The Electromagnetic Properties of Mesotrons

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(Received June 27, 1940)

A general theory describing particles of unit spin and arbitrary magnetic moment is developed and applied to the motion of such particles in a Coulomb field. In the particular case of magnetic moment unity (Proca theory), the exact equations for the radial components of the wave functions possess regular solutions only for those states ($j=l\neq 0$ and $j=0$, $l=1$) in which the orbital angular momentum l is a constant of the motion. For particles possessing a magnetic moment of two mesotron magnetons, the radial equations are free of singularities for all states but two: $j=1$, $l=0$ and $j=0$, $l=1$. The cross section for a fractional energy transfer to electrons by energetic mesotrons is calculated for the various simple possibilities of mesotron spin σ $(0, \frac{1}{2}, 1)$ and magnetic moment μ (arbitrary except for zero spin), and it is shown that only for $\sigma=\frac{1}{2}$, $\mu\neq 1$ (in particular, $\mu=0$) and $\sigma=1$, $\mu=1$ is the cross section of the correct magnitude and form (i.e., essentially independent of the mesotron energy) to account for observed burst phenomena at energies greater than 2×10^{10} ev. Criteria for the validity of these formulae indicate that the region of applicability for the theory $\sigma=1$, $\mu\neq 1$ is more limited than that for $\sigma=1$, $\mu=1$ (Proca).

I. INTRODUCTION

THE electromagnetic properties of nuclear systems depend on the interaction of mesotrons both with heavy particles and with the electromagnetic field. Indeed, so simple a quantity as the magnetic moment of the neutron is completely determined only by both of these interactions. Data obtained experimentally from electromagnetic nuclear properties do not therefore determine independently the mesotron-heavy particle coupling and the mesotron-radiation interaction, and it would be of advantage to study each of these forms of interaction directly. Information concerning the influence of the electromagnetic field on mesotrons may be obtained by the investigation of recoil electrons resulting from collisions with the mesotrons forming the penetrating component of cosmic radiation. This process presumably accounts for the fraction of the soft component (apart from secondaries arising from the disintegration of mesotrons in the atmosphere) which is in equilibrium with the hard component. The observed dependence of cosmic-ray bursts on their magnitude and the surrounding material indicates that mesotrons may, with appreciable probability, transfer a large fraction of their energy to the soft component. Phenomena associated with bursts of energy greater than 2×10^{10} ev may be adequately described by assuming that the cross section for a given fractional transfer of energy to the soft component is independent of the mesotron energy. A cross section of this nature possessing the correct magnitude would indeed be obtained[1,2] if the electromagnetic properties of the mesotron were those deduced from the Proca equations[3] by the application of the Born approximation.

To examine the validity of the Born approximation, one may consider the limiting case of scattering by an electron of infinite mass, the process involved then being the scattering of mesotrons by a static Coulomb field. An exact solution of this problem would enable one to judge the reliability of the Born approximation as applied to this type of process. Unfortunately, as will be shown in Section III, when one attempts to follow this program one encounters singular equations which admit of no complete set of regular solutions. This implies either that the theory is wrong or that the model of the scattering of a mesotron by a Coulomb field is too great an abstraction of the physical process to which it purports to approximate.

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[2] H. S. W. Massey and H. C. Corben, Proc. Camb. Phil. Soc. **35**, 463 (1939).
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The electromagnetic properties of a particle with given charge and mass are essentially determined by its spin and its magnetic moment. One may therefore use the experimental evidence available from cosmic-ray measurements to determine these two characteristics of the mesotron. Of the three simple possibilities for the spin (0, $\frac{1}{2}$, 1) the case of zero spin may be excluded, for the impossibility of associating an intrinsic magnetic moment with a scalar mesotron implies an energy transfer cross section varying inversely with the mesotron energy, thus contradicting the experimental evidence. Were the spin of the mesotron $\frac{1}{2}$, one could exclude the possibility of a magnetic moment of one mesotron magneton, as predicted by the Dirac theory, for this would lead to a cross section of similar character.[4] However, an alteration of the magnetic moment changes the energy dependence of the cross section to that demanded by experiment. On the basis of cosmic-ray evidence, therefore, one cannot exclude the possibility of a mesotron of spin $\frac{1}{2}$ and magnetic moment different from unity (in particular, zero) although such evidence as is available from nuclear phenomena indicates that this is not likely.

The current theory of mesotrons postulates a spin of unity and a magnetic moment of one mesotron magneton. The physical adequacy and the mathematical difficulties of this theory have already been mentioned. The sole remaining simple possibility is that of a mesotron of spin unity and magnetic moment different from that assumed in the Proca theory. The following pages are devoted to the development of a general Lagrangian theory of such particles (Section II) and the application of the theory to the motion of mesotrons in a Coulomb field (Section III) and the problem of electron recoils (Section IV).

II. General Theory

The most general bilinear Lagrangian density which involves only a four-vector ϕ_μ, and the electromagnetic potentials A_ν in the form of the first-order gauge covariant derivative

$$D_\nu\phi_\mu=[\partial_\nu+(ie/\hbar c)A_\nu]\phi_\mu,$$

together with the conjugate complex quantities $\bar{\phi}_\mu$, $\bar{D}_\nu\bar{\phi}_\mu$ is

$$L=A^{\mu\nu}_{\sigma\tau}(\bar{D}_\mu\bar{\phi}^\sigma)(D_\nu\phi^\tau)+\kappa^2\bar{\phi}^\sigma\phi_\sigma, \tag{1}$$

a quantity which automatically satisfies the condition of gauge invariance. Here the $A^{\mu\nu}_{\sigma\tau}$ are numerical tensors which can depend only on the values of the metric

$$\begin{aligned} g_{\mu\nu}&=0 \quad (\mu\neq\nu),\\ &=1 \quad (\mu=\nu=1,2,3),\\ &=-1 \quad (\mu=\nu=4). \end{aligned}$$

A general form of such a tensor is

$$A^{\mu\nu}_{\sigma\tau}=g^{\mu\nu}g_{\sigma\tau}+\beta g^{\mu}_{\tau}g^{\nu}_{\sigma}+\gamma g^{\mu}_{\sigma}g^{\nu}_{\tau}, \tag{2}$$

where β and γ are constants. The choice of unity for the first coefficient on the right-hand side of (2) is not an essential limitation. To guarantee the reality of L, one must impose the condition

$$\bar{A}^{\mu\nu}_{\sigma\tau}=A^{\nu\mu}_{\tau\sigma} \tag{3}$$

which implies that β and γ are real.

The equations of motion derived by variation of (1) are

$$A^{\mu\nu}_{\sigma\tau}D_\mu D_\nu\phi^\tau=\kappa^2\phi_\sigma \tag{4}$$

and the conjugate complex equations. Inserting the form (2) for the $A^{\mu\nu}_{\sigma\tau}$ one obtains

$$D^\mu D_\mu\phi_\sigma+\beta D_\nu D_\sigma\phi^\nu+\gamma D_\sigma D_\nu\phi^\nu=\kappa^2\phi_\sigma$$

or

$$D^\mu D_\mu\phi_\sigma+(\beta+\gamma)D_\sigma(D_\nu\phi^\nu)+(ie/\hbar c)\beta\phi^\nu H_{\nu\sigma}=\kappa^2\phi_\sigma, \tag{5}$$

where we have used the commutation relation

$$D_\mu D_\nu-D_\nu D_\mu=(ie/\hbar c)H_{\mu\nu}$$

and the $H_{\mu\nu}$ are the strengths of the electromagnetic field.

Specializing Eqs. (5) to the field-free case one obtains

$$\partial^\mu\partial_\mu\phi_\sigma+(\beta+\gamma)\partial_\sigma(\partial_\nu\phi^\nu)=\kappa^2\phi_\sigma. \tag{6}$$

Solutions of these equations may be written in the form

$$\phi_\sigma=\psi_\sigma+\partial_\sigma\varphi,$$

where ψ^σ satisfies the condition $\partial_\sigma\psi^\sigma=0$ and φ

[4] H. J. Bhabha, Proc. Roy. Soc. **A164**, 257 (1938).

is a scalar. Hence

$$(\partial^\mu\partial_\mu-\kappa^2)\psi_\sigma=0;\quad (\beta+\gamma+1)\partial^\mu\partial_\mu\varphi=\kappa^2\varphi. \tag{7}$$

Writing $\kappa=Mc/\hbar$ we may identify the first of Eqs. (7) as that satisfied by a particle of spin unity and mass M, and the second as that of a scalar particle of mass $M(\beta+\gamma+1)^{-\frac{1}{2}}$. One therefore has a theory of a particle which may exist in two states, with which are associated different spins and different masses. The theory may be restricted to particles of spin 1 by the choice

$$\beta+\gamma+1=0, \tag{8}$$

which implies $\varphi=0$. Were this restriction not imposed, there would be a finite probability for a particle to change its mass and spin in an external electromagnetic field, a phenomenon for which there is no experimental evidence.

With this restriction, the coefficients assume the form

$$\overset{\mu\nu}{A}_{\sigma\tau}=(\overset{\mu\nu}{g}g_{\sigma\tau}-\overset{\mu}{g}_\tau\overset{\nu}{g}_\sigma)+\gamma(\overset{\mu}{g}_\sigma\overset{\nu}{g}_\tau-\overset{\mu}{g}_\tau\overset{\nu}{g}_\sigma), \tag{9}$$

to which corresponds the Lagrangian density

$$L=\tfrac{1}{2}\bar{F}_{\mu\nu}F^{\mu\nu}+\kappa^2\bar{\phi}_\mu\phi^\mu+\gamma\{(\bar{D}_\mu\bar{\phi}^\mu)(D_\nu\phi^\nu)-(\bar{D}_\nu\bar{\phi}^\mu)(D_\mu\phi^\nu)\}, \tag{10}$$

where $F_{\mu\nu}=D_\mu\phi_\nu-D_\nu\phi_\mu$. This Lagrangian may be simplified by noting that the expression in brackets may be rewritten

$$-\partial_\mu(\bar{\phi}^\mu D_\mu\phi^\nu)+\partial_\mu(\bar{\phi}^\mu D_\nu\phi^\nu)-(ie/2\hbar c)H_{\mu\nu}(\bar{\phi}^\mu\phi^\nu-\bar{\phi}^\nu\phi^\mu). \tag{11}$$

The first two terms of (11) may be disregarded, for, possessing the form of divergences, they contribute nothing to the variation of the Lagrangian. We have therefore arrived at a convenient expression for the Lagrangian density,

$$L=\tfrac{1}{2}\bar{F}_{\mu\nu}F^{\mu\nu}+\kappa^2\bar{\phi}_\mu\phi^\mu-(ie/2\hbar c)\gamma H_{\mu\nu}(\bar{\phi}^\mu\phi^\nu-\bar{\phi}^\nu\phi^\mu), \tag{12}$$

which differs from that of current theory by the addition of a term involving the explicit appearance of the electromagnetic field strengths. This term may be interpreted as corresponding to an additional magnetic moment for the mesotron.

The linear equations of motion which follow from (12) are

$$D^\mu F_{\mu\nu}=\kappa^2\phi_\nu+(ie/\hbar c)\gamma\phi^\mu H_{\mu\nu};\quad D_\mu\phi_\nu-D_\nu\phi_\mu=F_{\mu\nu}. \tag{13}$$

The electromagnetic current four-vector of the mesotrons which follows by variation of the Lagrangian with respect to $-A_\mu$ then assumes the form

$$j^\mu=(ie/\hbar c)[\bar{\phi}_\nu F^{\mu\nu}-\bar{F}^{\mu\nu}\phi_\nu+\gamma\partial_\nu(\bar{\phi}^\mu\phi^\nu-\bar{\phi}^\nu\phi^\mu)], \tag{14}$$

which differs from that of the usual theory by the term in γ which is independently conserved. To show the interpretation of this additional term, we rewrite the expression for the current in the form

$$j^\mu=(ie/\hbar c)[(\bar{\phi}_\nu D^\mu\phi^\nu-\phi_\nu\bar{D}^\mu\bar{\phi}^\nu)+(\gamma+1)\partial_\nu(\bar{\phi}^\mu\phi^\nu-\bar{\phi}^\nu\phi^\mu)+\phi^\mu(\bar{D}_\nu\bar{\phi}^\nu)-\bar{\phi}^\mu(D_\nu\phi^\nu)]. \tag{15}$$

The second term represents a polarization current arising from the magnetic moment. Indeed, its coefficient is to be identified as the magnetic moment of the mesotron, which thus has on this theory the value $1+\gamma$ mesotron magnetons. The first term has the appearance of a convection current yet may not receive this interpretation for it is not by itself conserved. It is, in fact, conserved only upon the addition of the terms involving $D_\mu\phi^\mu$—terms which have no simple physical interpretation. The importance of the proposed generalization now becomes apparent, for, by proper choice of γ these terms may be made to vanish under certain conditions.

One may evaluate $D_\nu\phi^\nu$ by applying the operator D_ν to the first of Eqs. (13), with the result

$$\kappa^2 D_\nu\phi^\nu=(ie/2\hbar c)(\gamma-1)H_{\mu\nu}F^{\mu\nu}-(4\pi ie/\hbar c^2)\gamma J_\nu\phi^\nu, \tag{16}$$

where J_ν, the current-vector of the external charges, has been introduced by means of the equation

$$\partial^\mu H_{\mu\nu}=-(4\pi/c)J_\nu.$$

Thus in regions outside those occupied by the external charges (i.e., $J_\nu=0$) the choice $\gamma=1$ implies $D_\nu\phi^\nu=0$. It would then appear that the choice of $\gamma=1$, which in the absence of external charges guarantees the subdivision of the current into independently conserved conduction and convection currents and imparts to the particle a gyromagnetic ratio of 2, places the theory in close analogy with the Dirac theory of the electron.

This simplification finds its counterpart in the equations of motion

$$D^\mu D_\mu\phi_\nu - D_\nu(D_\mu\phi^\mu)=\kappa^2\phi_\nu+(ie/\hbar c)(\gamma+1)\phi^\mu H_{\mu\nu}, \tag{17}$$

which are obtained from (5) by writing $\beta=-(\gamma+1)$. In comparison with the rather complicated equations of the current theory ($\gamma=0$) *viz.:*

$$(D^\mu D_\mu-\kappa^2)\phi_\nu=(ie/\hbar c)\phi^\mu H_{\mu\nu}-(ie/2\hbar c)(1/\kappa^2)D_\nu(H_{\mu\sigma}F^{\mu\sigma}) \tag{18}$$

one obtains, for $\gamma=1$,

$$(D^\mu D_\mu-\kappa^2)\phi_\nu=(2ie/\hbar c)\phi^\mu H_{\mu\nu}-(4\pi ie/\hbar c^2\kappa^2)D_\nu(J_\mu\phi^\mu), \tag{19}$$

which in regions external to charge and current distributions reduce to

$$(D^\mu D_\mu-\kappa^2)\phi_\nu=(2ie/\hbar c)\phi^\mu H_{\mu\nu}. \tag{20}$$

To complete the theory, it is necessary to construct the stress-energy-momentum tensor and show that it satisfies all conditions that may be reasonably imposed on it. The real tensor constructed by the recipe

$$T^\mu{}_\nu=2\{(\partial L/\partial\bar F_{\mu\sigma})\bar D_\nu\bar\phi_\sigma+(\partial L/\partial F_{\mu\sigma})D_\nu\phi_\sigma+(\partial L/\partial H_{\mu\sigma})H_{\nu\sigma}\}-\delta_\nu{}^\mu L \tag{21}$$

satisfies the conservation equation

$$\partial_\mu T^\mu{}_\nu=-j^\mu H_{\mu\nu} \tag{22}$$

and thus may be interpreted as a stress tensor. For the Lagrangian (12), it assumes the explicit form:

$$T_{\mu\nu}=F_{\mu\sigma}\bar D_\nu\bar\phi^\sigma+\bar F_{\mu\sigma}D_\nu\phi^\sigma-(ie/\hbar c)\gamma H_{\nu\sigma}(\bar\phi_\mu\phi^\sigma-\bar\phi^\sigma\phi_\mu)-g_{\mu\nu}L. \tag{23}$$

In order to guarantee the conservation of angular momentum it is necessary that a theory yield a symmetric stress tensor. Although (23) is not symmetric, it may be replaced by the symmetric tensor[5]

$$\Theta_{\mu\nu}=\bar F_{\mu\sigma}F_\nu{}^{\cdot\sigma}+\bar F_{\nu\sigma}F_\mu{}^{\cdot\sigma}+\kappa^2(\bar\phi_\mu\phi_\nu+\bar\phi_\nu\phi_\mu)+(ie/\hbar c)\gamma\{H_{\sigma\mu}(\bar\phi_\nu\phi^\sigma-\bar\phi^\sigma\phi_\nu)+H_{\sigma\nu}(\bar\phi_\mu\phi^\sigma-\bar\phi^\sigma\phi_\mu)\}-g_{\mu\nu}L, \tag{24}$$

which also satisfies the conservation equation

$$\partial_\mu\Theta^\mu{}_\nu=-j^\mu H_{\mu\nu}$$

since

$$\Theta^\mu{}_\nu=T^\mu{}_\nu-\partial_\sigma\{F^{\mu\sigma}\bar\phi_\nu+\bar F^{\mu\sigma}\phi_\nu\}. \tag{25}$$

Although the energy density derived from this stress tensor is not positive definite, it does become so in the absence of an external electromagnetic field, which is all that may reasonably be required of the energy density derived from a theory of particles with integral spin.

III. Stationary States in a Coulomb Field

The importance of a rigorous treatment of the equations representing the mesotron in a Coulomb field has been stressed in the Introduction. An attempt to construct exact solutions of these equations forms the basis of this section.

The spherical symmetry of the Coulomb problem introduces the possibility of defining conserved

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[5] This symmetric tensor may be also derived directly from the Lagrangian by the prescription

$$\Theta^\mu{}_\nu=2\{(\partial L/\partial\bar F_{\mu\sigma})\bar F_{\nu\sigma}+(\partial L/\partial\bar\phi_\mu)\bar\phi_\nu+(\partial L/\partial F_{\mu\sigma})F_{\nu\sigma}+(\partial L/\partial\phi_\mu)\phi_\nu+(\partial L/\partial H_{\mu\sigma})H_{\nu\sigma}\}-\delta_\nu{}^\mu L.$$

angular momentum integrals. Appropriate definitions, in terms of the stress tensor $\Theta_{\mu\nu}$ are:

$$M_{jk}=\frac{1}{c^2}\int(x_j\Theta_{k4}-x_k\Theta_{j4})d\tau, \tag{26}$$

which, by employing the definition (24) of $\Theta_{\mu\nu}$ may be written

$$M_{jk}=(i/c^2)\int\{\bar{F}_{4l}(-i)(x_j\partial_k-x_k\partial_j)\phi^l-F_{4l}\,i(x_j\partial_k-x_k\partial_j)\bar{\phi}^l\}d\tau$$
$$+(i/c^2)\int\{(-i)(\bar{F}_{4j}\phi_k-\bar{F}_{4k}\phi_j)-i(F_{4j}\bar{\phi}_k-F_{4k}\bar{\phi}_j)\}d\tau. \tag{27}$$

The first term, with its associated operator

$$L_{jk}=-i(x_j\partial_k-x_k\partial_j)$$

is obviously to be interpreted as the orbital angular momentum, thus endowing the additional term with the properties of spin angular momentum. The operator representing the total angular momentum may therefore be defined by the equation

$$J_{jk}\phi^l=L_{jk}\phi^l-i(\delta_j{}^l\delta_{km}-\delta_k{}^l\delta_{jm})\phi^m, \tag{28}$$

from which one infers the operational definition of J^2, the square of the total angular momentum, *viz.*:

$$J^2\phi_l=(L^2+2)\phi_l-2iL_{lj}\phi^j. \tag{29}$$

To express these formulae in a more amenable form, we may introduce space-vector notation, representing the space-components of ϕ^μ by the vector $\mathbf{\Phi}$ and the time-component ϕ^4 by the scalar function $i\varphi$. The angular momentum definitions thus assume the guise of vector formulae:

$$J_z\mathbf{\Phi}=L_z\mathbf{\Phi}+i\mathbf{e}_z\times\mathbf{\Phi},\quad \text{etc.},\quad J^2\mathbf{\Phi}=(L^2+2)\mathbf{\Phi}+2i\mathbf{L}\times\mathbf{\Phi}, \tag{30}$$

wherein $\mathbf{e}_z$ denotes a unit vector in the z direction. The analogous definitions involving operations on the scalar φ are simply

$$J_z\varphi=L_z\varphi;\quad J^2\varphi=L^2\varphi. \tag{31}$$

Stationary states of the system may be characterized by m and $j(j+1)$, the proper values of J_z and J^2, respectively, and by the parity, the eigenvalue of the reflection operator R which is defined by

$$R\mathbf{\Phi}(\mathbf{r})=\mathbf{\Phi}(-\mathbf{r});\quad R\varphi(\mathbf{r})=-\varphi(-\mathbf{r}). \tag{32}$$

This latter equation is a consequence of the opposite behavior of vector and scalar functions under spatial reflection. The construction of stationary state wave functions is facilitated by the consideration of auxiliary $\mathbf{\Phi}$-functions which are also eigenfunctions of L^2 associated with the proper value $l(l+1)$. By virtue of Eq. (30), such auxiliary wave functions satisfy the condition:

$$(j(j+1)-l(l+1)-2)\mathbf{\Phi}=2i(\mathbf{L}\times\mathbf{\Phi}). \tag{33}$$

The detailed consideration of this equation is deferred to the Appendix, wherein it is shown that there exist three types of auxiliary $\mathbf{\Phi}$-functions which may be symbolically expressed in the form:

$$\begin{aligned} l=j:&\quad \mathbf{\Phi}=\mathbf{L}F(r)P_j{}^m,\\ l=j+1:&\quad \mathbf{\Phi}=\{(j+1)(\mathbf{r}/r)+i(\mathbf{r}/r)\times\mathbf{L}\}F^{(1)}(r)P_j{}^m,\\ l=j-1:&\quad \mathbf{\Phi}=\{-j(\mathbf{r}/r)+i(\mathbf{r}/r)\times\mathbf{L}\}F^{(2)}(r)P_j{}^m,\end{aligned} \tag{34}$$

involving three arbitrary radial functions. The general solution associated with the eigenvalue j is obtained by linear combination of these auxiliary $\mathbf{\Phi}$-functions. The existence of the parity quantum number permits a further classification of these solutions, for the parity associated with a state of orbital angular momentum l is $(-1)^l$, thus assigning to the state $l=j$ a parity opposite to that of the state $l=j\pm1$. The state $l=j$ and a linear combination of the states $l=j\pm1$ therefore constitute independent stationary states. No further sub-division will be possible in general, for the spin-orbit coupling introduced by relativistic requirements destroys the constancy of orbital angular momentum. Turning to the scalar function φ, we observe from (31) that

$$\varphi=G(r)P_j^m,$$

introducing an additional arbitrary radial function $G(r)$. For $l=j$ the requirement that φ and $\mathbf{\Phi}$ have opposite reflection characteristics can be made compatible with the equal parities of these two functions only by demanding that $\varphi\equiv0$. These remarks find their complete expression in the following formulae for the two distinct types of admissible solutions:

$$l=j:\quad \begin{cases}\mathbf{\Phi}=\mathbf{L}F(r)P_j^m,\\ \varphi=0,\end{cases}\qquad l=j\pm1:\quad \begin{cases}\mathbf{\Phi}=\{(\mathbf{r}/r)F_1(r)+i[(\mathbf{r}/r)\times\mathbf{L}]F_2(r)\}P_j^m,\\ \varphi=G(r)P_j^m,\end{cases}\tag{35}$$

which are written in terms of the linear combinations

$$F_1(r)=(j+1)F^{(1)}(r)-jF^{(2)}(r),\quad F_2(r)=F^{(1)}(r)+F^{(2)}(r).\tag{36}$$

These formulae express all the information available from the general symmetry properties of the system; to proceed further we must resort to the specific dynamical equations. The equations (18) of current theory, specialized to represent the motion of a mesotron with energy W in a static field described by a scalar potential $V(r)$, read

$$\begin{aligned}\left\{\nabla^2+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}\mathbf{\Phi}&=-\frac{e}{\hbar c}\frac{\mathbf{r}}{r}\frac{dV}{dr}\varphi+\frac{e\hbar}{M^2c^3}\nabla\left[\frac{dV}{dr}\left\{\frac{W+eV}{\hbar c}\frac{\mathbf{r}}{r}\cdot\mathbf{\Phi}-\frac{\partial\varphi}{\partial r}\right\}\right],\\ \left\{\nabla^2+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}\varphi&=\frac{e}{\hbar c}\frac{dV}{dr}\frac{\mathbf{r}}{r}\cdot\mathbf{\Phi}+\frac{e\hbar}{M^2c^3}\frac{W+eV}{\hbar c}\frac{dV}{dr}\left\{\frac{W+eV}{\hbar c}\frac{\mathbf{r}}{r}\cdot\mathbf{\Phi}-\frac{\partial\varphi}{\partial r}\right\},\end{aligned}\tag{37}$$

when translated into vector notation. These equations must be supplemented by the subsidiary condition contained in (16), *viz.:*

$$\nabla\cdot\mathbf{\Phi}+\frac{W+eV}{\hbar c}\varphi=\frac{e\hbar}{M^2c^3}\frac{dV}{dr}\left\{\frac{W+eV}{\hbar c}\frac{\mathbf{r}}{r}\cdot\mathbf{\Phi}-\frac{\partial\varphi}{\partial r}\right\}.\tag{38}$$

This set of equations then yields the radial equations appropriate to the two types of solutions (35). The treatment of the state $l=j$ affords no difficulty, for $\varphi=0$, $(\mathbf{r}/r)\cdot\mathbf{\Phi}=0$, $\nabla\cdot\mathbf{\Phi}=0$ $(\mathbf{L}=-i\mathbf{r}\times\nabla)$, thus effectively reducing the set of Eqs. (37), (38) to

$$\left\{\nabla^2+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}\mathbf{\Phi}=0,$$

or, in radial form

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}F(r)=0.\tag{39}$$

This is simply the canonical form of the Klein-Gordon equation, which requires no further attention. The radial equations for the second type of solution $(l=j\pm1)$ are readily derived with the aid of a few elementary lemmata. The gradient operator finds its most convenient expression in the form:

$$\nabla=\frac{\mathbf{r}}{r}\frac{\partial}{\partial r}-i\frac{\mathbf{r}\times\mathbf{L}}{r^2}\tag{40}$$

with the derivative relation

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{L^2}{r^2}. \tag{40a}$$

Further

$$L^2\mathbf{\Phi}=\left\{\frac{\mathbf{r}}{r}(j(j+1)F_1+2F_1+2j(j+1)F_2)+i\left(\frac{\mathbf{r}}{r}\times\mathbf{L}\right)(j(j+1)F_2+2F_1)\right\}P_j^m,$$

which is most easily obtained by reverting to the original expression of $\mathbf{\Phi}$ in terms of eigenfunctions of L^2. It is now possible to write the terms of the first of Eqs. (37) as linear combinations of the two angular functions $(\mathbf{r}/r)P_j^m$ and $i[(\mathbf{r}/r)\times\mathbf{L}]P_j^m$, the coefficients of which yield the desired radial equations for F_1 and F_2. The radial equation corresponding to the latter part of (37) may be written down by inspection, while to perform a similar operation for (38) requires only the observation that

$$\nabla\cdot\mathbf{\Phi}=\left\{\nabla\cdot\frac{\mathbf{r}}{r}F_1+i\nabla\cdot\frac{\mathbf{r}}{r}\times\mathbf{L}F_2\right\}P_j^m=\left\{\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)F_1+\frac{j(j+1)}{r}F_2\right\}P_j^m.$$

The complete set of radial equations thus obtained is:

$$\begin{gathered}\Omega F_1=\frac{2}{r^2}(F_1+j(j+1)F_2)-\frac{e}{\hbar c}\frac{dV}{dr}G-\frac{e}{\hbar c}\frac{d}{dr}\left(Q\frac{dV}{dr}\right),\\ \Omega F_2=\frac{2}{r^2}F_1+\frac{e}{\hbar c}\frac{1}{r}Q\frac{dV}{dr},\quad \Omega G=\frac{e}{\hbar c}\frac{dV}{dr}F_1-\frac{e}{\hbar c}\frac{W+eV}{\hbar c}Q\frac{dV}{dr},\\ \frac{dF_1}{dr}+\frac{2}{r}F_1+\frac{j(j+1)}{r}F_2+\frac{W+eV}{\hbar c}G=-\frac{e}{\hbar c}Q\frac{dV}{dr},\end{gathered} \tag{41}$$

where Ω symbolizes the Klein-Gordon operator

$$\Omega=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{j(j+1)}{r^2}+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2 \tag{42}$$

and Q is defined by

$$\left(\frac{Mc}{\hbar}\right)^2Q=\frac{dG}{dr}-\frac{W+eV}{\hbar c}F_1. \tag{43}$$

It should be noted that if $j=0$ the second equation of the set (41) disappears, for this equation was inferred from the coefficients of $i[(\mathbf{r}/r)\times\mathbf{L}]P_j^m$, which identically vanishes for zero angular momentum.

In order that regular solutions of these equations exist for the Coulomb field, $V=Ze/r$, it is necessary that the functions F_1, F_2 and G vanish at the origin at least as rapidly as some power of r, e.g., r^ζ, so that, from Eqs. (41), Q must exhibit a radial dependence proportional to $r^{\zeta+1}$. To test the consistency of these requirements with the definition of Q, we may employ the equations obtained by substituting the fourth member of (41) into the first and third members, namely:

$$\begin{gathered}-\frac{W+eV}{\hbar c}Q=F_1+\left(\frac{\hbar}{Mc}\right)^2\frac{j(j+1)}{r^2}\left\{F_1+\frac{d}{dr}(rF_2)\right\},\\ \frac{dQ}{dr}+\frac{2}{r}Q=G+\left(\frac{\hbar}{Mc}\right)^2\frac{j(j+1)}{r^2}\left\{G+\frac{W+eV}{\hbar c}(rF_2)\right\}.\end{gathered} \tag{44}$$

In states of nonvanishing angular momentum ($j\neq 0$), the second terms of these expressions for Q are dominant near the origin, thus imparting to Q a radial dependence two powers of r less than that of the functions F_1, F_2, and G, contrary to the mathematical regularity requirements. Therefore, no regular solutions exist for states possessing finite angular momentum. Whether *any* solutions compatible with the physical requirements exist remains an open question. For $j=0$, these Eqs. (44) become simply

$$F_1=-\frac{W+eV}{\hbar c}Q;\quad G=\frac{dQ}{dr}+\frac{2}{r}Q, \tag{45}$$

which, coupled with the definition (43) of Q, imply that

$$\frac{d^2Q}{dr^2}+\frac{2}{r}\frac{dQ}{dr}-\frac{2}{r^2}Q+\left\{\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}Q=0, \tag{46}$$

the Klein-Gordon equation for a state of unit angular momentum. This result is quite satisfactory, for a state of zero total angular momentum is rigorously a state of unit orbital angular momentum. The general conclusion to be drawn from these considerations based on the Proca theory may be summarized by the statement that regular solutions exist for the Coulomb field only for those states in which the orbital angular momentum is a rigorous constant of the motion, namely $l=j\neq 0$ and $l=1$, $j=0$.

The technique developed for the expression of the Proca equations in radial form is immediately applicable to the equations of the modified theory ($\gamma=1$). In vectorial notation these equations comprise the following set:

$$\begin{aligned}
&\left\{\nabla^2+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}\mathbf{\Phi}=-\frac{2e}{\hbar c}\frac{\mathbf{r}}{r}\frac{dV}{dr}\varphi-\frac{4\pi e\hbar}{M^2c^3}\nabla(\Gamma(r)\varphi),\\
&\left\{\nabla^2+\left(\frac{W+eV}{\hbar c}\right)^2-\left(\frac{Mc}{\hbar}\right)^2\right\}\varphi=\frac{2e}{\hbar c}\frac{dV}{dr}\frac{\mathbf{r}}{r}\cdot\mathbf{\Phi}-\frac{4\pi e}{M^2c^4}(W+eV)\Gamma(r)\varphi,\\
&\nabla\cdot\mathbf{\Phi}+\frac{W+eV}{\hbar c}\varphi=-\frac{4\pi e\hbar}{M^2c^3}\Gamma(r)\varphi,
\end{aligned} \tag{47}$$

wherein $\Gamma(r)=J^4/c$ denotes the density of the static, spherically symmetric charge distribution under whose influence the mesotron moves. The treatment of the state $l=j$ need not detain us, the Klein-Gordon equation being obtained as before. The radial equations describing the coupled states $l=j\pm 1$, *viz.*:

$$\begin{aligned}
&\Omega F_1=\frac{2}{r^2}(F_1+j(j+1)F_2)-\frac{2e}{\hbar c}\frac{dV}{dr}G-\frac{4\pi e\hbar}{M^2c^3}\frac{d}{dr}(\Gamma(r)G),\\
&\Omega F_2=\frac{2}{r^2}F_1+\frac{4\pi e\hbar}{M^2c^3}\frac{1}{r}\Gamma(r)G,\quad \Omega G=\frac{2e}{\hbar c}\frac{dV}{dr}F_1-\frac{4\pi e}{M^2c^4}(W+eV)\Gamma(r)G,\\
&\frac{dF_1}{dr}+\frac{2}{r}F_1+\frac{j(j+1)}{r}F_2+\frac{W+eV}{\hbar c}G=-\frac{4\pi e\hbar}{M^2c^3}\Gamma(r)G,
\end{aligned} \tag{48}$$

may be obtained, *mut. mutand.*, from the previous set (41). A complete solution of these equations can be effected in regions of space unoccupied by external charges ($\Gamma(r)=0$). Within such domains, the radial equations assume a particularly simple aspect:

$$\Omega F_1=(2/r^2)[F_1+j(j+1)F_2+Z\alpha G],\quad \Omega F_2=(2/r^2)F_1,\quad \Omega G=(2/r^2)(-Z\alpha F_1),$$
$$\frac{dF_1}{dr}+\frac{2}{r}F_1+\frac{j(j+1)}{r}F_2+\frac{W+eV}{\hbar c}G=0 \tag{49}$$

when the Coulomb field ($eV=Ze^2/r=Z\alpha\hbar c/r$) is introduced explicitly. A process of diagonalization reduces these equations to a canonical form.

The linear combination

$$\lambda F_1+j(j+1)F_2+Z\alpha G \tag{50}$$

provides a solution of the differential equation

$$\Omega\mathfrak{F}=(2\lambda/r^2)\mathfrak{F}$$

if λ is chosen to represent one of the roots of the eigenvalue equation

$$\lambda(\lambda-1)=j(j+1)-Z^2\alpha^2 \tag{51}$$

viz.:

$$\lambda_1=\tfrac{1}{2}+[(j+\tfrac{1}{2})^2-Z^2\alpha^2]^{\frac{1}{2}};\quad \lambda_2=\tfrac{1}{2}-[(j+\tfrac{1}{2})^2-Z^2\alpha^2]^{\frac{1}{2}}. \tag{52}$$

Associated with these roots, therefore, are two functions:

$$(\lambda_1-\lambda_2)\mathfrak{F}_1=\lambda_1F_1+j(j+1)F_2+Z\alpha G,\quad (\lambda_2-\lambda_1)\mathfrak{F}_2=\lambda_2F_1+j(j+1)F_2+Z\alpha G, \tag{53}$$

which satisfy the differential equation

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{W^2-M^2c^4}{\hbar^2c^2}+\frac{2W}{\hbar c}\frac{Z\alpha}{r}-\frac{\lambda(\lambda+1)}{r^2}\right\}\mathfrak{F}_\lambda=0$$
$$(\mathfrak{F}_1=\mathfrak{F}_{\lambda_1},\quad \mathfrak{F}_2=\mathfrak{F}_{\lambda_2}), \tag{54}$$

the form assumed by the Klein-Gordon equation in a Coulomb field. The two Eqs. (53), supplemented by the fourth member of (49), *viz.:*

$$F_1=\mathfrak{F}_1+\mathfrak{F}_2,\quad j(j+1)F_2+Z\alpha G=-(\lambda_2\mathfrak{F}_1+\lambda_1\mathfrak{F}_2)$$
$$G=-\frac{\hbar c}{W}\left\{\frac{d}{dr}\mathfrak{F}_1+\frac{1+\lambda_1}{r}\mathfrak{F}_1+\frac{d}{dr}\mathfrak{F}_2+\frac{1+\lambda_2}{r}\mathfrak{F}_2\right\}, \tag{55}$$

constitute a complete solution of the problem in terms of $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

In states of nonvanishing angular momentum, there exist two linearly independent solutions associated with $\mathfrak{F}_1$ and with $\mathfrak{F}_2$, which we label by a quantum number "l" $=j+1,\ j-1$, respectively. In the nonrelativistic limit, "l" becomes the orbital quantum number. $\mathfrak{F}_1$ and $\mathfrak{F}_2$ may not be considered as independent in states of zero angular momentum, for a relation between them is implied by the Eqs. (55) specialized to the state $j=0$. The last two equations—equations for G—now serve to determine the arbitrary constants in $\mathfrak{F}_2$ in terms of those in $\mathfrak{F}_1$ thus providing only one independent solution. To show this in detail, we observe that, if $\mathfrak{F}_\lambda$ is a solution of (54),

$$\mathfrak{F}_{\lambda-1}=\frac{d}{dr}\mathfrak{F}_\lambda+\frac{1+\lambda}{r}\mathfrak{F}_\lambda-\frac{W}{\hbar c}\frac{Z\alpha}{\lambda}\mathfrak{F}_\lambda, \tag{56}$$

whence

$$-\frac{\hbar c}{W}\left\{\frac{d}{dr}\mathfrak{F}_1+\frac{1+\lambda_1}{r}\mathfrak{F}_1+\frac{d}{dr}\mathfrak{F}_2+\frac{1+\lambda_2}{r}\mathfrak{F}_2\right\}=-\frac{1}{Z\alpha}(\lambda_2\mathfrak{F}_1+\lambda_1\mathfrak{F}_2)-\frac{\hbar c}{W}(\mathfrak{F}_{\lambda_1-1}+\mathfrak{F}_{\lambda_2-1}), \tag{57}$$

in which the relation $\lambda_1\lambda_2 = Z^2\alpha^2$ $(j=0)$ has been utilized. Since $\lambda_1(\lambda_1-1)=\lambda_2(\lambda_2-1)$, $\mathfrak{F}_{\lambda_1-1}$ and $\mathfrak{F}_{\lambda_2-1}$ satisfy the same differential equation, thus enabling one to impose consistently the condition

$$\mathfrak{F}_{\lambda_1-1}+\mathfrak{F}_{\lambda_2-1}\equiv 0, \tag{58}$$

which exhibits the relation between $\mathfrak{F}_1$ and $\mathfrak{F}_2$ demanded by the coexistence of Eqs. (55) for $j=0$.

Although the equations of the modified theory $(\gamma=1)$ are completely soluble in regions devoid of external charges, a comparable situation does not exist within charge distributions $(\Gamma(r)\neq 0)$. While the equations presumably possess regular solutions in regions of smoothly varying charge density, no serious attempt has been made to obtain them. In the limiting case of a point charge, the presence of the singular terms in $\Gamma(r)$ imposes boundary conditions, which may be inferred from the requirement of the existence of physically admissible charge distributions. According to Eqs. (15) and (16), the mesotron charge density ρ is given by

$$\rho = -\frac{2e}{\hbar c^2}\left\{\frac{W+eV}{\hbar c}(\bar{\mathbf{\Phi}}\cdot\mathbf{\Phi}+\bar{\varphi}\varphi)-(\mathbf{\Phi}\cdot\nabla)\bar{\varphi}-(\bar{\mathbf{\Phi}}\cdot\nabla)\varphi+\frac{4\pi e\hbar}{M^2c^3}\Gamma(r)\bar{\varphi}\varphi\right\},$$

which would predict an amount of charge

$$-2Z\alpha(e\hbar/M^2c^4)G^2(0) \tag{59}$$

to be contained within an arbitrarily small neighborhood of a point charge Ze. For solutions of the type $j=``l"+1$, $G(r)\sim r^{\lambda_1-2}$ in the vicinity of the origin, while for those of type $j=``l"-1$, $G(r)\sim r^{\lambda_1-1}$ Since λ_1 is not an integer, $G(r)$ tends to either zero or infinity as r is diminished. In the latter case, which occurs for $j=0$, $``l"=1$, and $j=1$, $``l"=0$, the amount of charge (59) in the immediate neighborhood of the origin, and hence also the total charge of the system, becomes infinite. This meaningless situation implies that these two states are completely forbidden.

It is of some interest to note that the energy levels of the allowed discrete states are represented by the Sommerfeld formula, all states of given total angular momentum and principal quantum number coalescing as in the Dirac theory. This result is inexplicable on the basis of the formula derived from the Larmor and Thomas precessions, which would predict in addition a dependence on the orbital angular momentum quantum number l. The origin of this discrepancy is to be sought in the inadequacy of considering the spin as a simple vector, for the coupling energy between spin and orbit consists not only of the usual Larmor-Thomas terms but in addition terms which are nonlinear in the spin. The equivalence between the Larmor-Thomas formula and the corresponding prediction of the Dirac theory is to be attributed to the necessary linearity of the spin terms for a particle of spin $\frac{1}{2}$.

The general program of this section has been an attempt to obtain rigorous solutions of the equations for a particle of unit spin in a Coulomb field. While for the equations describing Proca particles this attempt has been abortive, the modified theory discussed in this paper provides regular solutions for all states save two, these being completely forbidden. The net result of the modification has been, therefore, the concentration rather than the removal of the singularities. The lack of a complete set of regular solutions thus compels us to have resort to the Born approximation in order to give meaning to these two theories.

IV. Electron-Mesotron Collisions

Information concerning the electromagnetic properties of mesotrons may be obtained by the study of various scattering processes. However, it would be difficult to infer such information from processes such as mesotron-proton scattering and radiative emission induced by mesotrons, which involve in addition the mesotron-heavy particle coupling. For collisions between mesotrons and electrons, however, this latter type of coupling is presumably absent, and the energy transfer is dependent only on the electromagnetic interaction between the particles. Using various mesotron models, we shall calculate the cross section for this energy transfer, developing a general analysis, based on the Born approximation, without use of the explicit form of the mesotron charge and current densities. The results are applied to mesotrons of spin $\sigma=0$, magnetic moment $\mu=0$, $\sigma=\frac{1}{2}$, μ arbitrary (in particular $\mu=0$) and $\sigma=1$, μ arbitrary (in particular $\mu=1$ and $\mu=2$).

Let us consider a transition of the mesotron from a state of energy W^0 and momentum $\mathbf{P}^0$ to a state of energy W and momentum $\mathbf{P}$. With this transition are associated charge and current densities

$$\rho=\rho_{if}\exp\{i[(\mathbf{P}^0-\mathbf{P})\cdot\mathbf{r}-(W^0-W)t]/\hbar\},\quad \mathbf{j}=\mathbf{j}_{if}\exp\{i[(\mathbf{P}^0-\mathbf{P})\cdot\mathbf{r}-(W^0-W)t]/\hbar\} \tag{60}$$

which produce a transition electromagnetic field

$$\mathbf{A}=\frac{4\pi\hbar^2}{c}\mathbf{j}_{if}\frac{\exp\{i[(\mathbf{P}^0-\mathbf{P})\cdot\mathbf{r}-(W^0-W)t]/\hbar\}}{|\mathbf{P}^0-\mathbf{P}|^2-c^{-2}(W^0-W)^2},\quad V=4\pi\hbar^2\rho_{if}\frac{\exp\{i[(\mathbf{P}^0-\mathbf{P})\cdot\mathbf{r}-(W^0-W)t]/\hbar\}}{|\mathbf{P}^0-\mathbf{P}|^2-c^{-2}(W^0-W)^2}. \tag{61}$$

This field induces transitions of the electron from a state of energy and momentum E^0, $\mathbf{p}^0$ to the state E, $\mathbf{p}$. The spin direction of the electron in these positive energy states may be characterized by the indices λ^0, $\lambda=1, 2$. If the electron wave functions are taken to be

$$\text{initially: } u_{\mathbf{p}^0}{}^{\lambda^0}\exp[i(\mathbf{p}^0\cdot\mathbf{r}-E^0t)/\hbar],\quad \text{finally: } u_{\mathbf{p}}{}^{\lambda}\exp[i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar], \tag{62}$$

the condition $(u_{\mathbf{p}^0}{}^{\lambda^0}, u_{\mathbf{p}^0}{}^{\lambda^0})=(u_{\mathbf{p}}{}^{\lambda}, u_{\mathbf{p}}{}^{\lambda})=1$ corresponds to one electron per unit volume. The transition probability per unit time from the state $(\lambda^0, \mathbf{p}^0)$ to the state $(\lambda, \mathbf{p})$ of the electron is then given by

$$\frac{2\pi}{\hbar}\left[\frac{4\pi\hbar^2 e}{|\mathbf{P}^0-\mathbf{P}|^2-c^{-2}(W^0-W)^2}\right]^2\left|\left(u_{\mathbf{p}}{}^{\lambda},\ (\rho_{if}-\frac{1}{c}\boldsymbol{\alpha}\cdot\mathbf{j}_{if})u_{\mathbf{p}^0}{}^{\lambda^0}\right)\right|^2\cdot\delta(W+E-W^0-E^0)\delta_{\mathbf{P}+\mathbf{p},\,\mathbf{P}^0+\mathbf{p}^0}. \tag{63}$$

To find the total electron transition probability, we sum over the final spin and momentum states of the electron. The summation over the final momentum states has merely the effect of imposing the condition of momentum conservation. Summing over the final spin directions, one has

$$\sum_{\lambda=1,2}\left|\left(u_{\mathbf{p}}{}^{\lambda},\ (\rho_{if}-\frac{1}{c}\boldsymbol{\alpha}\cdot\mathbf{j}_{if})u_{\mathbf{p}^0}{}^{\lambda^0}\right)\right|^2=\frac{1}{E^0E}\Big[(E^0\rho_{if}-\mathbf{p}^0\cdot\mathbf{j}_{if})(E\rho_{if}-\mathbf{p}\cdot\mathbf{j}_{if}) -\tfrac{1}{2}(E^0E-c^2\mathbf{p}^0\cdot\mathbf{p}-m^2c^4)(\rho_{if}{}^2-\frac{1}{c^2}\mathbf{j}_{if}\cdot\mathbf{j}_{if})\Big], \tag{64}$$

which is independent of the initial spin of the electron. To calculate the cross section for a given incident mesotron, we sum the transition probability (63) over all relevant final states of the mesotron and divide by the product of the mesotron particle density and the relative velocity of the electron and mesotron. The cross section thus defined is relativistically invariant and is most easily computed in the center of mass system $(\mathbf{P}^0+\mathbf{p}^0=0)$, since in this system the energy of either particle does not change during the collision. The resulting cross section for the scattering of mesotrons through an angle ϑ into a solid angle $d\omega$ is

$$d\sigma=\frac{e^2}{4c^4P^{0^4}}\left(\frac{W^0}{W^0+E^0}\right)^2\left[(E^0\rho_{if}+\mathbf{P}^0\cdot\mathbf{j}_{if})^2-c^2P^{0^2}\sin^2\tfrac{1}{2}\vartheta(\rho_{if}{}^2-\frac{1}{c^2}\mathbf{j}_{if}\cdot\mathbf{j}_{if})\right]\frac{d\omega}{\sin^4\frac{1}{2}\vartheta} \tag{65}$$

if the mesotron particle density is also normalized to represent one particle per unit volume. Here we have used the facts that the relative velocity is $(c^2P^0/W^0)+(c^2P^0/E^0)$ in this coordinate system and that the number of mesotron states per unit volume and per unit total energy range is

$$\frac{P^0}{8\pi^3\hbar^3c^2}\frac{E^0W^0}{E^0+W^0}d\omega.$$

For a particle of spin one-half and magnetic moment unity the charge and current densities are given, according to the usual Dirac theory, by the four-vector

$$j^{\mu}=-(e\hbar/2M)\{(\phi\partial^{\mu}\psi-(\partial^{\mu}\phi)\psi)+\tfrac{1}{2}\partial_{\nu}(\phi(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})\psi)\}. \tag{66}$$

Here ϕ is defined by $\phi=\psi^{+}\gamma^{4}$, and γ^{μ} is given in terms of the more familiar Dirac matrices $\boldsymbol{\alpha}$, β by $\gamma^{4}=-i\beta$, $\gamma^{i}=-i\beta\alpha^{i}$. The two terms on the right-hand side of (66) are independently conserved and correspond, respectively, to a convection current and a polarization current arising from the magnetic moment. In close analogy with the theory for particles of spin unity, outlined in Section II, one may develop a theory for which the coefficient of the polarization current is arbitrary, corresponding to a particle of spin $\frac{1}{2}$ and of arbitrary magnetic moment. We consider here the particular case for which this magnetic moment is zero. The charge-current four-vector then assumes the simple form

$$j^{\mu}=(ie\hbar/2M)[\psi^{+}\beta\partial^{\mu}\psi-(\partial^{\mu}\psi^{+})\beta\psi], \tag{67}$$

an expression which differs from the corresponding result for particles of spin zero only by the presence of the matrix β. Using plane waves for the mesotron wave functions, *viz.:*

$$\psi_{i}=u_{\mathbf{P}^{0}}{}^{\lambda^{0}}\exp[i(\mathbf{P}^{0}\cdot\mathbf{r}-W^{0}t)/\hbar],\quad \psi_{f}=u_{\mathbf{P}}{}^{\lambda}\exp[i(\mathbf{P}\cdot\mathbf{r}-Wt)/\hbar],$$

one finds that the transition charge and current densities corresponding to a mesotron of spin $\frac{1}{2}$ and magnetic moment zero are

$$\rho_{if}=-e(W^{0}/Mc^{2})(u_{\mathbf{P}}{}^{\lambda},\beta u_{\mathbf{P}^{0}}{}^{\lambda^{0}}),\quad \mathbf{j}_{if}=-e\frac{c^{2}(\mathbf{P}^{0}+\mathbf{P})}{2W^{0}}\frac{W^{0}}{Mc^{2}}(u_{\mathbf{P}}{}^{\lambda},\beta u_{\mathbf{P}^{0}}{}^{\lambda^{0}}). \tag{68}$$

The wave functions are normalized by the condition

$$(u_{\mathbf{P}^{0}}{}^{\lambda^{0}},u_{\mathbf{P}^{0}}{}^{\lambda^{0}})=(u_{\mathbf{P}}{}^{\lambda},u_{\mathbf{P}}{}^{\lambda})=1,$$

which corresponds to one mesotron per unit volume.

Using the relation

$$\sum_{\lambda=1,2}|(u_{\mathbf{P}}{}^{\lambda},\beta u_{\mathbf{P}^{0}}{}^{\lambda^{0}})|^{2}=\frac{M^{2}c^{4}}{W^{0 2}}\left(1+\frac{P^{0 2}}{M^{2}c^{2}}\sin^{2}\tfrac{1}{2}\vartheta\right) \tag{69}$$

and substituting (68) in (65), one finds that the cross section for the scattering of a mesotron through an angle ϑ into a solid angle $d\omega$ is

$$d\sigma=\frac{e^{4}}{4c^{4}P^{0 4}}\left(\frac{W^{0}}{W^{0}+E^{0}}\right)^{2}\frac{d\omega}{\sin^{4}\frac{1}{2}\vartheta}\left[\left(E^{0}+\frac{c^{2}P^{0 2}}{W^{0}}\right)^{2}-c^{2}P^{0 2}\sin^{2}\tfrac{1}{2}\vartheta\left\{\frac{2}{W^{0}}\left(E^{0}+\frac{c^{2}P^{0 2}}{W^{0}}\right)+\frac{M^{2}c^{4}}{W^{0 2}}\right\}\right]\cdot\left[1+\frac{P^{0 2}}{M^{2}c^{2}}\sin^{2}\tfrac{1}{2}\vartheta\right]. \tag{70}$$

The corresponding cross section for a particle of zero spin—and hence zero magnetic moment—differs from the above expression only by the absence of the last factor, which, for the case of spin $\frac{1}{2}$, arises from the spin summation. It is to be noted that at high energies this factor entirely alters the dependence of the cross section on the mesotron energy.

To obtain from (70) the cross section for Coulomb scattering it is only necessary to consider the limiting case of infinite E^{0}, which yields

$$d\sigma=\frac{1}{4}\left(\frac{e^{2}}{Mc^{2}}\right)^{2}\frac{\epsilon^{2}}{(\epsilon^{2}-1)^{2}}[1+(\epsilon^{2}-1)\sin^{2}\tfrac{1}{2}\vartheta]\frac{d\omega}{\sin^{4}\frac{1}{2}\vartheta}, \tag{71}$$

in which we have set $W^0=\epsilon Mc^2$. At high mesotron energies (71) becomes

$$d\sigma=\frac{1}{4}\left(\frac{e^2}{Mc^2}\right)^2\frac{d\omega}{\sin^2\frac{1}{2}\vartheta},\tag{72}$$

which differs from the corresponding formula deduced from the Proca theory[6] by the absence of a factor $\frac{2}{3}\cos^2\frac{1}{2}\vartheta$.

The above formulae represent the angular scattering cross section in the barycentric system; we are primarily interested in the cross section for a given fractional energy transfer to an initially stationary electron. It is therefore expedient to introduce the ratio, f, of the kinetic energy gained by the electron to $W_0=\epsilon Mc^2$, the total energy of the incident mesotron, measured in the laboratory system. The relation between f and the angle of scattering in the center of mass system is simply

$$f=f_{\max}\sin^2\tfrac{1}{2}\vartheta,\quad\text{where}\quad f_{\max}=\frac{2\omega(\epsilon^2-1)}{\epsilon(1+\omega^2+2\omega\epsilon)}\quad(\omega=m/M)\tag{73}$$

represents the maximum fraction of the mesotron energy which may be transferred to an electron by impact. Employing the relation between W_0 and the electron and mesotron energies in the barycentric system, *viz.:*

$$E^0=\frac{\omega(\epsilon+\omega)}{(1+\omega^2+2\omega\epsilon)^{\frac{1}{2}}}Mc^2,\quad W^0=\frac{1+\omega\epsilon}{(1+\omega^2+2\omega\epsilon)^{\frac{1}{2}}}Mc^2,\tag{74}$$

we may rewrite the cross section as follows:

$$d\sigma=\pi\left(\frac{e^2}{Mc^2}\right)^2\frac{\epsilon^2}{\epsilon^2-1}\frac{df}{f^2}\left[f(1-f)+\frac{2}{\omega\epsilon}(1-f-\tfrac{1}{4}f^2)-\frac{1}{\omega^2\epsilon^2}f\right].\tag{75}$$

At high energies this is independent of the energy of the incident mesotron. For a mesotron of arbitrary magnetic moment $\mu\neq1$ and spin $\frac{1}{2}$ the leading term of this expression is multiplied by a factor $(\mu-1)^2$. The corresponding result for mesotrons of spin zero is obtained by omitting the last factor in (70):

$$d\sigma=\pi\left(\frac{e^2}{Mc^2}\right)^2\frac{\epsilon^2}{\epsilon^2-1}\frac{df}{f^2}\left[\frac{2}{\omega\epsilon}(1-f)-\frac{1}{\omega^2\epsilon^2}f\right],\tag{76}$$

which for high mesotron energies ϵ decreases inversely with ϵ.

For mesotrons of spin unity, one again uses plane waves for the mesotron wave functions, *viz.:*

$$\mathbf{\Phi}_i=\hbar c\left(\frac{c}{2W^0}\right)^{\frac{1}{2}}\mathbf{e}^0\exp\left[i(\mathbf{P}^0\cdot\mathbf{r}-W^0t)/\hbar\right],\quad\varphi_i=-i\hbar c\left(\frac{c}{2W^0}\right)^{\frac{1}{2}}\frac{c\mathbf{P}^0\cdot\mathbf{e}^0}{W^0}\exp\left[i(\mathbf{P}^0\cdot\mathbf{r}-W^0t)/\hbar\right]$$

with similar expressions for the final state. The normalizing condition

$$\mathbf{e}^0\cdot\mathbf{e}^0-(c\mathbf{P}^0\cdot\mathbf{e}^0/W^0)^2=1$$

then corresponds to one mesotron per unit volume.

The transition current and charge densities are given by

$$\begin{aligned}\rho_{if}&=-e\left\{\left(\mathbf{e}^0\cdot\mathbf{e}-\frac{c\mathbf{P}^0\cdot\mathbf{e}^0}{W^0}\frac{c\mathbf{P}\cdot\mathbf{e}}{W^0}\right)+\tfrac{1}{2}(\gamma+1)\left(\frac{c\mathbf{P}\cdot\mathbf{e}}{W^0}\frac{c(\mathbf{P}^0-\mathbf{P})\cdot\mathbf{e}^0}{W^0}-\frac{c\mathbf{P}^0\cdot\mathbf{e}^0}{W^0}\frac{c(\mathbf{P}^0-\mathbf{P})\cdot\mathbf{e}}{W^0}\right)\right\},\\ \mathbf{j}_{if}&=-e\frac{c^2}{2W^0}\left\{(\mathbf{P}^0+\mathbf{P})\left(\mathbf{e}^0\cdot\mathbf{e}-\frac{c\mathbf{P}^0\cdot\mathbf{e}^0}{W^0}\frac{c\mathbf{P}\cdot\mathbf{e}}{W^0}\right)+(\gamma+1)(\mathbf{e}^0\times\mathbf{e})\times(\mathbf{P}^0-\mathbf{P})\right\},\end{aligned}\tag{77}$$

[6] O. Laporte, Phys. Rev. **54**, 905 (1938).

TABLE I. *Values of $f\kappa(f)$ for particles with various spins and magnetic moments which collide with the electron.*

TYPE	SPIN	MAGNETIC MOMENT	$f\kappa(f)$
I	0	0	$2\eta^{-1}(1-f)$
II	$\frac{1}{2}$	1	$2\eta^{-1}(1-f+\frac{1}{2}f^2)$
III	$\frac{1}{2}$	$\mu\neq 1$	$(\mu-1)^2(1-f)$
IV	1	1	$\frac{2}{3}(1-f+\frac{1}{2}f^2)$
V	1	$\mu\neq 1$	$\frac{2}{3}(\mu-1)^2\eta(1-f)$

and on substitution in (65) one may obtain the cross section. For general γ and ϵ the expression obtained is complicated but for large ϵ and $\gamma\neq 0$ the major contribution to the cross section arises from the terms proportional to γ^2 which correspond to longitudinal-longitudinal transitions. One finds, for $f\omega\epsilon\gg 1$, $f<f_{\max}\doteq 1-(1/2\omega\epsilon)$, and $\gamma\neq 0$,

$$d\sigma=\tfrac{2}{3}\pi(e^2/Mc^2)^2\gamma^2\omega\epsilon(1-f)df, \qquad (78)$$

which is to be compared with the corresponding cross section for $\gamma=0$

$$d\sigma=\tfrac{2}{3}\pi(e^2/Mc^2)^2(1-f+\tfrac{1}{2}f^2)df. \qquad (79)$$

It is to be noted that for $\gamma\neq 0$ the cross section increases linearly with the energy of the incident mesotron.

Writing

$$d\sigma=\sigma_0\kappa(f)df, \quad \sigma_0=\pi(e^2/Mc^2)^2 \qquad (80)$$

for the cross section for a particle of energy ϵMc^2 and mass M to transfer energy $f\epsilon Mc^2$ to an electron of mass $m=\omega M$, we may summarize the above results for $\eta=f\epsilon\omega\gg 1$ in Table I.

Only cases III and IV lead to cross sections essentially independent of the mesotron energy, as required by experiment. In addition, it has been shown[1] that in order to account for the observed size[7] of bursts of energy $>2\times 10^{10}$ ev it is necessary for the cross section to be such that

$$\kappa=\int_0^1\kappa(f)f^{1.8}df\sim\tfrac{1}{2}.$$

For the case of spin $\frac{1}{2}$ and magnetic moment $\mu\neq 1$

$$\kappa=0.2(\mu-1)^2,$$

which is of the correct order for $|\mu-1|<2$. Because of the uncertainty in the experimental evidence the value of μ cannot be assigned more definitely, but it is to be noted that, for the particular case $\mu=0$ the value of κ is 0.2, which is very close to the corresponding value (0.22) obtained from the Proca theory. It is therefore impossible to distinguish between cases III and IV from cosmic-ray evidence alone.

All these results having been obtained by use of the Born approximation, it is important to examine the conditions under which they may be expected to have validity. A necessary condition for the validity of the Born approximation is the smallness of the coupling between the mesotron and the fields arising from both the electron and the zero-point fluctuations of the electromagnetic field. A previous investigation of these questions by Oppenheimer, Snyder and Serber[1] has disclosed that the more incisive condition arises from the requirement that the coupling between the mesotron and the zero-point fluctuations be small. To express this requirement in quantitative form, we consider the mesotron-electron collision in the center of mass system, employing wave packets of approximate linear dimensions $\hbar/P$. The kinetic energy density $\sim Pc(P/\hbar)^3$ must then be large compared with the coupling energy density $-(1/c)\mathbf{j}\cdot\mathbf{A}$. The potential $\mathbf{A}$ describes the zero-point fluctuations of wave-length $\sim\hbar/P$ and possessses the order of magnitude $P(c/\hbar)^{\frac{1}{2}}$. The form of the current density $\mathbf{j}$ depends on the assumed mesotron model and is of the order of magnitude $ec(P/\hbar)^3\times(P/Mc)^n$, where n assumes the value zero for particles of type I, II of Table I, $n=1$ for types III, IV, and $n=2$ for type V. This varied momentum dependence of the current density is of course intimately related to the different energy dependence of the cross sections for these several types of particles. The condition of smallness of the coupling energy density compared with the kinetic energy density is then expressed quantitatively by

$$\alpha^{\frac{1}{2}}(P/Mc)^n\ll 1.$$

For the various types of particles this condition assumes the form

$$\begin{aligned}&(\mathrm{I, II}): && \alpha^{\frac{1}{2}}\ll 1,\\ &(\mathrm{III, IV}): && W_0\ll M^2c^2/\alpha m\doteq 2\times 10^{12}\ \mathrm{ev},\\ &(\mathrm{V}): && W_0\ll M^2c^2/\alpha^{\frac{1}{2}}m\doteq 2\times 10^{11}\ \mathrm{ev}\end{aligned} \qquad (81)$$

[7] M. Schein and P. S. Gill, Phys. Rev. **55**, 1111 (1939); M. Schein and V. C. Wilson, Rev. Mod. Phys. **11**, 292 (1939); H. Carmichael and C. N. Chou, Nature **144**, 325 (1939); Bhabha, Carmichael and Chou, Proc. Ind. Acad. Sci. **A10**, 221 (1939).

in its limitation on the mesotron energy W_0. The domain of validity of the modified theory (case V) is thus more limited than that of the Proca theory, but it is still of sufficient extent to include most of the experimental region.

The authors wish to express their gratitude to Professor J. R. Oppenheimer for his continued advice and encouragement.

APPENDIX

The basic equation determining the simultaneous eigenfunctions of L^2 and J^2, Eq. (33) of the text:

$$((j(j+1)-l(l+1)-2)\mathbf{\Phi}=2i(\mathbf{L}\times\mathbf{\Phi}), \qquad \text{(A1)}$$

may be readily solved by elementary quantum algebra. The operations of scalar and vector multiplication with $\mathbf{L}$ yield, respectively,

$$\begin{aligned}&(j(j+1)-l(l+1))\mathbf{L}\cdot\mathbf{\Phi}=0,\\ &\quad l(l+1)\mathbf{\Phi}-\mathbf{L}(\mathbf{L}\cdot\mathbf{\Phi})=\tfrac{1}{4}(j(j+1)-l(l+1))\\ &\qquad\qquad\times(j(j+1)-l(l+1)-2)\mathbf{\Phi},\end{aligned} \qquad \text{(A2)}$$

from which one obtains the characteristic equation

$$\begin{aligned}(j(j+1)-l(l+1))((j(j+1)-l(l+1))^2\\ -2j(j+1)-2l(l+1))=0\end{aligned} \qquad \text{(A3)}$$

by elimination of $\mathbf{L}\cdot\mathbf{\Phi}$. The admissible values of the orbital angular momentum are thereby restricted to

$$l=j-1,\ j,\ j+1, \qquad \text{(A4)}$$

a not altogether unexpected result.

Sufficient information is available from the relations (A2) to determine the properties of the solution $l=j$. The equation thus obtained:

$$j(j+1)\mathbf{\Phi}=\mathbf{L}(\mathbf{L}\cdot\mathbf{\Phi}), \qquad \text{(A5)}$$

when reformulated in terms of the scalar function u, defined by

$$\mathbf{L}\cdot\mathbf{\Phi}=j(j+1)u \qquad \text{(A6)}$$

yields the general form of $\mathbf{\Phi}$, *viz.*:

$$\mathbf{\Phi}=\mathbf{L}u,\quad L^2u=j(j+1)u. \qquad \text{(A7)}$$

The second equation of the set (A7) implies that u is a spherical harmonic of order j, P_j^m, multiplied by an arbitrary radial function $F(r)$. The spherical harmonics are assumed normalized to provide the conventional matrix representation of angular momenta. One therefore obtains

$$l=j\!:\ \mathbf{\Phi}=\mathbf{L}F(r)P_j^m. \qquad \text{(A8)}$$

The choice of the spherical harmonic P_j^m guarantees that J_z shall have the eigenvalue m, for

$$J_z\Phi_z=L_z\Phi_z=m\Phi_z. \qquad \text{(A9)}$$

The relations thus far developed (A2) provide only the information that

$$\mathbf{L}\cdot\mathbf{\Phi}=0 \qquad \text{(A10)}$$

when restricted to the states $l=j\pm1$. To proceed further one may utilize the equations

$$\begin{aligned}&(j(j+1)-l(l+1)+2)(\mathbf{r}/r)\cdot\mathbf{\Phi}=-2i\mathbf{L}\cdot((\mathbf{r}/r)\times\mathbf{\Phi}),\\ &\quad(j(j+1)-l(l+1))(\mathbf{r}/r)\times\mathbf{\Phi}=2i\mathbf{L}((\mathbf{r}/r)\cdot\mathbf{\Phi}),\end{aligned} \qquad \text{(A11)}$$

obtained by applying the operations of scalar and vector multiplication with $\mathbf{r}/r$ to the basic equation (A1). The introduction of the scalar function v, defined by

$$(\mathbf{r}/r)\cdot\mathbf{\Phi}=-\tfrac{1}{2}(j(j+1)-l(l+1))v, \qquad \text{(A12)}$$

permits a more convenient expression of Eq. (A11), namely $(\mathbf{r}/r)\times\mathbf{\Phi}=-i\mathbf{L}v$,

$$\begin{aligned}L^2v&=\tfrac{1}{4}(j(j+1)-l(l+1))\\ &\qquad\times(j(j+1)-l(l+1)+2)v\\ &=j(j+1)v.\end{aligned} \qquad \text{(A13)}$$

The relations (A12) and (A13) suffice to determine the general form of $\mathbf{\Phi}$, for

$$\begin{aligned}\mathbf{\Phi}&\equiv(\mathbf{r}/r)((\mathbf{r}/r)\cdot\mathbf{\Phi})-(\mathbf{r}/r)\times[(\mathbf{r}/r)\times\mathbf{\Phi}]\\ &=\{\tfrac{1}{2}(l(l+1)-j(j+1))(\mathbf{r}/r)+i((\mathbf{r}/r)\times\mathbf{L})\}v.\end{aligned} \qquad \text{(A14)}$$

In virtue of the second of Eqs. (A13), the scalar function v may be represented as the product of the spherical harmonic P_j^m with an arbitrary radial function which is independently assignable for the two states $l=j\pm1$. As before, the choice of the harmonic P_j^m guarantees that J_z shall have the eigenvalue m. The $\mathbf{\Phi}$-functions associated with these two states, expressed in terms of the two arbitrary functions $F^{(1)}(r)$, $F^{(2)}(r)$ then assume the symbolic form

$$\begin{aligned}&l=j+1\!:\ \mathbf{\Phi}=\{(j+1)(\mathbf{r}/r)+i(\mathbf{r}/r)\times\mathbf{L}\}F^{(1)}(r)P_j^m,\\ &l=j-1\!:\ \mathbf{\Phi}=\{-j(\mathbf{r}/r)+i(\mathbf{r}/r)\times\mathbf{L}\}F^{(2)}(r)P_j^m.\end{aligned} \qquad \text{(A15)}$$

To exhibit clearly the angular dependence of the $\mathbf{\Phi}$-functions for the several states, we shall write in detail the behavior of the components

$$\left\{\begin{matrix}\Phi_x+i\Phi_y\\ \Phi_x-i\Phi_y\\ \Phi_z\end{matrix}\right\},$$

$$l=j\!:\quad \left\{\begin{matrix}[(j-m)(j+m+1)]^{\frac{1}{2}}P_j^{m+1}\\ [(j+m)(j-m+1)]^{\frac{1}{2}}P_j^{m-1}\\ m\qquad\qquad P_j^m\end{matrix}\right\}F(r),$$

$$l=j+1\!:\quad \left(\frac{2j+1}{2j+3}\right)^{\frac{1}{2}}\left\{\begin{matrix}-[(j+1+m)(j+2+m)]^{\frac{1}{2}}P_{j+1}^{m+1}\\ [(j+1-m)(j+2+m)]^{\frac{1}{2}}P_{j+1}^{m-1}\\ [(j+1)^2-m^2]^{\frac{1}{2}}\qquad P_j^m\end{matrix}\right\}F^{(1)}(r), \qquad \text{(A16)}$$

$$l=j-1\!:\quad \left(\frac{2j+1}{2j-1}\right)^{\frac{1}{2}}\left\{\begin{matrix}[(j-m)(j-m-1)]^{\frac{1}{2}}P_{j-1}^{m+1}\\ -[(j+m)(j+m-1)]^{\frac{1}{2}}P_{j-1}^{m-1}\\ [j^2-m^2]^{\frac{1}{2}}\qquad P_{j-1}^{m}\end{matrix}\right\}F^{(2)}(r).$$

These results are not completely novel, being intimately related to the angular representation of electromagnetic multipole fields.[8] The electromagnetic equations are identical with the field-free Proca equations in the limit of zero rest mass, the quantities $\mathbf{\Phi}$ and $i\varphi$ being identified with the vector and scalar potentials in this limit. The vector $\mathbf{\Phi}$ satisfies the differential equation

$$(\nabla^2+k^2)\mathbf{\Phi}=0\quad(k=W/\hbar c), \qquad \text{(A17)}$$

[8] W. Heitler, Proc. Camb. Phil. Soc. **32**, 112 (1936); W. W. Hansen, Phys. Rev. **47**, 139 (1935); S. M. Dancoff and P. Morrison, Phys. Rev. **55**, 122 (1939).

which, for our purposes, is more conveniently expressed as

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{L^2}{r^2}+k^2\right)\mathbf{\Phi}=0,$$

employing Eq. (40a) of the text. All three possible types of radial functions associated with a given orbital angular momentum l thus satisfy the differential equation

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{l(l+1)}{r^2}\right\}f_l(r)=0. \quad \text{(A18)}$$

The solution $\mathbf{\Phi}_j$ for $l=j$ (or more exactly the real part of $\mathbf{\Phi}_j e^{-ikct}$):

$$\mathbf{\Phi}_j=\mathbf{L}f_j(r)P_j^m$$

represents a transverse wave field ($\nabla\cdot\mathbf{\Phi}_j=0$) which is to be identified as the vector potential describing the radiation emitted by a magnetic multipole of order $2j$. Appropriate linear combinations of the solutions for $l=j\pm1$

$$\mathbf{\Phi}_{j+1}=\{(j+1)(\mathbf{r}/r)+i((\mathbf{r}/r)\times\mathbf{L})\}f_{j+1}P_j^m,$$
$$\mathbf{\Phi}_{j-1}=\{-j(\mathbf{r}/r)+i((\mathbf{r}/r)\times\mathbf{L})\}f_{j-1}P_j^m,$$

may be found to represent the longitudinal and transverse parts of the vector potential associated with an electric multipole of order $2j$. Using the recursion relations

$$f_{j+1}+f_{j-1}=(2j+1/kr)f_j, \quad \text{(A19)}$$
$$(j+1)f_{j+1}-jf_{j-1}=-(2j+1/k)(d/dr)f_j,$$

one may verify that

$$\mathbf{\Phi}_1=\mathbf{\Phi}_{j+1}+\mathbf{\Phi}_{j-1}=-\frac{2j+1}{k}\left\{\frac{\mathbf{r}}{r}\frac{d}{dr}f_j-i\frac{\mathbf{r}\times\mathbf{L}}{r^2}f_j\right\}P_j^m \quad \text{(A20)}$$
$$=-(2j+1/k)\nabla f_jP_j^m,$$

by Eq. (40) of the text. Similarly

$$\mathbf{\Phi}_2=\frac{j}{2j+1}\mathbf{\Phi}_{j+1}-\frac{j+1}{2j+1}\mathbf{\Phi}_{j-1}$$
$$=\left\{\frac{j(j+1)}{kr}\frac{\mathbf{r}}{r}f_j-i\frac{\mathbf{r}\times\mathbf{L}}{r}\frac{1}{kr}\frac{d}{dr}(rf_j)\right\}P_j^m \quad \text{(A21)}$$
$$=-(i/k)\nabla\times(\mathbf{L}f_jP_j^m).$$

These functions therefore satisfy

$$\nabla\times\mathbf{\Phi}_1=0,\quad \nabla\cdot\mathbf{\Phi}_2=0,$$

and are thus to be identified as the longitudinal and transverse fields of an electric $2j$-pole.

DECEMBER 1, 1940 PHYSICAL REVIEW VOLUME 58

On the Theory of Recombination

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(Received October 5, 1940)

In Section I a general formula for α, the coefficient of recombination of ions in gases, is developed. It covers all ranges of pressures and temperatures and can be made to include all types of recombination (preferential, initial and volume ionization). The evaluation of the general formula, (1.1), depends on the relative values of three linear quantities: the mean free path of the ions λ, the mean distance between ions of different signs $\bar{r}$ and the well-known parameter $a_0=e^2/(kKT)$. In Section II the case $\lambda\ll a_0$ is treated. The mechanism of recombination then depends on the ratio $a_0/\bar{r}$. If $a_0/\bar{r}$ is large the migration of the ions under their mutual attraction prevails and (1.1) leads to Langevin's formula; in the opposite case, $a_0/\bar{r}\ll1$, diffusion is the decisive feature and 1.1 leads to a formula which is practically identical with that of Harper. In Section III the case $\lambda\gg a_0$ is treated by a method previously developed by the author. Under certain restrictions (1.1) reduces to Thomson's formula, but in general α depends on the concentration of the ions. In Section IV it is shown that (1.1) is in fair agreement with such experimental data as are available for the region of transition between the cases treated in Sections II and III.

THE problem of recombination of ions in gases under varying conditions of ionization, pressure and temperature has proved to be much more complex than was originally anticipated. Various types, such as preferential, initial and volume recombination are involved.[1] It has been recognized in recent years that the two most important theoretical formulae for the coefficient of recombination, that of Langevin and that of Thomson, have separate domains of applicability. Harper[2] and Loeb[3] have given formulae which bridge the expressions of Langevin and Thomson in a formal way, but it

[1] For the definition of these types and a survey of the whole subject see the excellent treatment in: L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (John Wiley and Sons, New York, 1939).

[2] W. R. Harper, Phil. Mag. **18**, 97 (1934); **20**, 740 (1935).

[3] L. B. Loeb, Phys. Rev. **51**, 1110 (1937).