On the Scattering of Mesons of Spin $\frac{1}{2}\hbar$ by Atomic Nuclei

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The non-electric scattering of charged mesons by protons and neutrons is calculated as a first-order effect in the heavy electron pair theory of nuclear forces. The latter theory regards the mesons as heavy electrons, that is, as particles with spin $\hbar/2$, obeying the Dirac equation and differing from usual electrons only in their rest mass μ. An upper limit for the scattering cross section of mesons by nuclear particles is derived; it is found that the cross section is less than 4×10^{-30} cm² for mesons of energy $E=\mu c^2$ and less than 1.6×10^{-29} for $E=3\mu c^2$. These values are about 1000 times smaller than the corresponding cross sections obtained on the basis of meson theories of nuclear forces which ascribe to the meson a spin $\hbar$. In contrast to the latter theories the values obtained in the present paper are in agreement with the upper limits of the scattering cross section found experimentally by Wilson and others. For meson energies large compared to the rest energy of the proton or neutron, the scattering cross section increases linearly with the energy; it first attains the value 10^{-26} cm² for meson energies of the order of 10^{11} ev.

§1. INTRODUCTION AND DISCUSSION OF RESULTS

THE scattering of mesons by neutrons or protons has recently been investigated both experimentally and theoretically. The experiments by Blackett and Wilson[1] have shown that the scattering of mesons by nuclei consists mostly of Rutherford scattering due to the Coulomb forces between the electric charges of the nucleus and of the meson. More recently, J. G. Wilson[1a] observed the scattering of mesons by metal plates in a cloud chamber and found that only one track out of 185 showed a scattering angle appreciably outside the range expected from multiple Rutherford scattering; this particular meson had an energy of 1.6×10^9 ev. Wilson also quotes four other tracks of particles with still higher energy, previously measured by Wilson and by Brode and Starr, and which could not be attributed to Rutherford scattering. The cross section for this large angle scattering (due to short range non-electric forces) computed from the three sets of experiments is of the order of 10^{-28} cm² per proton. The possibility that the tracks were due to protons scattered by the specifically nuclear forces cannot be excluded. Thus the available experimental material indicates that the cross section of meson scattering by nuclear particles is at most 10^{-28} cm² in the energy region: 2×10^8 to 2×10^9 ev.

This result, however, is in striking contrast to the theoretical calculations carried out by Heitler,[2] Bhabha,[3] and Wilson[4] on the basis of the current meson theories of nuclear forces. These theories regard the mesons as particles with spin $\hbar$ obeying Bose statistics, and interacting with nuclear particles (proton, neutron) in a way similar to the interaction between charged particles and light quanta. However, in order to describe the actual nuclear forces by means of a meson field it was necessary (a) to introduce an additional interaction of a kind that does not have its analog in electrodynamics—depending on the spin of the nuclear particles, (b) to introduce neutral mesons besides charged mesons ("symmetric" theory) or even to ascribe the nuclear forces entirely to the virtual emission and reabsorption of neutral mesons alone ("neutral" theory).

These theories (with the exception of Bethe's neutral theory which of course has nothing to say about charged mesons) give rise to a scattering of charged mesons by nuclear particles in virtue of a second-order process, e.g.:

$$N+m_1^{(+)}\rightarrow P\rightarrow N+m_2^{(+)}.$$

Here N stands for neutron, P for proton, $m_{1,2}^{(+)}$ designates a positive meson in two states 1 and 2, respectively. The cross section of this process was first calculated by Heitler[2] with the result

$$\sigma=\frac{4\pi}{3}(g_1^2+2g_2^2)^2\frac{p^4}{E^2\mu^4c^4}. \tag{1}$$

[1] P. M. S. Blackett and J. G. Wilson, Proc. Roy. Soc. **165**, 209 (1938).
[1a] J. G. Wilson, Proc. Roy. Soc. **174**, 73 (1940).
[2] W. Heitler, Proc. Roy. Soc. **166**, 529 (1938).
[3] H. J. Bhabha, Proc. Roy. Soc. **166**, 501 (1938).
[4] A. H. Wilson, Proc. Camb. Phil. Soc. **36**, 363 (1940).

TABLE I. *Values for the cross section computed from Eqs. (4), (5) and (6) for three characteristic energies.*

Energy (Mev)	89	267	1780
Energy (μc^2)	1	3	20
σ (cm^2) for mesons of spin $\frac{1}{2}$	$<4\times10^{-30}$	$<1.6\times10^{-29}$	$<5.5\times10^{-28}$
σ (cm^2) for mesons of spin 1	1.5×10^{-27}	2×10^{-25}	2×10^{-24}

The quantities g_1 and g_2 are the two factors determining the meson-nuclear particle interaction, both having the dimension of a charge, μ is the meson mass ($\sim$175 electron masses), p its momentum, E its energy; the nuclear particle has been assumed infinitely heavy. The calculations of Wilson[4] taking into account the finite mass of the nuclear particle give in addition to (1) a term proportional to p^2 and a term which does not vanish in the limit of zero p; namely,

$$\sigma_0=\frac{12\pi M^2}{(\mu+M)^2}\frac{(g_1+g_2)^4}{(\mu+2M)^2c^4}. \tag{2}$$

M is the mass of the proton or neutron. The factors g_1, g_2 have been determined by identifying the nuclear forces derived from the meson field with the observed nuclear forces and it is found that $g_{1,2}^2/\hbar c\approx 0.16$ and that the order of magnitude does not depend on the particular theory used. The numerical values of the scattering cross section for three characteristic energies can then be calculated and are listed in Table I.

Several authors have tried to remove this large discrepancy with experiment either by introducing new quantum states of the nuclear particle with higher multiples of electric charge or by investigating the possibility that higher approximations would reduce the cross section. The first hypothesis[5] reduces the cross section by introducing new intermediate states which give rise to interference terms of the same sort which occur in the expression for the scattering of neutral mesons by nuclear particles. As regards the second suggestion, Wentzel[6] has shown that the same large cross section $2\pi(\hbar/\mu c)^2$ is obtained in an expansion of the meson-nuclear interaction in powers of $(1/g)$—which represents the limit of strong interaction. It seems therefore improbable that a more exact approximation method would give rise to smaller values for the cross section.

[5] W. Heitler, Nature **175**, 69 (1940).

[6] G. Wentzel, Helv. Phys. Acta **13**, 269 (1940).

In view of these serious difficulties we have tried to calculate the scattering on the basis of a theory of mesons with spin $\frac{1}{2}$ and obeying Fermi statistics. The mesons are supposed to differ from ordinary electrons by the value of the rest mass alone. Thus they are to be described by the Dirac equation, and by a "hole" theory to account for the two signs of charge. We use the term "heavy electrons" for these hypothetical particles, which tentatively may be identified with the observed mesons in cosmic rays.

The interaction between heavy electrons and nuclear particles obviously must be different from the interaction in the theories of the Bose meson. In the present theory the interaction gives rise to a change of state of a heavy electron under the influence of a nuclear particle. This is equivalent to the creation of a heavy electron pair if the change of state consists of a transition from a negative to a positive energy state. It has been shown by one of us[7] that nuclear forces having a suitable spin and spatial dependence can be derived from an interaction of the form:

$$G(\Psi^*\sigma^{(n)}\Psi)(\psi^*\beta^{(e)}\sigma^{(e)}\psi). \tag{3}$$

In (3) Ψ and ψ are the wave functions, $\sigma^{(n)}$ and $\beta^{(e)}\sigma^{(e)}$ the well-known Dirac operators of the nuclear particle and the heavy electron, respectively. The interaction constant G can be computed from the nuclear forces. An upper bound for G is derived in §3 of this paper, namely

$$G<7.8\times10^{-40}\text{ Mev cm}^3. \tag{4}$$

It is likely that G is considerably smaller.

The scattering of a heavy electron by a nuclear particle is then a first-order process in this theory. The cross section is calculated in §2 and the result for heavy electrons whose energies are small compared to Mc^2 (M is the proton mass) is:

$$\sigma=\frac{3}{2\pi}\frac{G^2}{\hbar^4c^4}(E^2+\mu^2c^4). \tag{5}$$

The cross section for higher energies ($E\gg\mu c^2$) is given by ($E_0=Mc^2$):

$$\sigma=\frac{G^2}{\pi\hbar^4c^4}\left[\frac{EE_0(E+\frac{1}{4}E_0)}{E+\frac{1}{2}E_0}-\tfrac{1}{4}E_0^2\log\frac{2E+E_0}{E_0}\right]. \tag{6}$$

[7] R. E. Marshak, Phys. Rev. **57**, 1101 (1940); this paper will be referred to as I.

The values for the cross section computed on the basis of (4), (5) and (6) for three characteristic energies are given in Table I. They are much smaller than the cross sections calculated using the theories of the meson with spin one. The values quoted are upper bounds, since the actual G is certainly smaller than the value (4) used here. The results are, therefore, smaller than the observed upper limit for the cross section.

The cross section increases for high energies ($E \gg Mc^2$) with the first power of E, just as the corresponding expression for the scattering of mesons with spin 1. In our theory the value 10^{-26} cm^2 is first attained for energies $E \approx 10^{11}$ ev and is therefore not in conflict with any experiment performed thus far. There is, however, reason to doubt the validity of (6) for extremely high energies. It is known that the evaluation of the nuclear forces necessitates a "cut-off" of the potential energy between nuclear particles at small distances. Whatever the reason for this "cut-off" is—it may be caused either by the effect of higher approximations or by the breakdown of the interaction (3) at high energies—it represents a strong argument against the application of our theory to processes involving high energy heavy electrons.

The result that the cross section for the scattering of heavy electrons of spin $\frac{1}{2}$ is so much smaller than of mesons of spin 1 can be traced to the circumstance that the present theory has a much smaller interaction parameter. If we express G by a dimensionless number,

$$G = \Gamma \mu c^2 (\hbar/\mu c)^3, \tag{7}$$

the value of Γ is 0.95×10^{-2}, which is 40 times smaller than the corresponding dimensionless constant $g/(\hbar c)^{\frac{1}{2}}$ of the meson theory. This difference is due to the fact that the nuclear forces are represented by a field of two heavy electrons rather than one meson. Because of the larger statistical factors involved, a field of heavy electron pairs gives rise to relatively stronger nuclear forces than a field of single Bose mesons if the interaction parameters are equated to each other.

The calculations presented here show that a theory which describes the nuclear forces as arising from the emission and reabsorption of pairs of "mesons" of spin $\hbar/2$ is quite compatible with the experiments on the scattering of mesons by nuclei. It must be remembered, however, that the above result is derived by using the first approximation only of the perturbation method. Because of the divergences involved in the higher approximations, it is difficult to decide whether the same conclusion would be borne out by a more rigorous treatment.

§2. The Calculation of the Scattering Cross Section

The differential cross section $d\sigma$ for the scattering of a heavy electron into the solid angle $d\Omega$ by a free neutron or proton at rest is given by the well-known formula:

$$d\sigma = (2\pi/hv)\,|V_{ab}|^2 \rho d\Omega. \tag{8}$$

In Eq. (8) v is the velocity of the incident heavy electron, V_{ab} is the matrix element of the "tensor" interaction V between the heavy electron and the nuclear particle for the scattering of the meson; ρ is the density of final states of the heavy electron per unit volume, per unit energy and per unit solid angle. The process is of first order. Since we wish to examine the behavior of the scattering cross section for very energetic heavy electrons, we use the relativistic expressions for both nuclear particles and heavy electrons. At the same time the recoil of the nuclear particle and the consequent change of momentum of the heavy electron is taken into account. We may write[8] for V_{ab}:

$$V_{ab} = G\int dr^{(e)} \int dr^{(n)} \{ (\Psi_Q^* \beta^{(n)} \sigma^{(n)} \Psi_0)(\psi_q^* \beta^{(e)} \sigma^{(e)} \psi_p) + (\Psi_Q^* \beta^{(n)} \alpha^{(n)} \Psi_0)(\psi_q^* \beta^{(e)} \alpha^{(e)} \psi_p) \}. \tag{9}$$

In (9) the quantities Ψ_0, Ψ_Q are, respectively, the initial and final wave functions of the nuclear particle while ψ_p, ψ_q are the corresponding wave functions of the heavy electron; these wave functions are normalized per unit volume. Also $\sigma^{(n)}$, $\sigma^{(e)}$ are the Dirac four-component spin operators of the nuclear particle and heavy

[8] In I the nonrelativistic approximation (3) was used; this sufficed for the calculation of nuclear forces. Eq. (9) is the corresponding relativistic expression; cf. H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 82 (1936).

electron, respectively, whereas $\beta^{(n)}$, $\beta^{(e)}$; $\alpha^{(n)}$, $\alpha^{(e)}$ are the well-known Dirac operators. In the initial state a the nuclear particle is at rest so that its energy is[9] $E_0 = M/\mu$, while the heavy electron has momentum p and energy $E_i = (p^2+1)^{\frac{1}{2}}$; in the final state b the heavy electron has momentum q and energy $E_f = (q^2+1)^{\frac{1}{2}}$, the nuclear particle acquiring momentum $\mathbf{Q} = \mathbf{p} - \mathbf{q}$ and energy $E = E_0 + E_f - E_i = (Q^2 + (M/\mu)^2)^{\frac{1}{2}}$. In the notation used the density of states is given by:

$$\rho = qE_f/8\pi^3c^2\hbar^3. \tag{10}$$

In evaluating $|V_{ab}|^2$ we must average over the spins of both heavy electron and nuclear particle in the state a and sum over the spins of both particles in the state b. These operations and integration over the coordinate space of both particles leads to

$$\begin{aligned} |V_{ab}|^2 = \frac{G^2}{4}\sum_{k,\,l=1}^{3}\Bigg[&\frac{Spur}{4}\left\{\beta^{(e)}\sigma_k^{(e)}\left(1+\frac{\boldsymbol{\alpha}^{(e)}\cdot\mathbf{p}+\beta^{(e)}}{E_i}\right)\beta^{(e)}\sigma_l^{(e)}\left(1+\frac{\boldsymbol{\alpha}^{(e)}\cdot\mathbf{q}+\beta^{(e)}}{E_f}\right)\right\} \\ &\cdot\frac{Spur}{4}\left\{\beta^{(n)}\sigma_k^{(n)}(1+\beta^{(n)})\beta^{(n)}\sigma_l^{(n)}\left(1+\frac{\boldsymbol{\alpha}^{(n)}\cdot\mathbf{Q}+\beta^{(n)}(M/\mu)}{E}\right)\right\} \\ &+\frac{Spur}{4}\left\{\beta^{(e)}\alpha_k^{(e)}\left(1+\frac{\boldsymbol{\alpha}^{(e)}\cdot\mathbf{p}+\beta^{(e)}}{E_i}\right)\beta^{(e)}\alpha_l^{(e)}\left(1+\frac{\boldsymbol{\alpha}^{(e)}\cdot\mathbf{q}+\beta^{(e)}}{E_f}\right)\right\} \\ &\cdot\frac{Spur}{4}\left\{\beta^{(n)}\alpha_k^{(n)}(1+\beta^{(n)})\beta^{(n)}\alpha_l^{(n)}\left(1+\frac{\boldsymbol{\alpha}^{(n)}\cdot\mathbf{Q}+\beta^{(n)}(M/\mu)}{E}\right)\right\}\Bigg]. \end{aligned} \tag{11}$$

The cross terms which arise from (9) do not contribute to $|V_{ab}|^2$. Evaluation of (11) by means of the usual rules for taking products of Dirac operators gives:

$$\begin{aligned} |V_{ab}|^2 = \tfrac{3}{4}G^2\Bigg\{&\left(1+\frac{1+\frac{1}{3}\mathbf{p}\cdot\mathbf{q}}{(p^2+1)^{\frac{1}{2}}(q^2+1)^{\frac{1}{2}}}\right)\left(1+\frac{M/\mu}{(Q^2+(M/\mu)^2)^{\frac{1}{2}}}\right) \\ &+\left(1-\frac{1-\frac{1}{3}\mathbf{p}\cdot\mathbf{q}}{(p^2+1)^{\frac{1}{2}}(q^2+1)^{\frac{1}{2}}}\right)\left(1-\frac{M/\mu}{(Q^2+(M/\mu)^2)^{\frac{1}{2}}}\right)\Bigg\}. \end{aligned} \tag{12}$$

Of course, the second term in brackets which arises from the Dirac current operator, vanishes in the nonrelativistic approximation for the nuclear particle, i.e., in the limit $M\rightarrow\infty$. The first term in brackets, arising from the spin operators, does not vanish in the limit $M\rightarrow\infty$. We finally obtain the differential cross section $d\sigma$ as a function of the angle of scattering Θ by putting (12) and (10) into (8) and by observing that $v = cp/(p^2+1)^{\frac{1}{2}}$:

$$d\sigma = \frac{3\mu^2G^2}{8\pi^2\hbar^4}\left[(p^2+1)^{\frac{1}{2}}(q^2+1)^{\frac{1}{2}}+\frac{pq}{3}\cos\Theta + \frac{M/\mu}{(p^2+q^2-2pq\cos\Theta+(M/\mu)^2)^{\frac{1}{2}}}\right]\frac{q}{p}d\Omega. \tag{13}$$

[9] Just as in I, the unit of energy is μc^2, of momentum μc and of length $\hbar/\mu c$.

The momenta p and q are still measured in units of μc. When p is small compared to M/μ then $q\approx p$ for all angles Θ and we get for the total cross section:

$$\sigma = \frac{3\mu^2G^2}{2\pi\hbar^4}(p^2+2) \quad \text{for} \quad p \ll M/\mu. \tag{14}$$

The cross section is therefore finite in the limit of zero velocity. The integration over Θ can also be performed for $p\gg 1$; in this case the rest mass of the heavy electron can be neglected and in virtue of the conservation laws of momentum and energy it follows that

$$q = p(M/\mu)/((M/\mu)+p(1-\cos\Theta)).$$

Putting this value for q into (13) leads to an expression which can be integrated exactly with the result:

$$\sigma=\frac{1}{\pi}\frac{G^2\mu^2}{\hbar^4}\left[\frac{pE_0(p+\frac{1}{4}E_0)}{p+\frac{1}{2}E_0}-\frac{E_0^2}{4}\log\frac{2p+E_0}{E_0}\right]. \quad (15)$$

For energies large compared to M/μ, Eq. (15) reduces to

$$\sigma=\frac{1}{\pi}\frac{G^2\mu^2}{\hbar^4}pE_0. \quad (16)$$

It is seen from (16) that under extreme relativistic conditions the total cross section for the non-electric scattering of heavy electrons by nuclear particles increases linearly with the incident energy of the heavy electron.

§3. The Determination of the Interaction Constant

We follow Bethe's procedure[10] in order to evaluate the interaction constant G from the nuclear forces. Since the experimental material only provides an upper value for the scattering cross section, it will suffice for our purposes to find an upper bound for G; the actual value may be several times smaller.

The strong singularity of the nuclear forces which results from the heavy electron pair theory necessitates a "cut-off" at small distances. The "cut-off" may be made in different ways; in the following we use the "zero cut-off" method which assumes the nuclear potential to vanish within a certain "cut-off" distance r_0. In this way the constant G will be overestimated since we may safely assume that the actual nuclear force does not vanish completely for small distances. There are now two magnitudes to be determined: the constant G and the "cut-off" distance r_0. However, values for G and r_0 may be obtained by making use of the singlet scattering of neutrons by protons and of the fact that the triplet ground state of the deuteron is lower than the singlet state.

The radial part $u(r)$ of the wave function describing the scattering of a neutron of zero energy by a proton is given by the equation:

$$(d^2u(r))/(dr^2)=-V(r)u(r). \quad (17)$$

Here r is measured in units of $\hbar/\mu c$ and $V(r)$ in units of $\mu^2c^4/Mc^2=8.7$ Mev. $V(r)$ is the singlet potential given in I (cf. Eq. (8)):

$$V(r)=aF(r), \quad a=16\pi^3G^2, \quad (18a)$$

with

$$\begin{cases} F(r)=\left(\dfrac{K_0(2r)}{r^3}+\dfrac{4K_1(2r)}{r^2}+\dfrac{K_1(2r)}{r^4}\right) & \text{for } r>r_0, \\ F(r)=0 & \text{for } r<r_0. \end{cases} \quad (18b)$$

In (18) $K_\nu(z)$ is a Bessel function of order ν and related to the well-known Hankel function of the first kind:

$$K_\nu(z)=\frac{\pi}{2}ie^{\pi i\nu/2}H_\nu^{(1)}(iz).$$

Bethe[10] finds from the observed neutron-proton scattering cross section that, for large r, $u(r)$ is given by

$$u=1+0.1r. \quad (19)$$

The constant a in (18a) is to be adjusted so that $u(0)=0$. It is possible to get an upper limit for a by replacing $F(r)$ by another function $F_0(r)$ for which $F(r)\geqslant F_0(r)$ for all values of r. If $u(0)=0$ for the potential $V(r)=b\cdot F_0(r)$, b is an upper limit to the constant a. We have chosen for $F_0(r)$ the function $F_0(r)=Ce^{-\alpha r}$ which coincides in value and first derivative with $F(r)$ at the "cut-off" point; i.e.,

$$F(r_0)=F_0(r_0), \quad F'(r_0)=F_0'(r_0);$$

these two equations determine C and α. The differential equation (17) is now soluble and we obtain with the potential $V=bF_0(r)$ the solution:

$$\begin{cases} u=J_0(se^{-\alpha r/2})-0.1\left(\dfrac{\pi}{\alpha}N_0(se^{-\alpha r/2})-\dfrac{2}{\alpha}\log\dfrac{\gamma s}{2}\right) & \text{for } r>r_0, \\ u=Kr & \text{for } r<r_0. \end{cases} \quad (20)$$

In (20) $s=2(bC)^{\frac{1}{2}}/\alpha$, J_0, N_0 are Bessel functions and γ is the Bernouilli number. The quantity K should be adjusted so that u and du/dr are continuous at $r=r_0$. This is only possible for a suitable value of s, the lowest one of which finally determines b. Several values of b for different "cut-off" radii r_0 are listed in Table II.

[10] H. A. Bethe, Phys. Rev. 57, 390 (1940).

TABLE II. *Values of b for different "cut-off" radii r_0.*

r_0	0.2	0.4	0.6	0.8
b	0.041	0.27	0.56	0.90

TABLE III. *Values of the distance p for several values of a.*

a	1.5	0.75	0.35	0.12	0.04
p	0.63	0.56	0.50	0.39	0.29

At this point we make use of the fact that the triplet state of the deuteron lies lower than the singlet state. The triplet state is given by two simultaneous differential equations:

$$\begin{cases} (d^2\chi)/(dr^2)-\epsilon\chi=A\chi-D\varphi, \\ (d^2\varphi)/(dr^2)-\epsilon\varphi=-D\chi+C\varphi. \end{cases} \quad (21)$$

ϵ is the energy of the triplet state of the deuteron. In (21) χ is the S wave function and φ the D wave function; also we have:

$$A=\frac{a}{3}F(r);\quad C=\frac{1}{3}aF(r)+\frac{6}{r^2}+\frac{2a}{3}G(r);$$

$$D=\frac{2^{\frac{3}{2}}}{3}aG(r);$$

$$G(r)=\begin{cases} \left(\dfrac{5K_0(2r)}{r^3}+\dfrac{2K_1(2r)}{r^2}+\dfrac{5K_1(2r)}{r^4}\right) & \text{for } r>r_0, \\ 0 & \text{for } r<r_0. \end{cases}$$

According to Bethe[10] these equations are almost equivalent to a single Schrödinger equation with the potential:

$$W(r)=\tfrac{1}{2}(A+C)\pm\left[\tfrac{1}{4}(A-C)^2+D^2\right]^{\frac{1}{2}}$$
$$=\frac{a}{3}F(r)+\frac{3}{r^2}+aG(r)\pm\left[\left(aG+\frac{1}{r^2}\right)^2+\frac{8}{r^4}\right]^{\frac{1}{2}}. \quad (22)$$

The lower sign holds for the ground state of the deuteron, and should therefore lead to an eigenvalue lower than the lowest singlet state given by Eq. (17). This state of affairs, however, would be impossible if $[W(r)-V(r)]$ were positive or zero for all values of r. It is easily seen that $W-V\geqslant 0$ for all values of r greater than a certain value p. The distance p is a function of a and several values of $p(a)$ are tabulated in Table III. In order to depress the triplet state below the singlet state, the cut-off radius r_0 must be smaller than p, thereby providing a region in which $W-V$ is negative. We therefore obtain an upper limit for r_0 by the condition $r_0<p$.[11] Let us now compare $p(a)$, which is a monotonically increasing function of a, and $b(r_0)$ which is an upper bound of a and determined in the preceding paragraph; $b(r_0)$ is also a monotonically increasing function of r_0. Supposing there is a value b^* of the second function belonging to a value r_0^*, so that r_0^* in turn is equal to the first function $p(a)$ for $a=b^*$; this is actually the case for $b^*=0.46$. The following inequalities are then valid:

$$b^*>b(r_0)>a.$$

Here r_0 is the actual cutting off radius ($r_0<r_0^*$) and a the actual value of the force constant. Thus b^* is an upper bound of a.

The relation between a and G is given in (18) and we get:

$$G^2=a/(16\pi^3)<b^*/(16\pi^3).$$

Therefore after introducing other units, we have:[12]

$$G^2<0.9\times10^{-4}\mu^2c^4(\hbar/(\mu c))^6. \quad (23)$$

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[11] We have checked this procedure by applying it to Bethe's theory; according to Bethe the correct value of a is 1.66 for $r_0=0.3$. For this value of a we find $p=0.8$.

[12] It is interesting to note that one gets a value of G^2 close to (23) by the following approximate method: one takes the experimental proton potential from the paper of Hoisington, Share and Breit (cf. Phys. Rev. **56**, 884 (1939)), i.e., $J(r)=90mc^2e^{-r/\alpha}/(r/\alpha)$, $(\alpha=\hbar/2\mu c)$ and sets it equal to the singlet potential (18) at $r=\alpha$; one finds $G^2=1.6\times10^{-5}\mu^2c^4(\hbar/\mu c)^6$. If one sets the two equal at $r=2\alpha$, one finds $G^2=8.5\times10^{-5}\mu^2c^4(\hbar/\mu c)^6$.