On an Asymmetrical Metric in the Four-Space of General Relativity

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Physical space-time allows of a metric of asymmetrical properties because of the unidirection of time. The simplest possible asymmetrical generalization of Riemannian metric is considered. The study of this metric is here mainly confined to the mathematical aspects. However, the physical consequences by application to space-time are obvious, and may be of interest by leading directly to a description of the electromagnetic field. A close formal connection with five-dimensional Riemannian geometry is shown, which might carry interest for the physical interpretation of five-dimensional relativity.

1. INTRODUCTION

IN the geometry of affinely connected spaces the metrical and the affine properties are completely independent. The characteristics of the space described by the curvature tensor are derivable from the definition of parallel displacement, or covariant derivation. Only when we want to compare vectors of different directions, the need for a metric arises. We are, however, quite free in the choice of metric. That is, we can choose at will measuring units in every direction at each point of the space. The locus of the end points of all the unit lengths radiating from a certain point x_0, is called the indicatrix. In Euclidian geometry this locus is a (hyper-) sphere around x_0. In Riemannian geometry the indicatrix is a quadratic surface around x_0, with coefficients equal to the fundamental tensor $g_{\mu\nu}$, which already exists in each point by the definition of the Riemannian parallel displacement.

There would be nothing to prevent the choice of another tensor, $h_{\mu\nu}$, to define the metric. As far as the application to physical space-time goes, however, there is no indication of the need for a new tensor, because there is no need for twenty independent potentials. However, the Riemannian metric has one property which does not seem quite appropriate for the application to physical space-time, and that is the perfect symmetry between opposite directions for any coordinate interval. Perhaps the most characteristic property of the physical world is the unidirection of time-like intervals. Since there is no obvious reason why this asymmetry should disappear in the mathematical description it is of interest to consider the possibility of a metric with asymmetrical properties.

2. THE ECCENTRIC METRIC

It is known that many reasons speak for the necessity of a quadratic indicatrix.[1] The only way of introducing an asymmetry while retaining the quadratic indicatrix, is to displace the center of the indicatrix. In other words, we adopt as indicatrix an eccentric quadratic (hyper-) surface. This involves the definition of a vector at each point of the space, determining the displacement of the center of the indicatrix. The formula for the length ds of a line-element dx^μ must necessarily be homogeneous of first degree in dx^μ. The simplest "eccentric" line-element possessing this property, and of course being invariant, is

$$ds = k_\mu dx^\mu + (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}, \tag{1}$$

where $g_{\mu\nu}$ is the fundamental tensor of the Riemannian affine connection, and k_μ is a covariant vector determining the displacement of the center of the indicatrix. If a space of Riemannian affine connection is given, we are, as mentioned earlier, completely free in our choice of metric, and consequently free in our choice of the vector k_μ. This vector, therefore, does not describe any properties of the Riemannian space considered, but only the properties of the units chosen for measuring intervals. To change from one vector field k_μ to another, only means to change from one system of asymmetrical units to another. By each of the measuring systems

[1] H. Weyl, "Die Einzigartigkeit der Pythagoreischen Massbestimmung," Math. Zeits. 12, 114–146.

we can determine certain paths in the space, defined by the condition

$$\delta \int ds = 0. \tag{2}$$

We may now divide the variety of arbitrary vector fields k_μ into classes giving the same paths (2). If we only allow changes of units within each class, the vector k_μ has attained a certain significance besides describing the units chosen, namely by defining a set of paths. The only change of k_μ which will not affect the path defined by (2) is the addition of an arbitrary gradient vector,

$$k_\mu' = k_\mu + \partial\phi/\partial x^\mu. \tag{3}$$

The change (3) will according to (1) result in the addition of a total differential $d\phi$ to ds. Hence

$$ds' = ds + (\partial\phi/\partial x^\mu)dx^\mu \tag{4}$$

and this addition does not affect Eq. (2). The change of units corresponding to the transformation (3) will be called a k transformation. The fundamental difference from the gauge-transformation of Weyl should be noticed. Weyl's transformation is a change of units at different points, while the k transformation is a change of units in different directions at the same point.

3. The Asymmetry of Time-Like Intervals

The quadratic form with coefficients $g_{\mu\nu}$ is indefinite. When the invariant

$$g_{\mu\nu}dx^\mu dx^\nu \tag{5}$$

is positive, the interval is "time-like," when negative, it is "space-like." For space-like intervals the square root in (1) becomes imaginary, and ds consequently complex. But the results of measurements are real numbers. We, therefore, assume that the result of measurement is given by the absolute value $|ds|$. Time-like intervals are purely real, so that we have simply

$$(\pm)|ds| = k_\mu dx^\mu + (g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}}. \tag{6}$$

Space-like intervals are complex, the square root being the imaginary part. The absolute value becomes

$$|ds| = [(k_\mu k_\nu - g_{\mu\nu})dx^\mu dx^\nu]^{\frac{1}{2}}. \tag{7}$$

It follows that a reversal of sign of a space-like interval dx^μ does not affect the measured length, because the form (7) is purely quadratic. For time-like intervals this is different. The linear term $k_\mu dx^\mu$ will change its sign with dx^μ, and we have different values of $|ds|$ for opposite directions.

It might seem as though our introduction of an asymmetrical line-element has also introduced an absolute preferred direction, at each point, defined by the vector k_μ. However, this preference arises only as a result of the special system of units chosen. We can change k_μ by changing to new units by a k transformation, without affecting the properties of space, or even of the paths of extreme length as measured by our system. The direction of k_μ is, therefore, not determined before we have specified the units we want to use. And as these units are a matter of convention, there is obviously no absolute direction given.

If k_μ is a given time-like vector, we can, of course, transform our coordinates so as to make the time-coordinate go in the k_μ direction. In this coordinate system k_μ has no components perpendicular to the time-axis, and measurements perpendicular to the time-axis will by (7) give the usual Riemannian length

$$|ds| = (g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}}. \tag{8}$$

However, since quite generally the asymmetry does only affect time-measurements, there is no reason for considering this system of coordinates as unique.

4. The Paths of Extreme Length

We shall study the type of extreme paths resulting from our choice of measuring units, as defined by the eccentric line-element (1). The equation for the path is

$$\delta \int [k_\mu dx^\mu + (g_{\mu\nu}dx^\mu d^\nu)^{\frac{1}{2}}] = 0. \tag{9}$$

We consider a path of time-like direction, i.e., ds is real. Now choose a parameter τ, and denote differentiation with respect to τ by a dot:

$$\delta \int [k_\mu \dot{x}^\mu + (g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu)^{\frac{1}{2}}]d\tau = 0.$$

The equivalent Eulerian equations are

$$\frac{d}{d\tau}\frac{\partial}{\partial \dot{x}^\nu}[k_\mu \dot{x}^\mu + (g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{\frac{1}{2}}] - \frac{\partial}{\partial x^\nu}[k_\mu \dot{x}^\mu + (g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{\frac{1}{2}}] = 0,$$

or

$$\frac{d}{d\tau}\left(k_\nu + \frac{1}{2}\frac{(g_{\mu\nu}+g_{\nu\mu})\dot{x}^\mu}{(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{\frac{1}{2}}}\right) - \frac{\partial k_\mu}{\partial x^\nu}\dot{x}^\mu - \frac{1}{2}\frac{\dot{x}^\mu\dot{x}^\alpha \partial g_{\mu\alpha}/\partial x^\nu}{(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{\frac{1}{2}}} = 0. \quad (10)$$

We now specify the arbitrary parameter by putting

$$d\tau = (g_{\alpha\beta}dx^\alpha dx^\beta)^{\frac{1}{2}}, \quad (11)$$

which makes the square root $(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{\frac{1}{2}}$ equal to unity. We can do so without inconsistency because the square root mentioned is an integral of the resulting Eq. (14).[2] Performing the differentiation in (10), and changing some summation indices, we get

$$g_{\mu\nu}\ddot{x}^\mu + \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\alpha + \frac{\partial g_{\alpha\nu}}{\partial x^\mu}\dot{x}^\mu\dot{x}^\alpha - \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\dot{x}^\mu\dot{x}^\alpha\right) + \left(\frac{\partial k_\nu}{\partial x^\alpha} - \frac{\partial k_\alpha}{\partial x^\nu}\right)\dot{x}^\alpha = 0.$$

Using the familiar notations

$$\{\mu\alpha, \sigma\} = g^{\sigma\nu}[\mu\alpha, \nu] = g^{\sigma\nu}\cdot\frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\right) \quad (12)$$

$$F^\sigma{}_\alpha = g^{\sigma\nu}F_{\nu\alpha} = g^{\sigma\nu}\left(\frac{\partial k_\nu}{\partial x^\alpha} - \frac{\partial k_\alpha}{\partial x^\nu}\right) \quad (13)$$

we obtain after multiplication by $g^{\nu\sigma}$

$$\ddot{x}^\sigma + \{\mu\alpha, \sigma\}\dot{x}^\mu\dot{x}^\alpha + F^\sigma{}_\alpha\dot{x}^\alpha = 0. \quad (14)$$

As we expected, Eq. (14) is invariant for k transformations. If k_μ is a gradient vector, the path reduces to the Riemannian geodesic.

It will be noted that starting a path (14) with initial conditions given in the form of "velocities" $(dx^1/dx^4, dx^2/dx^4, dx^3/dx^4, 1)$, there are two paths possible. This is because a reversal of the signs of all dx^μ does not affect the "velocities," while the effect on Eq. (14) is to reverse the sign of the term $F^\sigma{}_\alpha\dot{x}^\alpha$.

5. Connection with Five-Dimensional Riemannian Geometry

Suppose we assign to a space not only the affine properties, but also the property of possessing certain paths of extreme length. If there exist laws describing these properties, it is obvious that such laws must be covariant for coordinate transformations as well as for k transformations. In order to treat both kinds of transformations under one, we can proceed in the following way:

By the formula for the eccentric line-element the length of a vector A^μ is

$$A = k_\mu A^\mu + (g_{\mu\nu}A^\mu A^\nu)^{\frac{1}{2}}. \quad (15)$$

Moving $k_\mu A^\mu$ to the left-hand side and squaring the equation gives

$$(g_{\mu\nu} - k_\mu k_\nu)A^\mu A^\nu + 2k_\mu A^\mu A - A^2 = 0. \quad (16)$$

Consider now a 5-dimensional Riemannian space with fundamental tensor γ_{mn}, $(m, n = 1, 2, 3, 4, 5;$ $\mu, \nu = 1, 2, 3, 4)$

$$\gamma_{\mu\nu} = g_{\mu\nu} - k_\mu k_\nu;\ \gamma_{\mu 5} = \gamma_{5\mu} = k_\mu;\ \gamma_{55} = -1. \quad (17)$$

It follows immediately from (16) that a 5-vector in the space (17) with components A^1, A^2, A^3, A^4, and A, is a zero-length 5-vector. In other words, the fifth component A^5 of a null-vector in the 5-space (17) represents the measured length of the 4-vector A^μ composed of the first four components of the 5-vector. This property makes the 5-space (17) useful for expressing the metrical properties of our 4 space. It is obvious that any 5-space conformal with (17) would do, since it is only the ratio of the coefficients that count. However, the affine properties of our Riemannian 4-space are determined by $g_{\mu\nu}$, and not by this quantity multiplied by an arbitrary function. It is therefore necessary to specify the tensor γ_{mn} to the values (17) if we want to describe the properties of the 4-space by the properties of the corresponding 5-space. It should be remarked, however that a constant factor of proportionality may always be introduced in (17), because such

[2] Multiplying (14) by $g_{\sigma\beta}\dot{x}^\beta$ gives

$$g_{\sigma\beta}\dot{x}^\beta\frac{d}{d\tau}\dot{x}^\sigma + \tfrac{1}{2}\left(\frac{d}{d\tau}g_{\sigma\beta}\right)\dot{x}^\sigma\dot{x}^\beta = 0$$

or $g_{\sigma\beta}\dot{x}^\sigma\dot{x}^\beta = \text{const.}$

a factor does not affect the affine properties of space.

A 5-dimensional space of the type (17) has been used frequently for representing gravitational and electromagnetic laws by Riemannian geometry,[3] and also for introducing the quantum concept into the geometrical picture. The lack of physical interpretation of the fifth coordinate seems to be the main obstacle to the theory.

If the length A is denoted by A^5, the formula for the length of A^μ can be written

$$\gamma_{mn}A^m A^n = 0. \tag{18}$$

Similarly for the interval length

$$\gamma_{mn}dx^m dx^n = 0, \tag{19}$$

where $dx^5 = ds$. It must, however, be remembered that the interpretation $dx^5 = ds$ is only valid for null-directions and cannot be integrated to give $x^5 = s$. This is obvious because ds is not a perfect differential.

It is easily realized that a coordinate transformation in 5-space of the form

$$x^{\mu\prime} = x^{\mu\prime}(x^1, x^2, x^3, x^4); \quad x^{5\prime} = x^5 \tag{20}$$

has the same effect on the fundamental quantities $g_{\mu\nu}$, k_μ, and on 4-vectors A^μ as the corresponding coordinate transformation in 4-space. The length A^5 is left unchanged. Similarly it is realized that a 5-coordinate transformation

$$x^{\mu\prime} = x^\mu; \quad x^{5\prime} = x^5 + \phi(x^1, x^2, x^3, x^4) \tag{21}$$

has the same effect as a k transformation. By combining (20) and (21) we can therefore treat k transformations and coordinate-transformations simultaneously as a single coordinate-transformation in 5-space. This transformation leaves the "cylindricity" of the space (17) unaffected.

Because of the one-to-one correspondence between the 4-space of $g_{\mu\nu}$ and k_μ, and the Riemannian 5-space of γ_{mn}, the properties of one are determined by the properties of the other, and we can study the combined metrical and affine properties of the four space by studying the affine properties of the Riemannian 5-space. Instead of looking for equations in $g_{\mu\nu}$ and k_μ, being covariant for k and coordinate-transformations, we can use 5-dimensional tensor equations involving γ_{mn}.

It is easy to show that the x^μ-projection of a Riemannian null-geodesic in the 5-space (17) is identical with the extreme path (14) in our 4-space of eccentrical metric.

The role played by the Riemannian 5-space can now be described shortly as follows: By means of the fundamental quantities $g_{\mu\nu}$ and k_μ of a space of Riemannian affine connection and eccentric metric we can construct a 5-dimensional Riemannian space (17). There is a direct correspondence between simultaneous coordinate- and k transformations in 4-space and simple coordinate-transformations in 5-space. Only null-directions in 5-space have a correspondence with 4-space, and the fifth component of a null-vector or line-element represents the measured length of the vector. The advantage of using the 5-dimensional method of representation arises from the fact that tensor equations in the 5-space represent, in 4-space, equations that are covariant for k transformations as well as coordinate-transformations.

The 5-space is introduced merely as a mathematical auxiliary. The fifth coordinate does only enter by its differential, and its interpretation is then clear: it is the measured interval length of dx^μ. The transformation of the fifth coordinate represents the change to new measuring units in 4-space, giving the same extreme paths.

6. "Physical" Space

The *Einstein-Ricci* tensor of the 5-space (17), expressed in the quantities $g_{\mu\nu}$ and k_μ have components of the form[4]

$$G^{\mu\nu} = (R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R) + \tfrac{1}{2}E^{\mu\nu}; \tag{22}$$

$$G_5{}^\nu = \tfrac{1}{2}F^{\nu\alpha}{}_{;\alpha} \tag{23}$$

$$G_{55} = \tfrac{1}{2}(R - \tfrac{3}{4}F^{\alpha\beta}F_{\alpha\beta}). \tag{24}$$

Here $E^{\mu\nu} = -F^{\mu\alpha}F_\alpha{}^\nu + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$, and $R^{\mu\nu}$ is the 4-dimensional *curvature* tensor. The quantities G^{mn} can be calculated if one knows the fundamental quantities $g_{\mu\nu}$ and k_μ.

[3] Originally by T. Kaluza, Preuss. Akad. Wiss. **54**, 966 (1921); Also O. Klein, Zeits. f. Physik **46**, 188 (1927); L. Rosenfeld, Bull. Akad. Roy. Belg. **13**, 304 (1927). For further references see O. Veblen, *Projective Relativitätstheorie* (Springer, 1933).

[4] See L. Rosenfeld, reference 3, p. 310.

We recall the fact that only null-vectors in 5-space have an immediate significance in terms of measurements in 4-space. Multiplying G_l^m by an arbitrary null-vector A^l gives a new 5-vector

$$B^m = G_l^m A^l. \tag{25}$$

We will now demand that B^m has also an interpretation in 4-space, as a vector and its length, which means that B^m must be a null-vector. This gives the condition

$$\gamma_{mn}(G_l^m A^l)(G_q^n A^q) = 0$$

or

$$(\gamma_{mn} G_l^m G_q^n) A^l A^q = 0.$$

But A^l is an arbitrary null-vector,

$$\gamma_{lq} A^l A^q = 0.$$

Consequently

$$G_{nl} G_q^n = \theta \gamma_{lq} \tag{26}$$

where θ is a function of the coordinates. Equation (26) gives the condition to be satisfied by the fundamental quantities $g_{\mu\nu}$ and k_μ in our space. There are fifteen equations (symmetrical 5-tensor of second rank), and fifteen functions, $g_{\mu\nu}$, k_μ and θ. We shall consider some examples of such spaces.

(1) One possible way to satisfy (26) would obviously be to put

$$G^{mn} = \lambda \gamma^{mn}. \tag{27}$$

Here $\lambda = \theta^{\frac{1}{2}}$ must be a numerical constant because the divergence of G^{mn} is known to vanish identically. Expressed in 4-dimensional quantities (27) becomes

$$(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) + \tfrac{1}{2} E^{\mu\nu} = \lambda g^{\mu\nu}; \quad F^{\nu\alpha}{}_{;\alpha} = 0. \qquad (28), (29)$$

Contracting (28) gives $-R = 4\lambda$, which together with the (5,5)-component of (27) becomes

$$\lambda = -\tfrac{3}{8} F^{\alpha\beta} F_{\alpha\beta}. \tag{30}$$

If $\lambda = 0$ the above equations are those of Einstein and of Maxwell for space containing plane electromagnetic waves only. If $F^{\alpha\beta}$ is also zero, (28) is Einstein's law for empty space.

(2) Writing (26) in the form

$$G_n^l G_q^n = \delta_q^l \cdot \theta$$

it is obvious that this equation can also be satisfied by the following system, in which $\theta^{\frac{1}{2}}$ need not be constant:

$$\begin{aligned} &G_1^1 = \theta^{\frac{1}{2}}, \ \text{(a)} && G_4^4 = \theta^{\frac{1}{2}}, \ \text{(d)} \\ &G_2^2 = -\theta^{\frac{1}{2}}, \ \text{(b)} && \\ &G_3^3 = -\theta^{\frac{1}{2}}, \ \text{(c)} && G_{55} = \pm\theta^{\frac{1}{2}}, \ \text{(e)} \end{aligned} \tag{31}$$

while $G_{mn} = G^{mn} = G_n^m = 0$ when $m \neq n$. Putting $E_1^1 = -E_2^2 = -E_3^3 = E_4^4 = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} = \frac{1}{2} F^{14} F_{14}$ as in a static symmetrical field with $k_4 = \epsilon / r$ the equation (e) gives

$$3E_1^1/2 = \pm\theta^{\frac{1}{2}}$$

and (31) becomes

$$R_1^1 = R_4^4 = \begin{cases} -2E_1^1 \\ +\ E_1^1 \end{cases}, \quad R_2^2 = R_3^3 = \begin{cases} +2E_1^1 \\ -\ E_1^1 \end{cases}, \tag{32}$$

which gives solutions of the type known as Nordström's solution[5] for the field of a charged particle. There appear to be two possibilities, one in which the gravitational effect of the charge counteracts the term m/r of the Schwarzschild line-element, another where the two act in the same direction.

(3) The static, symmetrical field

$$ds^2 = -\frac{dr^2}{1 + \alpha/r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2 \tag{33}$$

where k_μ vanishes identically is also a possible solution of (26). It is a solution of the form (31) with

$$G_1^1 = -G_2^2 = -G_3^3 = -G_4^4 = -G_5^5 = \theta^{\frac{1}{2}} = \alpha/r^4.$$

If interpreted as a particle-solution, (33) represents a particle without any first-order interaction with other particles (because $g_{44} = \text{const.}$). A "neutrino" field might be the nearest suggestion.

The *fundamental equation* (26) is seen to be a generalization of Einstein's equation for empty space. Besides the well-known particle solutions of general relativity (26) appears to be satisfied by the field of an electromagnetic wave and by the "neutrino" field (33).

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[5] See A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge University Press, 1924), second edition, p. 185