Appendix 2.)

Table I gives a comparison of the widths $\Gamma_r^{(2)}$, lifetimes $\tau^{(2)} = \hbar/\Gamma_r^{(2)}$, and $C = \Gamma_r^{(2)}/E^5$ of excited states of natural radioactive nuclei[15] and those of In^{115} and of Pb. In view of the incompleteness of the experimental data and the resulting ambiguity in the assumed level scheme, it is somewhat encouraging to find the same orders of magnitude.

In order to calculate the f-factors for the level scheme of Fig. 2, one has to use formula (10) of Appendix 1. On account of the uncertainty about the intermediate level and about the validity of formula (10) we defer, however, the calculation of the width.

VIII. Doppler Effect and Self-Absorption

In connection with the above evaluation of the widths of excited states, it is important to point out that both for In^{115} and Pb, the Doppler widths are large compared with the radiation widths. The Doppler width of this level of In^{115} is about 0.7 ev. This explains the smallness of self-absorption of the line in question in In^{115} which was observed by Waldman and Collins.

Table I. *Widths of the excited states of* RaC′, ThC′, Pb, In^{115}.

Nucleus	Level	E (Mev)	$\Gamma_r^{(2)}$ (Millivolts)	$\tau_r^{(2)}$ (Sec.)	C (Mev)$^{-4}$
RaC′	1	0.61	2.	3×10^{-13}	2.4×10^{-8}
	3*	1.41	0.01	8×10^{-11}	—
ThC′	1	0.73	1.5	4×10^{-13}	7.3×10^{-9}
	2	1.80	1.0	6×10^{-13}	6.0×10^{-11}
Pb	2	0.65	0.28	2.34×10^{-12}	2.40×10^{-9}
In^{115}	2	1.10	4.09	1.55×10^{-13}	2.59×10^{-9}

* Only conversion electrons observed.

[13] These cross sections are upper limits, taken from thick targets approaching a thin one. The cross sections are taken for electron energies above the threshold. They are sensibly independent of the electron energies.

[14] The value of α for Pb is a plausible assumption, but needs verification by experiment.

[15] Cf. H. A. Bethe, Rev. Mod. Phys. 9, 229 (1937), Table XXXXIV.

For the Doppler width, one has

$$\Gamma_{\text{Doppler}}=h\nu_a\cdot(2kT/Mc^2)^{\frac{1}{2}};$$
$$M\text{: mass of nucleus.} \quad (8)$$

At low temperatures, kT in this formula has to be replaced[16] by $k(T+\frac{1}{2}\Theta)$, where Θ designates the Debye temperature of the nucleus. It would be possible to measure the width of the activation level more accurately by direct absorption measurements at low temperatures, were it not for the fact that the value for the Debye temperature for In is still too high for an effective reduction of the Doppler width. All other elements with known metastable levels also have Debye temperatures too high to allow an appreciable reduction of the Doppler width, e.g., lead ($\Theta_D=88$) Kr, Xe, Ag. The Θ_D-values for In, Kr, Xe were calculated by means of the Lindemann formula:

$$\Theta=123(T_m/Av^{\frac{2}{3}})^{\frac{1}{2}}, \quad (9)$$

T_m, melting point; A, atomic weight; v, atomic volume. No data on the specific heats at low temperatures are known for these three elements.

It is a pleasure for the author to thank Drs. B. Waldman and G. B. Collins for discussion of their experiments and Drs. H. A. Bethe, W. E. Lamb, Jr., J. A. Wheeler and especially Dr. V. F. Weisskopf for helpful discussions and suggestions.

Appendix 1

More complete formula for multipole matrix elements

A semi-empirical formula for the multipole transition probabilities like (2):

$$\Gamma(E)^{(l)}=c,\ E^{2l+1} \quad (2)$$

gives the dependency upon E but the dependency upon l is not contained completely. Now if one wants to relate the transition probabilities for transitions of different multipole order, one needs a formula giving the dependency upon l more accurately. It may be shown that such a more complete formula will have for electric 2^l- or magnetic 2^{l-1}-pole transitions the general form:

$$\Gamma^{(l)}=B\cdot\frac{2j_i+1}{2j_k+1}\left(\frac{l+1}{l}\right)\cdot[(2l+1)(2l-1)\cdots3.1]^{-2}E(RE)^{2l}. \quad (10)$$

Here j_i and j_k are the spins of lower and upper state with energies E_i and E_k, respectively; $l=|j_i-j_k|$; $E=E_k-E_i$ measured in units mc^2; R: nuclear radius in units of $\hbar/mc$; B is a constant, to be taken from experiment.[17] This formula, however, is based on less certain assumptions than (2) and may be used only with caution.

Using (10) and the value of the width in Table I for fixing B, one obtains for the lifetime of the metastable state of Pb, assuming again the level scheme of Fig. 3: $\tau^{(4)}\sim67$ sec. in amazing agreement with the observation (100 sec.). This agreement has, however, to be taken only with great caution, because of the uncertainties in formula (10), in the assumed level scheme, and in the experimental data, inasmuch as for In^{115} the agreement is less good using either the level scheme of Fig. 1 or of Fig. 2. We do not want to discuss this matter further here, but defer it to another paper, where some more applications to isomers will also be treated.

Appendix 2

Sum formula for quadrupole radiation

An upper limit for the dipole matrix element

$$D_{ik}=\sum_{j=1}^{n}\int r_j\psi_i\bar{\psi}_k dr$$

may be derived, as is well-known, from the sum formula:

$$\sum_{i=0}^{k}|D_{ik}|^2\cdot2m_p\cdot E_{ik}=n\text{; namely: }|D_{ik}|^2<\frac{n}{2mE_{ik}}. \quad (11)$$

n designates the number of radiating particles (protons) with mass m_p.

Similarly, an upper limit for the quadrupole matrix element

$$Q_{ik}=\sum_{j=1}^{n}\int r_j^2\psi_i\bar{\psi}_k dr$$

may be derived from the sum formula

$$\sum_{i=0}^{k}|Q_{ik}|^2\cdot\frac{m_p}{2\hbar}=|Q_{kk}|^2;$$

namely:

$$|Q_{ik}|^2<\frac{2\hbar}{m_p}\cdot\frac{1}{E_{ik}}|Q_{kk}|^2. \quad (12)$$

In contrast to the dipole formula, this last formula contains itself an average value. Assuming, however, $|\varphi_{kk}|^2\leqq10^{-48}$ and remembering that

$$\Gamma^{(2)}=\frac{4r}{15}\frac{l^2}{\lambda^5}|Q^2| \quad (13)$$

one sees that the values of Table I satisfy our inequality.

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[16] Cf. W. E. Lamb, Jr., Phys. Rev. **55**, 190 (1939). The above formula was recommended to the author by Dr. H. A. Bethe. According to Dr. Lamb (personal communication) it is a fair approximation at the two limits $T\gg\Theta$. Lamb replaces T by $\tilde{\epsilon}(T)$ and (Fig. 1, p. 198) gives a plot of $\tilde{\epsilon}(T)$ against T/Θ. For the above-mentioned limits, one obtains (O is the well-known "order of" symbol):

$$\tilde{\epsilon}\sim T+\Theta\cdot O(\Theta/T);\qquad T\gg\Theta,$$
$$\epsilon\sim\tfrac{3}{8}\Theta+T\cdot O(T^3/\Theta^3);\quad T\ll\Theta.$$

Therefore, the replacement of T by $(T+\frac{3}{8}\Theta)$ instead of $(T+\frac{1}{2}\Theta)$ is somewhat better.

[17] For an α-particle moving in a potential well, a formula of this type (with $\hbar^{-1}B(2j_i+1)/(2j_k+1)\to8/137$) was given by Hebb and Uhlenbeck, reference 7.