Perturbation of Boundary Conditions

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(Received November 18, 1940)

The perturbation method of solving boundary value problems with irregular boundary conditions has been reexamined and generalized. Problems are considered for which the boundary of the region is changed and the boundary conditions are unaltered, and also the inverse case where the boundary is fixed and the boundary conditions are changed. It is pointed out that the integrations involved in obtaining the perturbed eigenvalues will be in general very difficult, but that this difficulty can be avoided in two important cases. The perturbed eigenfunctions, however, can always be simply obtained. An application of the method was made to a problem in room acoustics which requires the calculation of second-order terms in the perturbation, and the results obtained for the pressure distribution and frequency were found to check well with experiment. A simple formula is found for the problem of the absorbing wall. The method is useful in calculating electromagnetic resonators of the "rhumbatron" type, and may also be useful in calculating electronic wave functions for metals. In Appendix III the method is applied to the scattering problem, and the results are related to Huygens' principle.

INTRODUCTION

MANY physical problems reduce to finding the eigenvalues and eigenfunctions of an equation of the type $H\phi+K\phi=0$ where H is some differential operator. It often happens that solutions are desired under circumstances differing slightly from conditions for which the solutions are well known. This perturbation can be introduced into the problem in two ways. First, small additional terms may enter the differential operator, whereupon the usual methods of perturbation theory are employed. Secondly, it may happen that the boundary conditions for the eigenfunctions are slightly different than in the case of the problem whose solutions are known. Problems of the second type arise in the Wigner-Seitz theory of metals, in problems in room acoustics, in electromagnetic radiation problems and in certain problems in quantum mechanics. In many cases problems of the second type can be reduced to the first kind as has been shown by Brillouin[1] and Cabrerra.[2] A method of obtaining first-order corrections to the unperturbed eigenvalues for problems of the second type has been indicated by Froelich,[3] but this method is not essentially different from that of Brillouin and Cabrerra (Appendix I). The method that is to be employed in this paper is nearly the same as that of Brillouin and Cabrerra. It is desired to bring out various aspects of the problem that have not been considered, and to show how solutions are actually to be obtained in certain cases.

METHOD

Let x represent some set of coordinates x_1, x_2, x_3, and let S_0 be a surface enclosing a region R_0. Suppose that the eigenvalue equation

$$H^0(x,\ \partial/\partial x)\phi_n{}^0(x)+K_n{}^0\phi_n{}^0(x)=0 \qquad (1)$$

has been solved with the $\phi_n{}^0$ satisfying boundary conditions I on S_0. Suppose that the boundary conditions are such that the solutions are orthonormal. Two cases will be considered in this paper: case 1, $\phi_n{}^0=0$ on S_0; case 2, $\partial\phi_n{}^0/\partial n=0$ on S_0. Let S be a surface enclosing a region R. It is required to find the solutions of the equation

$$H^0(x,\ \partial/\partial x)\psi_n(x)+K_n\psi_n(x)=0 \qquad (2)$$

with ψ_n satisfying boundary conditions II on S. It will be supposed that ψ_n does not differ very much from $\phi_n{}^0$. Let T be some operator, and define a function ϕ_n by $\phi_n=T\psi_n$. T must be such an operator that ϕ_n satisfies boundary conditions I on S_0.[4] Suppose that T may be written as $1+\sigma+\rho+\cdots$ where σ is a quantity small in first order and ρ is a quantity small in second order. The equation satisfied by ϕ_n is

$$(H^0+H^1+H^2+\cdots)\phi_n+K_n\phi_n=0, \qquad (3)$$

[1] L. Brillouin, Comptes rendus **204**, 1863 (1937).
[2] N. Cabrerra, Comptes rendus **207**, 1175 (1938).
[3] H. Froelich, Phys. Rev. **54**, 945 (1938).

[4] Brillouin and Cabrerra introduce an operator P such that $P\phi_n{}^0$ satisfies boundary conditions II on S. This is a more stringent condition than is necessary.

where

$$H^1=\sigma H^0-H^0\sigma, \tag{4}$$

$$H^2=(\rho-\sigma^2)H^0-H^0(\rho-\sigma^2)+\sigma H^1. \tag{5}$$

The problem has thus been reduced to one in which ordinary perturbation methods may be applied. One obtains the formulas

$$K_n=K_n{}^0-\overset{1}{H}_{nn}-\overset{2}{H}_{nn}+\sum_k{}'\frac{\overset{1}{H}_{nk}\overset{1}{H}_{kn}}{K_n{}^0-K_k{}^0}+\cdots, \tag{6}$$

$$\phi_n=\phi_n{}^0+\sum_k{}'\frac{\overset{1}{H}_{kn}}{K_k{}^0-K_n{}^0}\phi_k{}^0+\cdots. \tag{7}$$

The quantities $\overset{1}{H}_{kn}$ and $\overset{2}{H}_{nn}$ are given by

$$\overset{1}{H}_{kn}=\int_{R_0}\overset{*}{\phi}_k{}^0H^1\phi_n{}^0dV, \tag{8}$$

$$\overset{2}{H}_{nn}=\int_{R_0}\overset{*}{\phi}_n{}^0H^2\phi_n{}^0dV. \tag{9}$$

The prime on the summation sign indicates omission of the term with k equal to n. For these perturbation formulas to be valid, it is necessary that H^1 and H^2 do not introduce singularities into the problem. For ψ_n one may write to first order,

$$\psi_n=\phi_n{}^0-\sigma\phi_n{}^0+\sum_k{}'\frac{\overset{1}{H}_{kn}}{K_k{}^0-K_n{}^0}\phi_k{}^0. \tag{10}$$

It will now be assumed that H^0 is given by $H^0=\nabla^2+V(x)$. One then obtains the following results:

$$\overset{1}{H}_{kn}=(K_k{}^0-K_n{}^0)\sigma_{kn}-A_{kn}, \tag{11}$$

$$\overset{2}{H}_{nn}=(\sigma H^1)_{nn}-B_{nn}, \tag{12}$$

where

$$A_{kn}=\int_{S_0}\{\overset{*}{\phi}_k{}^0(\partial/\partial n)(\sigma\phi_n{}^0)-\partial\overset{*}{\phi}_k{}^0/\partial n(\sigma\phi_n{}^0)\}dS, \tag{13}$$

$$B_{nn}=\int_{S_0}\{\overset{*}{\phi}_n{}^0(\partial/\partial n)(\rho-\sigma^2)\phi_n{}^0-\partial\overset{*}{\phi}_n{}^0/\partial n(\rho-\sigma^2)\phi_n{}^0\}dS. \tag{14}$$

For ψ_n there can now be written,

$$\psi_n=(1-\sigma_{nn})\left[\phi_n{}^0+\{\sum_k\sigma_{kn}\phi_k{}^0-\sigma\phi_n{}^0\}-\sum_k{}'\frac{A_{kn}}{K_k{}^0-K_n{}^0}\phi_k{}^0\right]. \tag{15}$$

The expression in curly brackets needs some discussion. The sum $\sum_k\sigma_{kn}\phi_k{}^0$ is the expansion of $\sigma\phi_n{}^0$ and will be equal to it inside S_0. Aside from a normalization factor, ψ_n there depends only upon the surface integrals A_{kn}, and this means that ψ_n can always be easily obtained. This would not be the case, as will be seen later, if ψ_n really depended upon $\overset{1}{H}_{kn}$. In general, $\sigma\phi_n{}^0$ does not obey the same boundary conditions as $\phi_n{}^0$. If it did, A_{kn} would vanish, and there could be no first-order change of the wave pattern. This is seen from the fact that a necessary condition for the orthogonality of the $\phi_n{}^0$ is the vanishing of the integral

$$\int_{S_0}[\overset{*}{\phi}_k{}^0(\partial\phi_n{}^0/\partial n)-(\partial\overset{*}{\phi}_k{}^0/\partial n)\phi_n{}^0]dS. \tag{16}$$

Therefore, it is not necessarily true that the sum equals $\sigma\phi_n{}^0$ on S_0. If $\phi_n{}^0$ is zero on S_0, the expression in curly brackets will be discontinuous on S_0, while if $\partial\phi_n{}^0/\partial n=0$ on S_0, it will show merely a discontinuity in slope. However, ψ_n will not show any discontinuity. This discontinuity on S_0 presents difficulties in obtaining the eigenvalues in certain cases and will be mentioned again in discussing the eigenvalue formulas.

The eigenvalue (6) may be written

$$K_n=K_n{}^0+A_{nn}+B_{nn}+\sigma_{nn}A_{nn}+[\sum_k\sigma_{nk}\overset{1}{H}_{kn}-(\sigma H^1)_{nn}]+\sum_k{}'\frac{\overset{1}{H}_{kn}A_{nk}}{K_k{}^0-K_n{}^0}.$$

One can usually represent σ by a first-order differential operator. In this case, it is possible to show that

$$(\sigma H^1)_{nn}=\sum_k\sigma_{nk}\overset{1}{H}_{kn}.$$

This formula does not follow obviously from the usual rules of matrix multiplication, because the conditions for application of these rules are not always fulfilled in the present case. Equation (11) is an example of their failure. One can now write,

$$K_n = K_n^0 + A_{nn} + B_{nn} + \sigma_{nn} A_{nn} + \sum_k{}' \frac{\overset{1}{H}_{kn} A_{nk}}{K_k^0 - K_n^0}. \quad (17)$$

Transformation Operators

Let $X_i(x)$ be defined so that, if x_i represents the coordinates of a point on S_0, $x_i + X_i(x)$ gives the coordinates of a point on S. An operator R is defined by

$$Rf(x) = f(x+X). \quad (18)$$

If one writes $R = 1 + \mu + \nu + \cdots$, then one has

$$\mu = \sum_i X_i \partial/\partial x_i, \quad \nu = \tfrac{1}{2} \sum_{i,j} X_i X_j (\partial^2/\partial x_i \partial x_j). \quad (19)$$

Case 1. Suppose that $\phi_n = 0$ on S_0 and $\psi_n = F(x)$ on S, where F is a small quantity. Then one has,

$$\begin{aligned} \psi_n(x) &= F(x) \quad x \text{ on } S, \\ R\psi_n(x) &= RF(x) \quad x \text{ on } S_0, \\ RT^{-1}\phi_n(x) &= RF(x) \quad x \text{ on } S_0. \end{aligned} \quad (20)$$

Placing $\phi_n = \phi_n^0 + \phi_n^1 + \phi_n^2 + \cdots$ and separating into orders one finds,

first order: $\sigma\phi_n^0 = \mu\phi_n^0 - F$ (21)

second order: $\sigma\phi_n^1 + (\rho - \sigma^2)\phi_n^0 = \mu\phi_n^1 + \nu\phi_n^0 - \mu(\sigma\phi_n^0 + F)$. (22)

Some special cases are worth mentioning:

Case 1a. If F is zero, it follows at once from (20) that a solution for T is $T = R$.

Case 1b. If S coincides with S_0, $R = 1$ and the equations become,

$$\sigma\phi_n^0 = -F, \quad (21a)$$

$$\sigma\phi_n^1 + (\rho - \sigma^2)\phi_n^0 = 0. \quad (22a)$$

Case 2. Suppose that $\partial\phi_n/\partial n = 0$ on S_0, and $\partial\psi_n/\partial n = F$ on S, where F is again a small quantity. Let the surface S be determined by the equation $S(x) = 0$ for x on S. Then one has that $S(x+X) = 0$ for x on S_0. Hence the function $S_0(x)$ determining S_0 may be defined by $S_0(x) = RS(x)$. Thus one has

$$\begin{aligned} S &= S_0 - \mu S_0 + (\mu^2 - \nu) S_0 + \cdots\cdots, \\ \nabla S &= \nabla S_0 - \nabla \mu S_0 + \nabla(\mu^2 - \nu) S_0 + \cdots\cdots. \end{aligned}$$

If $S(x)$ is given the proper sign, one has

$$\begin{aligned} \nabla S \cdot \nabla \psi_n &= |\nabla S| F \quad x \text{ on } S, \\ R(\nabla S \cdot \nabla T^{-1}\phi_n) &= R(|\nabla S| F) \quad x \text{ on } S_0. \end{aligned} \quad (23)$$

This last condition can be worked out in general as before, but the equations are very complicated. Some special cases are of more interest:

Case 2a. If F is zero, the conditions can usually be satisfied as before with $T = R$. Under these circumstances, the conditions become,

first order: $\nabla S_0 \cdot [\mu\nabla]\phi_n^0 + \nabla\phi_n^0 \cdot [\mu\nabla] S_0 = 0$; (24)

second order: $\nabla S_0 \cdot [\mu\nabla]\phi_n^1 + \nabla\phi_n^1 \cdot [\mu\nabla] S_0 + \nabla S_0 \cdot [\nu\nabla]\phi_n^0 + \nabla\phi_n^0 \cdot [\nu\nabla] S_0 - \nabla S_0 \cdot [\mu\nabla]\mu\phi_n^0 - \nabla\phi_n^0 \cdot [\mu\nabla]\mu S_0 + [\mu\nabla] S_0 \cdot [\mu\nabla]\phi_n^0 = 0$, (25)

where $[\mu\nabla]$ is a symbol for $(\mu\nabla - \nabla\mu)$. In a particular two-dimensional case that has been worked out, these somewhat complicated conditions simply demanded that the X_i satisfy Cauchy conditions on S_0. This could be anticipated.

Case 2b. If S coincides with S_0, the conditions become,

$$(\partial/\partial n)(\sigma\phi_n^0) = -F, \quad (26)$$

$$(\partial/\partial n)(\sigma\phi_n^1) + (\partial/\partial n)(\rho - \sigma^2)\phi_n^0 = 0. \quad (27)$$

Case 3. Suppose that $\partial\phi_n/\partial n = 0$ on S_0, and $\partial\psi_n/\partial n = F\psi_n$ on S_0. Here F is again a small quantity. In this case the conditions are

$$(\partial/\partial n)(\sigma\phi_n^0) = -F\phi_n^0, \quad (28)$$

$$(\partial/\partial n)(\sigma\phi_n^1) + (\partial/\partial n)(\rho - \sigma^2)\phi_n^0 = -F\phi_n^1 + F\sigma\phi_n^0. \quad (29)$$

The operators σ and ρ that are determined by these conditions should really carry another index that has been suppressed for simplicity. This index would indicate to which solution ϕ_n the σ was appropriate. Thus, the perturbation introduced into the equation for ϕ_n may be different from that introduced into the equation for ϕ_k.

Further Discussion of the Eigenvalues

In the previous section there have been worked out the conditions that must be fulfilled by σ and ρ in various cases. The notable fact about these conditions is that they must hold only upon the surface S_0. Nevertheless, in the eigenvalue formulas there enter matrix components that involve integrals of σ over the region R_0. If σ is not specified in any way inside R_0, it would seem that any answer at all could be obtained by a suitable choice of σ. This certainly cannot be the case, for in any particular problem there will be introduced some parameter λ measuring the magnitude of the perturbation, and the eigenvalue will be a power expansion in this parameter. This expansion must be unique. Thus one comes to the conclusion that the results for each order must be independent of the volume behavior of σ, as long as it does not introduce singularities that will cause the expansions in the orthogonal set $\phi_n{}^0$ to diverge. To find a σ satisfying all the boundary conditions and of sufficient continuity may be, in all but extremely simple cases, very difficult. Even if σ has been found, the matrix components σ_{kn} may be impossible to obtain except by numerical integration. One is led to suspect that, as in the case of the eigenfunctions, the eigenvalues can always be obtained in terms of surface integrals and thus made independent of the volume behavior of σ. It has not been found possible to do this in general. The difficulty is illustrated by the cases 1b and 2b for which conditions were deduced in the last section. The eigenvalues may be written,

$$\text{case 1b:}\quad K_n=K_n{}^0+\int_{S_0}(\partial\overset{*}{\phi}_n{}^0/\partial n)FdS+\sigma_{nn}\int_{S_0}(\partial\overset{*}{\phi}_n{}^0/\partial n)FdS;$$

$$\text{case 2b:}\quad K_n=K_n{}^0-\int_{S_0}\overset{*}{\phi}_n{}^0FdS-\sigma_{nn}\int_{S_0}\overset{*}{\phi}_n{}^0FdS,$$

where there is a direct dependence in second order upon σ_{nn}. The value of σ_{nn} must therefore be unique, but it has not been found possible to verify this in any way.

In certain cases, however, the eigenvalue can be written in a form depending only upon surface integrals. For case 3 where $\partial\phi_n/\partial n=0$ on S_0 and $\partial\psi_n/\partial n=F\psi_n$ also on S_0, the eigenvalue formula becomes

$$K_n=K_n{}^0+A_{nn}-\sum_k{}'\frac{A_{kn}A_{nk}}{K_k{}^0-K_n{}^0}\tag{30}$$

with

$$A_{kn}=-\int_{S_0}F\overset{*}{\phi}_k{}^0\phi_n{}^0dS.$$

This equation is indeed independent of anything but known quantities and is in a very simple form. This case is of interest because of its application to acoustic problems. The boundary condition $\partial\psi_n/\partial n=F\psi_n$ describes conditions at an absorbing wall, F being a complex function related to the acoustic impedance.

In one other case the eigenvalue formula may be reduced to a form involving only surface integrals. This is for the case 2a. From Eq. (9) in Appendix II, it can be deduced that

$$K_n=K_n{}^0+A_{nn}+\int_{S_0}(\sigma\phi_n{}^0)(\partial/\partial n)(\overset{*}{\sigma}\overset{*}{\phi}_n{}^0)dS-\int_{S_0}(\mathbf{n}\cdot\mathbf{D})\overset{*}{\phi}_n{}^0H^1\phi_n{}^0dS+B_{nn}-A_{nn}\int_{S_0}(\mathbf{n}\cdot\mathbf{D})\overset{*}{\phi}_n{}^0\phi_n{}^0dS-\sum_k{}'\frac{A_{kn}\overset{*}{A}_{kn}}{K_k{}^0-K_n{}^0}.\tag{31}$$

This formula has been applied as described below to calculating a certain problem in room acoustics.

The other very simple case 1a does not yield to this treatment. The eigenvalue expression may be written,

$$K_n=K_n{}^0+A_{nn}+B_{nn}-\int_{S_0}(\partial\overset{*}{\phi}_n{}^0/\partial n)(\sum_k\sigma_{kn}\sigma\phi_k{}^0)dS-\sum_k{}'\frac{A_{kn}A_{nk}}{K_k{}^0-K_n{}^0}.$$

But since $\sigma\phi_n{}^0\neq0$ on S_0, the series $\sum_k\sigma_{kn}\sigma\phi_k{}^0$ will diverge on S_0, and the series

$$\sum_k{}'\frac{A_{kn}A_{nk}}{K_k{}^0-K_n{}^0}$$

must therefore also diverge. Since from Eq. (9) in Appendix II, $A_{nk}=\overset{*}{A}_{kn}$, an expression like (31) cannot be obtained. No other way of writing the eigenvalue has been found.

If it is assumed that the formulas are independent of the volume behavior of σ, a procedure can be given by means of which a closed form for the eigenvalue may be obtained in all cases. It amounts essentially to choosing a particular σ. Suppose that σ is such that one has

$$H^1=\sigma H^0-H^0\sigma=cH^0\tag{32}$$

where c is some constant. It is seen at once that c must be $A_{nn}/K_n{}^0$. Thus one has $\overset{1}{H}_{kn}=-A_{nn}\delta_{kn}$.

The eigenvalue expression then becomes

$$K_n = K_n{}^0 + A_{nn} + B_{nn} + \sigma_{nn} A_{nn} \qquad (33)$$

and the problem is to find σ_{nn}. One has from (32)

$$H^0(\sigma\phi_n{}^0) + K_n{}^0(\sigma\phi_n{}^0) = A_{nn}\phi_n{}^0. \qquad (34)$$

Let $G(x, x')$ be the Green's function for the region R_0 and the equation $H^0\phi + K_n{}^0\phi = 0$. Then,

$$\sigma\phi_n{}^0 = \int_{R_0} G(x, x') A_{nn}\phi_n{}^0 dV' + \int_{S_0} \{\partial G(x, x')/\partial n' [\sigma\phi_n{}^0]_{x'} - G(x, x')[(\partial/\partial n)(\sigma\phi_n{}^0)]_{x'}\} dS' \qquad (35)$$

and

$$\sigma_{nn} = A_{nn} \int_{R_0}\int_{R_0} G(x, x') \overset{*}{\phi}_n{}^0(x)\phi_n{}^0(x') dV dV' + \int_{R_0} \overset{*}{\phi}_n{}^0 dV \int_{S_0} \left\{ \frac{\partial G(x, x')}{\partial n'} [\sigma\phi_n{}^0]_{x'} - G(x, x')[\partial/\partial n(\sigma\phi_n{}^0)]_{x'} \right\} dS'. \qquad (36)$$

The Green's function used must be appropriate to having $\sigma\phi_n{}^0$ or $(\partial/\partial n)(\sigma\phi_n{}^0)$ specified on S_0. The expression obtained for ψ_n is of interest.

$$\psi_n = \phi_n{}^0 - A_{nn} \int_{R_0} G(x, x')\phi_n{}^0(x') dv' + \int_{S_0} \{\partial G(x, x')/\partial n' [\sigma\phi_n{}^0]_{x'} - G(x, x')[\partial/\partial n(\sigma\phi_n{}^0)]_{x'}\} dS'. \qquad (37)$$

An application of formula (31) has been made to a problem in room acoustics. A rectangular box with perfectly reflecting walls was distorted into a trapezoidal cross section, and the distortion of the standing wave pattern and change of frequency computed for a particular mode of vibration. Experiments of this type have been made,[5] and it was possible to compare the results with experimental data. It was found that the pressure distribution was predicted closely, mostly within experimental error. The change of frequency from the unperturbed case was experimentally +4.9 percent while the calculated change was +4.2 percent.[6]

[5] R. H. Bolt, J. Acous. Soc. Am. **11**, 184 (1939).

[6] These calculations will be published in *The Journal of the Acoustical Society of America*.

Conclusion

It has been found that the perturbed eigenfunctions ψ_n can always be found by a surface integration over known quantities. The perturbed eigenvalues, however, as given by Eq. (17), may involve volume integrals of the transformation operator σ, and no general way has been found of eliminating these integrals. In two cases, 2a and 3, however, where the unperturbed eigenfunctions have normal boundary conditions, the eigenvalue can be so expressed as to depend only upon surface integrals and can therefore be easily calculated. Of these two cases, the first has direct application to the Wigner-Seitz theory of metals, to electromagnetic resonators and to acoustic problems, where it has been found to give accurate results. The second case can lead to complex eigenvalues and applies directly to the problem of the slightly absorbing wall. If it is assumed, as seems to be necessarily true, that the eigenvalues are independent of the volume behavior of σ, then for all cases formulae involving a Green's function of the unperturbed region can be obtained for the perturbed eigenfunctions and eigenvalues.

We wish to thank Professor P. M. Morse for reading this paper and for the interest he has taken in it during its preparation.

Appendix I

In this appendix it will be shown that the method employed by Froelich if extended to include second-order terms leads to results identical with the perturbation method. One has obviously,

$$K_n = K_n{}^0 + \frac{\int_{S_0} \{\psi_n \partial \overset{*}{\phi}_n{}^0/\partial n - \overset{*}{\phi}_n{}^0 \partial\psi_n/\partial n\} dS}{\int_{R_0} \psi_n \overset{*}{\phi}_n{}^0 dV}. \qquad (1)$$

Take for ϕ_n the expression

$$\phi_n = \phi_n{}^0 + \sum_k{}' B_{kn}\phi_k{}^0$$

where

$$B_{kn} = \frac{\overset{1}{H}_{kn}}{K_k{}^0 - K_n{}^0} + \text{second-order terms.}$$

Then for ψ_n to second order one obtains

$$\psi_n = \phi_n{}^0 + \sum_k{}' B_{kn}\phi_k{}^0 - \sigma\phi_n{}^0 - \sum_k{}' \frac{\overset{1}{H}_{kn}}{K_k{}^0 - K_n{}^0} \sigma\phi_k{}^0 + (\sigma^2 - \rho)\phi_n{}^0.$$

If this is employed in (1), there is obtained directly Eq. (17).

Appendix II

A formula used in Eq. (31) will be established in this appendix. First, the integral $\int_R f(x)dV$ taken over the region R will be expressed as an integral taken over the region R_0. Consider

$$\int_R f(x)dV=\int_R f(x)J(x)dx_1dx_2dx_3, \quad (1)$$

where J is a quantity dependent upon the coordinate system chosen. A new set of variables x_i' is now introduced by the equations

$$x_i=x_i'+X_i(x'). \quad (2)$$

One now has,

$$\int_R f(x)dV=\int_{R_0} R'(fJ)\frac{\partial(x_1, x_2, x_3)}{\partial(x_1', x_2', x_3')}dx_1'dx_2'dx_3'. \quad (3)$$

If the following quantities are defined,

$$\alpha=\sum_i\frac{\partial X_i}{\partial x_i}, \quad \beta=\tfrac{1}{2}\sum_{i,j}\left(\frac{\partial X_i}{\partial x_i}\frac{\partial X_j}{\partial x_j}-\frac{\partial X_i}{\partial x_j}\frac{\partial X_j}{\partial x_i}\right), \quad (4)$$

the integral may be written, dropping the primes,

$$\int_R f(x)dV=\int_{R_0} R(fJ)(1+\alpha+\beta+\cdots)dx_1dx_2dx_3. \quad (5)$$

If for $f(x)$ one takes $\overset{*}{\psi}_n(x)\psi_k(x)$, the integral (5) becomes on evaluation to first-order terms,

$$\int_R \overset{*}{\psi}_n\psi_k dV=\int_{R_0}\overset{*}{\phi}_n\phi_k dV +\int_{R_0}\sum_i(\partial/\partial x_i)(\overset{*}{\phi}_n{}^0\phi_k{}^0 JX_i)dx_1dx_2dx_3 -\int_{R_0}[\overset{*}{\phi}_n{}^0\sigma\phi_k{}^0+\phi_k{}^0\overset{*}{\sigma}\overset{*}{\phi}_n{}^0]dV. \quad (6)$$

From (6) there are obtained two results, one for $k\neq n$ and one for $k=n$. To write these, a vector $\mathbf{D}$ is defined by the equation

$$\mathbf{D}=\sum_i(\partial\mathbf{r}/\partial x_i)X_i, \quad (7)$$

where $\mathbf{r}$ is the radius vector. In terms of $\mathbf{D}$ one has,

$$\sigma=\mathbf{D}\cdot\nabla, \quad (1/J)\sum_i(\partial/\partial x_i)(JX_i)=\nabla\cdot\mathbf{D}. \quad (8)$$

Except for second-order terms, $\mathbf{D}$ is the vector displacement corresponding to the increments X_i. For $k\neq n$, one obtains, using Eq. (11),

$$\int_R\overset{*}{\psi}_n\psi_k dV=\left(\frac{A_{nk}-\overset{*}{A}_{kn}}{K_k{}^0-K_n{}^0}\right)+\int \mathbf{n}\cdot\mathbf{D}\overset{*}{\phi}_n{}^0\phi_k{}^0 dS. \quad (9)$$

For $k=n$, there is found

$$\int_R\overset{*}{\psi}_n\psi_n dv=1+\int_{S_0}\mathbf{n}\cdot\mathbf{D}\overset{*}{\phi}_n{}^0\phi_n{}^0dS-(\sigma_{nn}+\overset{*}{\sigma}_{nn}). \quad (10)$$

Equation (8) gives information about the relation of A_{nk} and $\overset{*}{A}_{kn}$. If ψ_n obeys boundary conditions that make the solutions orthogonal, and if $\phi_k{}^0$ is zero on S_0, then $A_{nk}=\overset{*}{A}_{kn}$. The surface integral also vanishes in cases where S coincides with S_0. The formula (9) relates the normalization of ψ_n and ϕ_n.

Appendix III

Here the method developed in this paper will be applied to the continuous eigenvalue case, the problem of scattering. The equation satisfied by ϕ_n is to first order,

$$H^0\phi_n+K_n\phi_n=-H^1\phi_n{}^0. \quad (1)$$

If Φ_n is the solution of the homogeneous equation, the solution of the inhomogeneous equation may be written,

$$\phi_n=\Phi_n+\int_{R_0}G(x, x')H^1\phi_n{}^0dV, \quad (2)$$

where it has been specified that ϕ_n behave as Φ_n plus a first-order term, and where $G(x, x')$ is the Green's function for the region R_0. From the properties of the Green's function, Eq. (2) becomes

$$\phi_n=\Phi_n+\sigma\phi_n{}^0+\int_{S_0}\{G(x, x')[(\partial/\partial n)(\sigma\phi_n{}^0)]_{x'} -[\sigma\phi_n{}^0]_{x'}(\partial/\partial n)G(x, x')\}dS'. \quad (3)$$

For ψ_n, one has to first order,

$$\psi_n=\Phi_n+\int \{G(x, x')[(\partial/\partial n)(\sigma\phi_n{}^0)]_{x'} -[\sigma\phi_n{}^0]_{x'}(\partial/\partial n)G(x, x')\}dS'. \quad (4)$$

This result is related to Huygens' principle. It is the result that would be obtained by solving the equation $H^0\psi_n+K_n\psi_n=0$ by means of a Green's function and using in the surface integral, as a first approximation, the Green's function for the region R_0 and the unperturbed functions $\phi_n{}^0$. This is the method employed by Sommerfeld.[7]

[7] P. Frank and R. von Mises, *Reimann-Weber Differential Gleichungen der Physik*, Vol. 2 (Fred. Vieweg & Sohn, 1927), p. 478.