A Note on the Density and Compressibility of Nuclear Matter

EUGENE FEENBERG
Washington Square College, New York University, New York, New York
(Received December 2, 1940)

The influence of the Coulomb force between protons on nuclear radii is calculated by using the minimum property of the energy eigenvalue. Several different methods of estimating the compressibility of nuclear matter yield qualitatively concordant results. A small correction to the Coulomb energy is required by the finite value of the compressibility. This correction, which may be called the Coulomb "expansion" energy, is proportional to the square of the Coulomb energy.

I. INTRODUCTION

It is generally assumed that the density of nuclear matter has approximately the same constant value in all nuclei. Support for this assumption has been found in the approximately linear relation between binding energy and number of particles, in the general trend of nuclear radii in the heavy radioactive elements[1] and in the values of the Coulomb energy required to account for the stability relations among light and intermediate nuclei.[2] The evidence in support of an approximately constant density does not exclude the possibility of small systematic or irregular variations both within a given nucleus[3] and from nucleus to nucleus.[1]

II. THE EFFECT OF COULOMB FORCES ON NUCLEAR DENSITY

One possibility for systematic variations is given by the Coulomb force between protons which might be expected to produce a trend toward decreasing density in heavy nuclei. By utilizing the minimum property of the energy eigenvalue, this effect can be discussed without the introduction of special hypotheses.

Let H_0 represent the nuclear Hamiltonian with the omission of the Coulomb interaction. The equation

$$H_0\psi_0 = E_0\psi_0 \tag{1}$$

determines the normal state eigenfunction $\psi_0(x_1, \cdots)$, eigenvalue E_0 and radius R_0 of a nuclear model in which there is no Coulomb force acting between protons. The eigenvalue E and radius R of the actual nucleus may be obtained by calculating an expectation value of the total Hamiltonian using as wave function $\psi_0(\lambda x_1\cdots)$ with λ a scale factor to be determined by minimizing E. Thus,

$$\begin{aligned} E(\lambda) &= \frac{\int\cdots\int \psi_0^*(\lambda x_1\cdots)H_0\psi_0(\lambda x_1\cdots)d\tau}{\int\cdots\int \psi_0^*(\lambda x_1\cdots)\psi_0(\lambda x_1\cdots)d\tau} + \lambda E_{0c} \\ &= \dot{E}_0(\lambda) + \lambda E_{0c} \\ &= E_0 + \tfrac{1}{2}(\lambda-1)^2 E_0'' + \lambda E_{0c}. \end{aligned} \tag{2}$$

In the last line use has been made of the fact that

$$E_0'(\lambda) = 0 \quad \text{for} \quad \lambda = 1 \tag{3}$$

and terms involving derivatives of $E_0(\lambda)$ beyond the second have been dropped. E_{0c} is the Coulomb energy for the radius R_0. From the condition for a minimum,

$$E'(\lambda) = 0 \tag{4}$$

we obtain

$$\lambda = 1 - E_{0c}/E_0'', \tag{5}$$

$$R = R_0(1 + E_{0c}/E_0''), \tag{6}$$

$$E = E_0 + E_{0c} - E_{0c}^2/2E_0''. \tag{7}$$

The last term in Eq. (7) may be called the Coulomb "expansion" energy.

III. ORDER OF MAGNITUDE ESTIMATE OF E_0''

The facts mentioned in the introduction are consistent with the relations

$$R_0 \sim 0.5A^{\frac{1}{3}}(e^2/mc^2), \tag{8}$$

$$E_0 \sim -16Amc^2. \tag{9}$$

To calculate E_0'' we must know the dependence

[1] G. Gamow, *Atomic Nuclei and Nuclear Transformations* (Oxford Press, 1937), p. 106.
[2] E. Wigner, Phys. Rev. **51**, 947 (1937).
[3] H. Euler, Zeits. f. Physik **105**, 553 (1937).

of E_0 on the nuclear radius R_0. We write

$$E_0 = A[T_0 - W_0(R_0)], \tag{10}$$

in which T_0 is the average kinetic energy per particle and $W_0(R_0)$ the average potential energy per particle. Since the kinetic energy varies inversely with the square of the radius Eq. (10) implies

$$E_0(\lambda) = A[\lambda^2 T_0 - W_0(R_0/\lambda)]. \tag{11}$$

The statistical model and Eq. (8) yield the value[2] $T_0 \sim 28mc^2$. A convenient unit in which to express E_0'' is provided by Eq. (11). We see that the contribution from the kinetic energy to E_0'' is simply $2AT_0 \sim 56mc^2$. The results of the next paragraph show that

$$E_0'' = (2AT_0)k \tag{12}$$

with k in the neighborhood of 2 or 3.

To proceed beyond this point an explicit form must be assumed for $W_0(R)$. Three assumptions are treated here:

$$\begin{aligned} &\text{(a)} \quad W_0(R) = B\exp(-R/a), \\ &\text{(b)} \quad W_0(R) = B\exp(-R^2/a^2), \\ &\text{(c)} \quad W_0(R) = B/(R^2+a^2)^{\frac{1}{2}}. \end{aligned} \tag{13}$$

Some theoretical basis exists for Case (c).[4]

Case (a)

$$\begin{aligned} &B\exp(-R_0/a) = 2T_0 a/R_0, \quad [\text{Eq. (3)}], \\ &E_0 = AT_0(1-2a/R_0), \\ &a/R_0 = 0.79, \quad (\text{Eq. 9}), \\ &E_0'' = 1.73(2AT_0) = 97Amc^2. \end{aligned} \tag{14}$$

In the same way we find

Case (b) $\quad E_0'' = 2.73(2AT_0) = 153Amc^2. \quad (15)$

Case (c) $\quad E_0'' = 1.88(2AT_0) = 105Amc^2. \quad (16)$

Another type of estimate is provided by using the Hartree approximation to calculate $W_0(R)$ from an assumed Hamiltonian operator. A calculation of this type, based on the exchange force Hamiltonian with error function potentials, has been made by Bethe.[5] He obtains

$$E_0'' = 2.6(2AT_0). \tag{17}$$

IV. Discussion

The numerical results in this section are based on the assumed value $E_0'' = 150Amc^2$. This gives

$$R = R_0(1 + Z^2/125A^{4/3}). \tag{18}$$

The following figures show how R/R_0 varies with charge number. Chemical atomic weights are substituted for A in order to obtain a single valued function of Z

Z	12	32	52	72	92
R/R_0	1.016	1.027	1.034	1.041	1.046

The most interesting aspect of the energy formula Eq. (7) is the way in which the Coulomb energy varies within an isobaric series. If Z is increased by one unit the Coulomb energy increases by the amount

$$\Delta E_c = E_{0c}(Z+1) - E_{0c}(Z) - E_{0c}(Z)2Z/125A^{4/3}. \tag{19}$$

If the Coulomb exchange energy is neglected Eq. (19) becomes

$$\Delta E_c = (6Ze^2/5R_0)(1 - Z^2/125A^{4/3}) = 6Ze^2/5R. \tag{20}$$

The effect of the term in brackets in Eq. (20) is to shift the region of stability toward smaller values of the isotopic number, $N-Z$, as compared with a theory in which nuclear matter is treated as incompressible. Wigner's theory of stability relations in isobaric series is of the latter type. However, the correction to Wigner's energy formula implied by Eq. (19) is not large enough to extend the region in which the theory agrees with experiment, although it is in the right direction. In fact, a considerable extension of the range in which theory and experiment agree could be obtained by arbitrarily taking E_0'' about one-fifth the value used here. Wigner has emphasized that the breakdown of his theory beyond $A=52$ probably results in large part from the neglect of the effect of the Coulomb interaction in coupling together states with different types of symmetry. Precisely this coupling or mixing effect is neglected in Eq. (2) since the symmetry properties of a wave function are not changed by a change of scale. Consequently it seems best to defer the application of Eq. (20) to the problem of nuclear stability until reliable estimates of the mixing effect and of the Coulomb exchange energy are available. Then it may become possible to determine E_0'' directly from stability relations among isobars.

[4] E. Feenberg, Phys. Rev. **52**, 758 (1937).
[5] H. Bethe, Rev. Mod. Phys. **9**, 69 (1937), Eq. (319).