The Scattering of Alpha-Particles in Helium*

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Deviations from the scattering to be expected on the basis of an inverse square law of force between alpha-particles receive a satisfactory interpretation in terms of the influence on particles of zero, two and four units of angular momentum of a specific nuclear interaction having a range less than 9×10^{-13} cm. The semistable state of the compound nucleus Be^8 formed in certain disintegration experiments is found from the scattering analysis to have an energy of about 3 Mev, a mean life of 0.8×10^{-21} sec., and *zero* angular momentum. There is also some evidence for a semistable state of still shorter life with two units of angular momentum and an energy of 4 to 5 Mev.

(1) INTRODUCTION

THE observations on the scattering of alpha-particles in helium accumulated from 1927 to the present[1–6] have led so far on the theoretical side to relatively little of positive value either for the understanding of nuclear structure in general or even for the description of the interaction between two alpha-particles.

The first serious attempt to analyze the anomalous scattering of alpha-particles on the basis of the wave description of matter was made by Taylor.[7,8] He assumed that exclusively those particles which collide with zero mutual angular momentum come close enough to experience departures from the inverse square law of force. In the hands of Breit and his collaborators this assumption has since proved a safe starting point for interpreting the proton-proton scattering,[9] where the classical distance of closest approach, $\lambda\!\!\!^{-}$, of particles with even one unit of angular momentum is already large compared to the extension of the forces ($\lambda\!\!\!^{-}=\hbar/\text{velocity}\cdot\text{reduced mass}=\text{wave-length}/2\pi=9\times10^{-13}$ cm for 1-Mev protons). For 7-Mev alpha-particles scattered in helium, however, this distance is sufficiently small in comparison with nuclear dimensions that particles of three or four units of angular momentum must be influenced by the departures from the inverse square law ($\lambda\!\!\!^{-}=1.7\times10^{-13}$ cm as against an estimated alpha-particle diameter of 5 or 6×10^{-13} cm). We therefore have to expect that the scattered alpha-particle wave is a complicated superposition of partial waves including the zero-order wave as well as waves corresponding to two and four units of angular momentum (waves of odd order not appearing because the alpha-particles obey the symmetric statistics). In fact, while Taylor found that an anomalous zero-order partial wave of suitable magnitude superposed on the normal (inverse square law) scattered wave gave a fair fit to the intensity of scattering at 45° and 37°, he encountered[8] a disagreement with subsequent observations[4] at 10° quite outside the experimental error. Moreover, the assumed interaction potential predicted not only a very stable Be^8 nucleus, in contradiction to all available evidence, but also, as is readily shown, anomalous partial waves of higher orders with amplitudes so great as to prove the analysis internally inconsistent.

An attempt was soon made to follow up Taylor's most suggestive beginning, to allow for the influence of higher order waves, and to

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* The present work was initiated in the winter of 1933–34 when the author was a National Research Fellow at New York University. He wishes to express his appreciation to Professor G. Breit for many suggestive discussions on this and other subjects at that time. Completion of the analysis of the alpha-particle scattering has only been made possible, as pointed out in the text, through the recent measurements of Mohr and Pringle and of Devons. The author is indebted to Dr. Mohr and Dr. Pringle for much helpful correspondence in connection with this work.

[1] E. Rutherford and J. Chadwick, Phil. Mag. **4**, 605 (1927).
[2] J. Chadwick, Proc. Roy. Soc. **A128**, 120 (1930).
[3] P. M. S. Blackett and F. C. Champion, Proc. Roy. Soc. **A130**, 380 (1931).
[4] P. Wright, Proc. Roy. Soc. **A137**, 677 (1932).
[5] C. B. O. Mohr and G. B. Pringle, Proc. Roy. Soc. **A160**, 193 (1937).
[6] S. Devons, Proc. Roy. Soc. **A172**, 564 (1939).
[7] H. M. Taylor, Proc. Roy. Soc. **A134**, 103 (1931).
[8] H. M. Taylor, Proc. Roy. Soc. **A136**, 605 (1932).
[9] See G. Breit, H. M. Thaxton and L. Eisenbud, Phys. Rev. **55**, 1018 (1939) and earlier work there cited.

determine their amplitudes.[10] The angular distribution of scattered particles was not known in sufficient detail, however, to lead to a unique or acceptable analysis of the data. Only now, with the additional and indispensable observations of Mohr and Pringle[5] and of Devons[6] available, does it become possible to narrow down the possible alternative interpretations of the scattering to one. This analysis, given below, is in reasonable accord not only with the observations but also with indirect information about the interaction between two alpha-particles provided by certain disintegration experiments in which the compound nucleus Be^8 is formed in an excited unstable state.

The information obtained from anomalous scattering in the present paper is combined in the following article[11] with other observational evidence (a) to determine as completely as possible the properties of the compound nucleus Be^8 and (b) particularly to test, and on the whole confirm, the predictions for this nucleus of a suitably formulated alpha-particle model of light nuclei based on symmetry arguments rather than on any assumed law of force between alpha-particles. The difficulties which in fact lie in the way of any attempt to give a *detailed* account of the interaction between two alpha-particles are especially emphasized by Professor H. Margenau[12] in an accompanying paper. There he shows that hopes of calculating the interaction between two alpha-particles, along the general lines he and others have followed so successfully in dealing with the van der Waals interaction between atoms, must in the last analysis be renounced because the range of the interactions between the constituent neutrons and protons is short in comparison with nuclear dimensions.

The following theoretical analysis of the scattering experiments takes a simple form through the employment of vectors in the complex number plane to represent the amplitudes of the various partial waves, the corresponding "phase shifts" being simply related to the direction of these vectors (Section 2). Geometrical arguments lead, in Section 3, to a simple procedure to determine the phase shifts or, equivalently, the strengths of the partial waves of orders zero, two, and four. Between the three alternative sets of values so found for the three phase shifts, a decision is not possible on the basis of the scattering observations alone. Additional experimental and theoretical information however, clearly singles out as correct a particular set of phase shifts from the behavior of which it follows that the semistable state of Be^8 obtained in certain disintegration experiments has an angular momentum, not of two units as previously generally believed, but of zero (Section 4). The strength of the fourth-order wave gives an approximate estimate of the range of the interaction between alpha-particles (Section 5). Section 6 gives a theoretical treatment of the variation of phase shift with energy near a resonance level, taking account in a semiclassical approximation the influence of the long range forces, and the results allow a determination of some of the properties of the semistable state of Be^8 (Section 7).

(2) Representation of the Scattered Amplitude as a Superposition of Partial Waves

The quantum formula for the ratio, R, of actual number of particles observed at an angle φ compared to the number given by Rutherford's law is[7, 8]

$$R(\varphi)=|A'(\varphi)+A''(\varphi)+\sum_{L=0}^{\infty}A_L(\varphi)(1-\exp 2iK_L)|^2. \tag{1}$$

Here the terms

$$A'(\varphi)=\cos^2\varphi(\sin^4\varphi+\cos^4\varphi)^{-\frac{1}{2}}\exp i\cdot(4e^2/\hbar v)\ln(1/\sin^2\varphi) \tag{2}$$

and

$$A''(\varphi)=A'(\tfrac{1}{2}\pi-\varphi)=\sin^2\varphi(\sin^4\varphi+\cos^4\varphi)^{-\frac{1}{2}}\exp i\cdot(4e^2/\hbar v)\ln(1/\cos^2\varphi) \tag{3}$$

[10] J. A. Wheeler, Phys. Rev. **45**, 746 (1934), and independently by E. Feenberg, according to a private communication of his results in April, 1934, for which the author wishes here to make appreciative acknowledgment.

[11] J. A. Wheeler, Phys. Rev. **59**, 27 (1941), following paper.

[12] H. Margenau, Phys. Rev. **59**, 37 (1941), this issue.

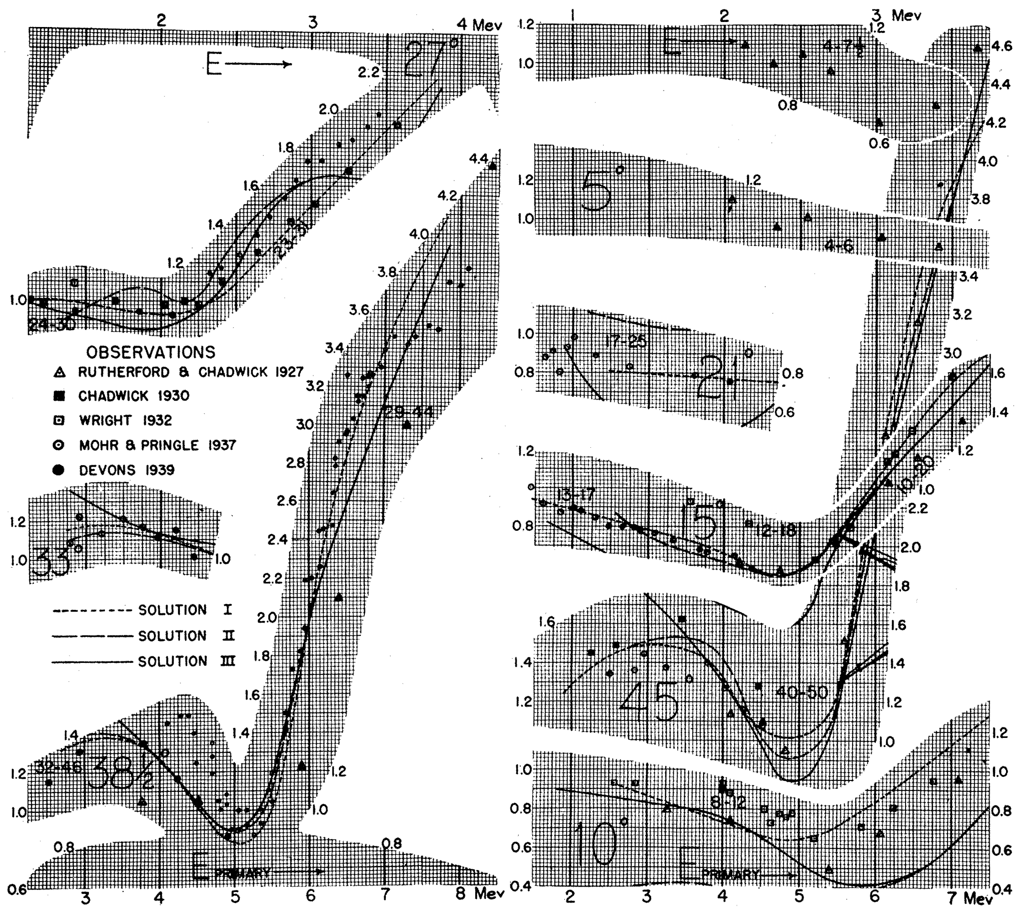


FIG. 1 (left). The square root of the ratio between the actual scattering and that predicted by the Rutherford law is plotted vertically as a function of the energy, E_{primary}, of the incident alpha-particle or of the excitation energy, E available for formation of a temporary compound nucleus Be^8. Included in the figure are all observations except early ones (reference 3) performed with a cloud chamber and without special attempt to reduce the statistical errors to a point where a measurement of anomalous scattering would result. Angular aperture of the diaphragms which defined the scattering volume is indicated for each angle by a notation near the corresponding set of experimental values. The curves are calculated from Eq. (1) and the three alternative sets of phase shifts shown in Figs. 4 and 5. Only solution III is in accord with additional experimental and theoretical evidence cited in the text.

FIG. 2 (right). Continuation of Fig. 1. The measurements at 15° indicated by the compass points are used in Fig. 3(A).

represent in relative magnitude and phase the amplitudes of the waves which would be scattered and knocked on, respectively, at an angle φ, and for a velocity v, if the inverse square law of force applied. The absolute values of the amplitudes are so defined that addition of the corresponding *intensities* gives unity: $|A'|^2+|A''|^2=1$. The addition of amplitudes rather than intensities in Eq. (1) brings about a quantum-mechanical interference effect and consequent deviation from classical scattering which was first noted by Mott.[13] Moreover, the specific nuclear interaction itself alters the scattering amplitude. Equation (1) expresses this alteration as the sum of terms of the type $A_L(\varphi)(1-\exp 2iK_L)$. Each term represents the modification of a partial wave of a particular order, L (L is an even integer). The first factor in the term does not depend on the nature of the forces and is given by the expression

$$A_L(\varphi)=-2i(hv/4e^2)(1/\sin^4\varphi+1/\cos^4\varphi)^{-\frac{1}{2}}(2L+1)P_L(\cos 2\varphi)$$
$$\times\exp 2i\{\arctan(4e^2/hv)+\arctan(1/2)(4e^2/hv)+\cdots+\arctan(1/L)(4e^2/hv)\}. \quad (4)$$

[13] H. M. Mott, Proc. Roy. Soc. **A126**, 259 (1929).

The second factor, $1-\exp 2iK_L$, is determined by the constant difference, K_L, which exists at large distances between (a) the phase of the actual radial wave function associated with the motion of two alpha-particles of mutual angular momentum L and (b) the phase calculated on the basis of inverse square forces (in which case the wave function varies asymptotically as $\sin\{(r/\lambda\!\!\!^{-})-(4e^2/\hbar v)\ln(2r/\lambda\!\!\!^{-})-L(\pi/2)+$the phase of the gamma-function of the quantity $L+1+i\cdot 4e^2/\hbar v\}$). An argument given elsewhere[14] shows that the interchange of neutrons and protons in close collisions between two alpha-particles does not impair the validity of Eq. (1). It is only required that no inelastic scattering occur, a condition which is certainly fulfilled at the energies available from natural alpha-ray sources.

Amplitudes are of more direct theoretical significance than are intensities, as we see from the form of Eq. (1). Therefore we designate as relative scattering amplitude the square root, $R^{\frac{1}{2}}$, of the ratio of observed to classical scattering. Figures 1 and 2 give the relative scattering amplitude as function of energy for all angles at which observations have been made.

Interpretation of complex numbers as vectors in a plane is the basis of a simple geometrical means to calculate the relative scattering amplitude (see Fig. 3A) when the phase shifts are known. The inverse of this procedure—determination by ruler and compass of one phase shift from a knowledge of the scattering amplitude and all other phase shifts—materially simplified the following phase shift analysis.

(3) Procedure for Determining the Strength of the Partial Waves

From the observations at a given energy we have to find the phase shifts K_0, K_2, $\cdots$. The arguments in the introduction indicate that the specific nuclear interaction does not appreciably distort partial waves of order higher than four. Our unknowns, therefore, number only three, K_0, K_2, and K_4. To determine them requires at the least a knowledge of the scattering at three angles. When φ is 15.27° or 35.07°, the scattering does not depend upon the fourth-order phase shift because at those angles the coefficient $A_4(\varphi)$ in Eq. (1) vanishes. Thus the observations at 15° and 35° suffice, in principle, to determine the phase shifts K_0 and K_2. The scattering at a third angle then fixes K_4. In particular, let this third angle be 27.37°; then the number of deflected particles depends only on K_0 and K_4. The former quantity being known, the latter is readily found.

Figure 3B illustrates the above outlined procedure to determine the three unknown phase shifts, and shows how due allowance is made for experimental error in measurement of the intensity of scattering. It is seen that there are several regions in the diagram which give reasonable accord with the observations at 15°, 27°, and 35°. The measurements at a fourth angle (45°) give enough additional information (a) to reduce somewhat the number of possible alternative solutions of the phase shift problem, and particularly (b) to determine more precisely for each solution the values of the phases which give the best attainable agreement with the observations at all four angles.

The analysis illustrated in Fig. 3 has been carried out for ten values of the energy and leads, in the end, to three alternative solutions (solutions I, II and III) of the phase shift problem[15] (see Figs. 4 and 5). We should like to decide between them by using all the observational material. With this aim in mind we use Eq. (1) to calculate as a function of energy the scattering to be expected for each choice of the phase shifts, with the results shown by the curves in Figs. 1 and 2.

Before comparing the calculated curves with the experimental points, we have to show that no important error comes into the observations due either to multiply scattered particles or to the finite opening of the apertures defining the angle of scattering. The mean angle of deviation due to multiple small angle deflections will be equal in order of magnitude to the square root of the average number of helium atoms contained in a cylindrical tube whose extension agrees with the length of path of the alpha-particle in the

[14] J. A. Wheeler, Phys. Rev. 52, 1111 (1937).

[15] Additional solutions appear for a few values of the energy but are excluded because they fail to join continuously on to any solution acceptable at the remaining values of the energy.

apparatus and whose radius is the distance of closest approach calculated classically for the inverse square law of force.[16] The latter distance is 2.9×10^{-13} cm for alpha-particles of 4 Mev energy. For a reasonable representation of the path lengths used in the experiments we adopt a figure of 1 cm at a pressure of one atmosphere. The angle characterizing the multiple scattering will be $\{\pi\times(3\times10^{-13})^2\times3\times10^{19}\}^{\frac{1}{2}}=3\times10^{-3}$ radian or ($\frac{1}{6}$) degree, in order of magnitude. The effect of multiple scattering is, therefore, too small to be appreciable.

Lack of definition of the angle of scattering results less from the finite size of source and counter in the usual experimental arrangement than from the finite aperture of the diaphragms which define the scattering volume. Assuming that all the spread in angle is due to the latter cause, we can easily calculate a factor $W(\varphi)$ which tells with what weight each angle in a given interval contributes to the observed value, R_{obs}, of the ratio of actual to Rutherford scattering:

$$R_{\text{obs}}(\varphi_1, \varphi_2)=\int_{\varphi_1}^{\varphi_2} R(\varphi)W(\varphi)d\varphi \bigg/ \int_{\varphi_1}^{\varphi_2} W(\varphi)d\varphi.$$

From the curves for $W(\varphi)$ in Fig. 6 and from the observed rate of variation of $R(\varphi)$ with angle we

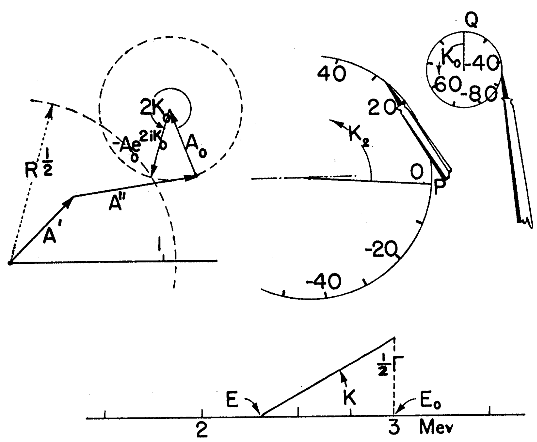


FIG. 3A. The distance, in the diagram at the left, from the origin to the end of the vector A'' represents the relative scattering amplitude, $R^{1/2}$, at a given angle of observation and a given energy, provided that the interaction between the two alpha-particles follows the inverse square law. Anomalous scattering signifies a different value for $R^{1/2}$, shown in the diagram. If it is known that the actual field of force modifies only the zero-order wave, the intersection of two circles of radii $R^{1/2}$ and A_0 determines the two possible values of the phase shift K_0, corresponding to the two points a and b. The left-hand diagram is purely schematic while the one at the right shows the construction, to the same scale, when two phase shifts K_0 and K_2 determine the scattering, as is the case when the angle of observation is 15°. The values of the vectors A are correct for a primary energy of 5.5 Mev ($E=2.8$ Mev) but the general geometrical relationships change only slowly with energy. The separation of the compass points corresponds to the value of $R^{1/2}$ shown in Fig. 2 and the distance from P to Q is determined by the vector sum of A' and A''. The lower diagram illustrates the variation of phase shift with energy, E, near a resonance level of energy E_0 and natural width Γ, provided the de Broglie wave-length is large compared to nuclear dimensions. The actual variation is more complicated.

[16] For a derivation of this result and a more detailed treatment of multiple scattering in general see E. J. Williams, Proc. Roy. Soc. **A169**, 531 (1939) or J. A. Wheeler, Phys. Rev. (to be published).

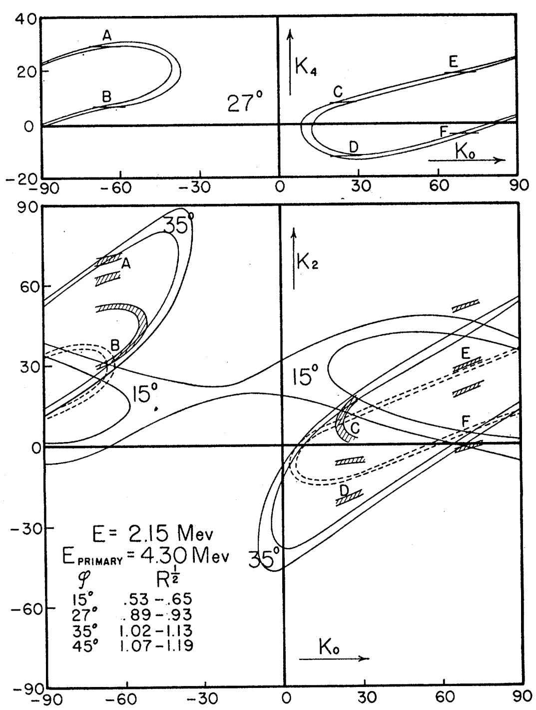
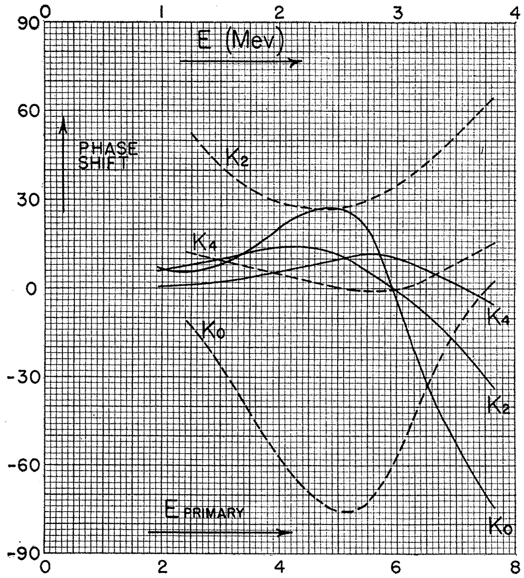


FIG. 3B. For a given energy the anomalous scattering at 15° determines by the construction of Fig. 3A a curve in the K_0, K_2 plane, every point on which is consistent with the observed value of $R^{1/2}$. The experimental uncertainties broaden out the line into a band. The intersection of this band with a similar band for 35° gives four sets of values for K_0 and K_2. For each of the four values of K_0 we can find in a similar way from the scattering at 27° two values of K_4, corresponding to the lettered regions in the upper diagram. K_4 being so determined, the scattering at 45°, which depends on all three phase shifts, gives a relation between K_0 and K_2 shown by the broken curves in the lower diagram. The solutions I, II and III of the phase shift problem mentioned in the text correspond, respectively, to regions C, B and F in the lower diagram. The dashed curve is obtained from the measurements at 45° on the assumption $K_4=0°$. The phases are measured in degrees and the diagrams repeat periodically every 180°.



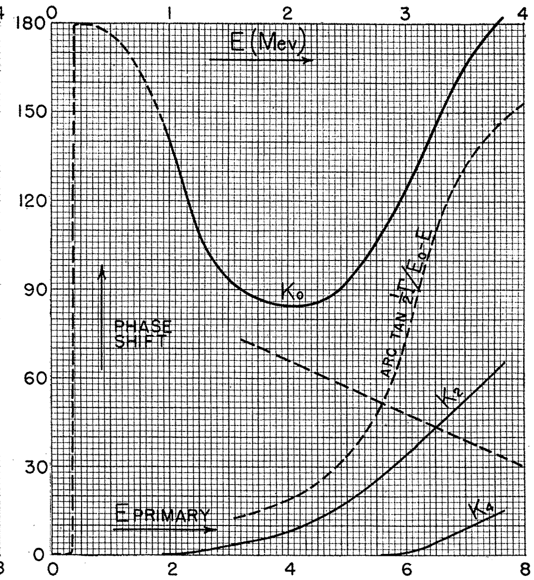


FIG. 4 (left). The problem of representing the observed scattering at seven angles in terms of three parameters, the phase shifts K_0, K_2 and K_4, leads, on account of the uncertainty in the measurements, to three alternative sets of values. Plotted here, in degrees, are solution I (smooth curves) and solution II (dashed curves), both of which are, however, excluded by additional experimental and theoretical evidence.

FIG. 5 (right). The only set of values for the phase shifts in acceptable accord at the same time with the scattering observations and other information about the interaction between two alpha-particles. Within the errors associated with the uncertainties in the measured scattering, the phase shift K_0 is representable as the sum of a contribution varying regularly with energy and a resonance term of the form arctan $(\frac{1}{2}\Gamma/E_0-E)$, with $E_0=3.1$ Mev and $\Gamma=0.8$ Mev. This decomposition of K_0 is shown by the two dashed curves in the region above $E=1.5$ Mev. The observations do not extend to the range of energies below $E_{\text{primary}}=2$ Mev. There the dashed curve represents only the probable course of the zero-order phase shift.

conclude that the corrections for angular spread are relatively unimportant except in the two cases when the angular spread is from 10° to 20° and from 29° to 44°. We dispense with the observations in these two cases because the recent measurements of Mohr and Pringle and of Devons give the desired information and were made with considerably narrower apertures.

Now that we can safely compare the calculated scattering curves in Figs. 1 and 2 with the experimental points, we come to the following conclusions: (a) Solution I of the phase shift problem leads to as good agreement as can be expected with the observations at the seven angles 10°, 15°, 21°, 27°, 33°, 38½° and 45°. (b) Solution II gives fairly good agreement except at 21°, where calculated and observed values are appreciably out of accord, and at 10°, where the discrepancy seems too great to be attributed to experimental error. (c) Solution III predicts scattering curves which agree reasonably well with the observations at 27°, 33°, 38½° and 45°. At 10° the calculated scattering amplitude agrees with the measurements except at the higher energies where the counts are few and the statistical errors greatest (Rutherford and Chadwick used the scintillation method). At 15° the predicted values are appreciably lower than those observed in the interval from 2 Mev to 5 Mev, but above 5 Mev there is good agreement. At 21° we find accord at the highest and lowest energies employed in the experiments, but the measurements are again higher than the calculated values near 3 Mev.

(4) Additional Evidence Leads to a Unique Solution

If the determination of the phase shifts for the various partial waves were a purely mathematical problem, based entirely on measurements of scattering, we should undoubtedly conclude that solution I is in the best agreement with experiment. However, there is good observational evidence[17] that two alpha-particles can form a semistable compound nucleus Be^8 of about 2.8 Mev excitation energy. The existence of a temporary state of this nature implies that one of the partial waves representing scattered alpha-particles of a particular angular momentum must show an anomalous behavior when the energy of the impinging particles is in the neighborhood of 5.6 Mev, and, indeed, the corresponding phase shift must increase through the resonance region by approximately 180°, according to the dispersion theory of nuclear reactions.[18] Solution I definitely fails to satisfy this requirement, and must be ruled out. Solutions II and III, on the other hand, show at the higher energies a rapid rise in the phase shift of the wave of zero angular momentum. This behavior is consistent with the existence of a semistable state of the nucleus Be^8 and, in fact, not only independently indicates the existence of such a state with just the properties required by disintegration experiments (as we shall see below) but also reveals the angular momentum of this state.

Extent of agreement with the scattering experiments is sufficiently comparable for solutions II and III as not to decide between them. These solutions are, however, distinguished by the behavior of the phase shifts of orders two and four. K_2 and K_4 decrease monotonically with diminishing energy for solution III while for solution II they decrease initially at the higher energies but then rise at lower energies. The latter behavior is contrary to every reasonable expectation based on the fact that the wavelength of the colliding particles becomes greater than the range of interaction in inverse proportion to the velocity. In particular, we have to expect for every phase shift a certain energy below which that phase falls off continuously; and in general this energy will lie not far below the first semistable level of the compound system, the angular momentum of which corresponds to the order of the given phase shift. Both the behavior of the phases K_2 and K_4 themselves and other experimental and theoretical considerations[17] supply arguments against

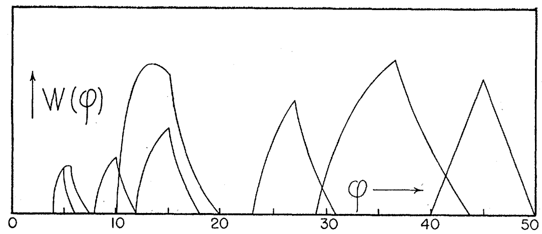


FIG. 6. Observations on the scattering from the region lying within a given angular aperture depend with the weight $W(\varphi)$ on the ratio, $R(\varphi)$, of actual to Rutherford scattering at all intermediate angles.

the existence of any semistable levels of Be^8 with angular momentum of two or four units lying sufficiently low to account for such a behavior of the second- and fourth-order phases as is shown by solution II. For solution III, on the other hand, the phases K_2 and K_4 vary with energy in accordance with the just-mentioned expectations. We therefore conclude that *solution III is the only solution of the phase shift problem* in acceptable accord at the same time with the scattering experiments, with experimental evidence for a semistable state of the nucleus Be^8, and with requirements as to the variation with energy of higher order phase shifts. Moreover, the magnitude of the fourth-order phase shift is sufficiently small to indicate that sixth and higher order phases are negligible and thus to justify the internal consistency of the above analysis.

Previous to the present scattering analysis, the level of the nucleus Be^8 at 2.8 Mev indicated by disintegration experiments has been generally believed to possess two units of angular momentum. A simple demonstration that this cannot be so is, therefore, of interest. The relative scattering amplitudes at 15° depends on the phase angles K_0 and K_2 in the manner illustrated by Fig. 3A, and is independent of K_4. Increase of $2K_2$ near resonance by approximately 360°, implied by the supposed existence of a level of two units of angular momentum, would (as is seen in the figure) cause the scattering to rise to a value much larger than permitted by the observations, whatever might be the assumed simultaneous variation with energy of K_0.

(5) Range of the Interaction

Principal direct conclusions from the above-determined phase shifts concern the range of the

[17] Summarized briefly in the following paper, J. A. Wheeler, Phys. Rev. **59**, 27 (1941).

[18] See in this connection G. Breit and F. L. Yost, Phys. Rev. **48**, 207 (1935) and R. Peierls and P. L. Kapur, Proc. Roy. Soc. **A166**, 277 (1938).

interaction between two alpha-particles and the properties of the semistable state of the compound nucleus Be^8. The behavior of the phase shift K_4 shows that particles with four units of mutual angular momentum first have appreciable nonelectrostatic interaction when the energy of the primary particle is above 6 Mev. At an energy of 6 Mev, the previously mentioned distance $\lambda\!\!\!^{-}$ (wave-length/2π), which characterizes the minimum dimensions of a wave packet, is sufficiently small (1.86×10^{-13} cm) to justify an approximate estimate of the distance between centers of the two alpha-particles on the basis of the classical formula:

Distance of closest approach = (product of charges/primary energy) + {(charge product/energy)^2 + (angular momentum)^2/(half-energy $\times$ mass of alpha-particle)}$^{\frac{1}{2}}$

$$=0.96\times 10^{-13}+\{(0.96\times 10^{-13})^2+20\lambda\!\!\!^{-2}\}^{\frac{1}{2}}=9\times 10^{-13}\text{ cm}. \quad (5)$$

This estimate undoubtedly represents an upper limit for the range of the interaction because we have inserted in the formula the lowest possible value for the energy at which there may be said to be appreciable forces of specific nuclear character between two alpha-particles of the given angular momentum. There does not appear to exist at present any method to obtain a well-defined value for the range of the interaction.

(6) Variation of Phase Shift Near Resonance

Existence of a semistable compound state has what consequences for the variation with energy of the corresponding phase shift? Conversely, a knowledge of the energy dependence of the phase leads to what information on the properties of the semistable state? Without a detailed knowledge of the interaction, only certain general conclusions can be drawn. Those already given by the dispersion theory[18] we wish to supplement here by taking into account in a semiclassical approximation the influence of electrostatic and centrifugal forces.

At a certain initial moment, $t=0$, there will be complete certainty that the two alpha-particles are in the given compound state. This state of affairs will be represented by a probability amplitude $\psi_c(r_1\sigma_1, r_2\sigma_2, \cdots, r_8\sigma_8)$, a function of the coordinates and spins of the eight constituent particles which has negligible magnitude outside of nuclear dimensions. The probability amplitude function at any later instant will have appreciable magnitude both in the region of nuclear dimensions, where it will have fallen off exponentially with time, and at greater distances, where it will represent a wave train moving outward with a group velocity equal to the natural velocity of separation of the alpha-particles. This wave train must be mathematically equivalent to a superposition of the proper functions which describe the probability amplitude for well-defined values of the energy. But as each such proper function is characterized by its phase at large distances, the just mentioned mathematical equivalence will imply certain conditions as to the variation of phase with energy.

The mean life of the radioactive state defines a certain energy, Γ (the natural width of the level), through the relation

$$\text{reciprocal of mean life} = \text{radioactive decay constant} = \Gamma/\hbar.$$

In a series of experiments with the same initial conditions, the energy of the emitted particles will scatter around a certain average energy, E_0, with a spread of the order of magnitude of Γ. The corresponding uncertainty in the velocity of separation will cause a smearing out of the general structure of the outgoing wave train after it has traveled for a time of the order of magnitude of (E_0/Γ) times the mean life of the radioactive state, or for a distance of the order of velocity times $(E_0/\Gamma)(\hbar/\Gamma)$, as may easily be shown. We shall concern ourselves in the following, however, with separations sufficiently small that we can neglect this spreading effect. Then the amplitude of the outgoing wave at a distance r and an instant t will be proportional to the probability amplitude of the compound state at a preceding moment $t-t_{ar}$. Here t_{ar} is the time required according to classical mechanics to bring about the given separation of the two alpha-particles starting with them at a distance a equal in order of magnitude to the range of the specific nuclear interaction:

$$t_{ar}=\int_a^r dr/v_r=\int_a^r dr$$

times the derivative of the relative momentum of the particles at a given point with respect to their energy, equals

$$\int_a^r (dp/dE)dr. \tag{6}$$

The probability that the compound nucleus shall disintegrate during a time interval dt at the instant $t-t_{ar}$ will be $(\Gamma dt/\hbar)\exp(-\Gamma(t-t_{ar})/\hbar)$. If it so disintegrates, the alpha-particles will have a separation between r and $r+v_r dt$ at the moment t, and, therefore, the absolute magnitude of the probability amplitude at that distance and time must be $(\Gamma/\hbar v_r)^{\frac{1}{2}}\exp\Gamma(t_{ar}-t)/2\hbar$. According to the semiclassical approximation treatment of Wentzel, Kramers and Brillouin, the phase of the probability amplitude will increase with distance at a rate given by the momentum p_0 at the point in question divided by $\hbar$, and will decrease with time at the rate $E_0/\hbar$. Thus, to the accuracy in which we are interested, the complete expression for the outgoing wave train will be[19]

$$(\Gamma/\hbar v_r)^{\frac{1}{2}}\exp\left(i\cdot\text{constant}+i\int_a^r p_0 dr/\hbar+\Gamma(t_{ar}-t)/2\hbar-iE_0t/\hbar\right). \tag{7}$$

It is supposed here that the difference $t-t_{ar}$ is positive. A negative value of this difference will signify, according to classical mechanics, that sufficient time has not elapsed for the alpha-particles to attain the given separation and, therefore, the probability amplitude will then be zero in the approximation in question.

The dependence upon time of the probability amplitude is not that of a proper state of a given circular frequency $E/\hbar$, but rather corresponds to a superposition of such states, of the form

$$\int dE\psi_E(r)\exp(-iEt/\hbar). \tag{8}$$

From a knowledge of the outgoing wave in its dependence upon position and time we are interested in obtaining a knowledge of how the individual proper states $\psi_E(r)$ depend upon r. Following the standard inversion procedure of Fourier analysis, we multiply the probability amplitude at r by the expression $(dt/2\pi\hbar)\exp(iEt/\hbar)$, integrate over all values of the time which contribute (extending from $t=t_{ar}$ to $t=\infty$ for a point r outside nuclear dimensions), and thereby find $\psi_E(r)$:[20]

$$\psi_E^{(1)}(r)=(\Gamma/\hbar v_r)^{\frac{1}{2}}\{1/2\pi i(E-E_0+\tfrac{1}{2}i\Gamma)\}\exp\left\{i\cdot\text{constant}+i\int_a^r p_0 dr/\hbar+i(E-E_0)t_{ar}/\hbar\right\}. \tag{9}$$

By use of Eq. (6) and trigonometry we rewrite this result in the form

$$-\pi\{(E-E_0)^2+(\tfrac{1}{2}\Gamma)^2\}^{\frac{1}{2}}\psi_E^{(1)}(r) = (\Gamma/\hbar v_r)^{\frac{1}{2}}(1/2i)\exp\left\{i\cdot\text{constant}+i\int_a^r p_E dr/\hbar+i\arctan(\tfrac{1}{2}\Gamma/E_0-E)\right\}. \tag{10}$$

We do not however, obtain the complete expression for the proper function in question by taking into account only the existence of a diverging wave train. A wave train converging upon the region of

[19] Actually expression (7) has to be multiplied by the internal wave functions for the two alpha-particles, and to the product have to be added, with the proper signs, similar products in which the coordinates of neutrons and protons have been suitably interchanged. This complication introduces, however, no essential new features in the treatment in the text, except to exclude altogether radial waves of odd angular momentum.

[20] No acquaintance with the mathematics of Fourier analysis is necessary to recognize that the above procedure is analogous to the determination for an electromagnetic wave train of the amplitude of a given monochromatic constituent. The retardation in time of the disturbances coming from different points of a grating corresponds to the multiplying factor exp $(iEt/\hbar)$ and the summation by a lens of the amplitudes from all parts of the grating is, in principle, the same as the above integration with respect to time.

specific nuclear interaction will exist at times preceding the initial instant $t=0$, according to the principle of reversibility of time. In its wave-mechanical formulation, this principle states that if the probability amplitude is represented by a real function at the time $t=0$ (as may be achieved in our case by a suitable choice of the phase of ψ_c), then the mathematical expression for the incoming wave will be obtained by reversing simultaneously the sign of t and the sign of i in (7). Therefore, we obtain $\psi_E(r)$ by adding to the outgoing wave $\psi_E^{(1)}(r)$ of Eq. (10) an incoming wave $\psi_E^{(2)}(r)$ which differs from it only through the change in sign of i. We conclude that the proper function corresponding to an energy E varies with distance as

$$(\Gamma/\hbar v_r)^{\frac{1}{2}} \sin\left\{\text{constant}+\int_a^r p_E dr/\hbar+\arctan\left(\tfrac{1}{2}\Gamma/E_0-E\right)\right\}. \tag{11}$$

This result will be valid to the same extent as any semiclassical representation of a wave function (provided that the value of the energy in question differs from E_0 by an amount not so great in comparison with the natural width Γ that the proper function fails to play an appreciable role in the representation of the disintegration process).

We are interested in the difference K between the actual phase of the proper function and the phase calculated on the basis of inverse square law forces. The latter phase will be $\pi/4$ plus the integral $\int p_E dr/\hbar$ extended from the classical distance r_E of closest approach to the point r in question,[21] according to the same semiclassical approximation upon which the above calculations are based. Comparison with expression (11) gives for the phase shift the approximate equation

$$K(E)=\text{constant}+\int_a^{r_1} p_E dr/\hbar-\int_{r_E}^{r_1} p_E dr/\hbar+\arctan\left(\tfrac{1}{2}\Gamma/E_0-E\right). \tag{12}$$

Here the momentum p_E is to be calculated classically (a) for the actual field of force in the first integral and (b) in the second integral for the inverse square law of interaction. The two fields of force are the same beyond a certain distance r_1, which appears as the upper limit in the integrals.

In applying Eq. (12) to problems of nuclear scattering we have to expect that the range of the specific nuclear forces will be of the same order of magnitude as a. By neglecting altogether the difference between r_1 and a we will, therefore, obtain a first approximation to the actual state of affairs: The variation with energy of the phase shift near resonance will be ascribed to two effects, of which one, represented by the term $\arctan(\frac{1}{2}\Gamma/E_0-E)$, is determined by the position and sharpness of the resonance level, while the second will depend upon the size of the compound nucleus, and be described by the integral

$$-\int_{r_E}^{a} p_E dr/\hbar.\,^{22}$$

The latter integral (according to a simple numerical evaluation for the case of two alpha-particles of zero angular momentum) is practically a linear function of energy above 1.5 Mev (that is, when the primary energy exceeds 3 Mev) provided that the distance a is 3×10^{-13} cm or greater. Thus, in the problem in question we have, to a first approximation, the result:

$$K(E)=\text{a smoothly varying function of energy nearly linear in the neighborhood of resonance} \\ +\arctan\left(\tfrac{1}{2}\Gamma/E_0-E\right). \tag{13}$$

In the next approximation, the unknown departure from the inverse square law outside a will make it

[21] Numerical evaluation shows that this semiclassical calculation gives for the phase at large distances of the zero-order Coulomb wave functions a value in error by at most 12° in the energy range of interest for the alpha-particle scattering.

[22] When the conditions are satisfied for applying the above semiclassical treatment to neutrons, the integral in question gives just the variation of phase with energy which follows from the treatment of H. A. Bethe and G. Placzek (Phys. Rev. **51**, 450 (1937)) without, however, requiring the introduction of a fictitious repulsive nuclear potential of range equal to a.

TABLE I. *The radial wave functions F_L^* oscillate at large distances as sine functions with unit amplitude and with the phase $\{(r/\lambdabar)-(4e^2/\hbar v)\ln(2r/\lambdabar)-L(\pi/2)+$ phase of gamma-function of $(L+1+i\cdot 4e^2/\hbar v)+$ the phase K_L of Fig. 5$\}$ and are calculated here for the point $r=9\times 10^{-13}$ cm on the assumption that inverse square forces hold down to that distance. Energies are given in Mev.*

E_{PRIMARY}	E	$\hbar v/4e^2$	K_0	F_0^*	rdF_0^*/F_0^*dr	K_2	F_2^*	rdF_2^*/F_2^*dr
2.48	1.24	1.25	118	0.39	6.22	2	0.62	4.47
3.57	1.79	1.50	92	0.65	4.13	6	1.02	1.28
4.86	2.43	1.75	89	1.03	1.00	16	1.20	−0.27
6.35	3.17	2.00	139	0.27	−16.7	40	0.69	−5.29
8.04	4.02	2.25	(185)	−0.99	1.63	(72)	−0.52	8.44

impossible to calculate the actual behavior of the first term in (13) but every reasonable law of force will still lead to a smoothly varying function of energy.

(7) PROPERTIES OF THE SEMISTABLE STATE OF Be^8

The zero-order phase shift shown in Fig. 5 is resolved satisfactorily into two parts in accordance with Eq. (13) when we insert for E_0 and Γ, respectively, the values 3.1 Mev and 0.8 Mev, figures for which, however, the variation of the phase shift will be unable to give very accurate values until the scattering has been measured for primary energies in the range 7 Mev to 10 Mev. The just stated values for the disintegration energy and natural breadth of the semistable level are in reasonable accord with the figures 2.8 Mev and 0.8 Mev obtained[23] from disintegration experiments in which the nucleus Be^8 is formed in a radioactive state. This agreement helps to confirm the phase shift analysis given above. The actual value of the natural width corresponds to a mean life, $\hbar/\Gamma$, amounting to 0.8×10^{-21} sec.

If departure from the inverse square law were negligible outside the distance a referred to above, we should obtain for this quantity a value of about 3×10^{-13} cm from the slope in the neighborhood of resonance of the first term in (13). The steepened slope at lower energies of this nonresonance component of the phase shift curve (see Fig. 5) indicates, however, that the specific nuclear interaction must extend to distances considerably greater than 3×10^{-13} cm.

Existence of a semistable state of mean life still shorter than 10^{-21} sec. and an angular momentum of two units is suggested by the nature of the rise of the phase shift K_2 at the higher energies in Fig. 5. The observations do not extend to energies sufficiently great to allow much more than a rough estimate that such a state, if it exists, has an energy of the order of 4 or 5 Mev.

Values of the wave functions near the nucleus furnish what is, in some ways, a more significant representation of the observations than that provided by the phase shifts. The knowledge of the asymptotic behavior of the wave functions implied by the phase shifts in Fig. 5; the negligible effect of the specific nuclear forces down to a distance of $r=9\times 10^{-13}$ cm; and formulae and tables of wave functions in an inverse square field of force[24] allow us to calculate the values of the wave functions and their derivatives listed in Table I. From the values in the table it does not seem possible to deduce in a simple way any straightforward conclusions about the interaction between two alpha-particles over and above what has already been revealed by the behavior of the phase shifts.

(8) SUMMARY

Anomalies in the scattering of alpha-particles in helium receive a satisfactory interpretation in terms of the distortion by a specific nuclear interaction of the partial waves of zero, two and four units of angular momentum. The actual magnitude of the distortion of the fourth-order wave sets an upper limit of about 9×10^{-13} cm for the range of the interaction. The behavior of the zero-order wave gives information about a semistable state of the nucleus Be^8 formed in certain disintegration experiments. The variation with energy of the second-order wave suggests the existence of another state of energy 4 to 5 Mev and still shorter lifetime.

[23] P. I. Dee and C. W. Gilbert, Proc. Roy. Soc. **A154**, 279 (1936); H. A. Bethe, Rev. Mod. Phys. **9**, 218 (1937).

[24] J. A. Wheeler, Phys. Rev. **52**, 1123 (1937).