Theory of the Resonance Scattering of Protons and Neutrons on Helium

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A treatment of the general dispersion problem is presented in which the wave function, describing a resonance process appears as a monochromatic Fourier component of a wave packet, which is built in such a way that it represents for $t>0$ the decay of a compound state and for $t<0$ the building up of the same compound state. The method is applied to the resonance P scattering of protons and neutrons on helium.

THE existence of an unstable p state of He^5 and Li^5 and the correlated anomalies[1] in the scattering cross sections of protons and neutrons on He^4 represent a particularly interesting case of nuclear dispersion. It seems worth while to work out this case separately both for its simplicity and for the fact that here the finer features of the theory, arising from two levels with a splitting comparable to their width and from the interference of the so-called "potential scattering" with the resonance scattering are essential in the discussion of the experiments.

1. The Dispersion Formula

The general theory of nuclear dispersion has been well worked out by several authors.[2] Among these the treatment of Kapur and Peierls deserves special attention insofar as it requires a minimum of assumptions, the compound state of the system being introduced merely by the condition that at sufficient distance one will observe only out-going but no incoming waves. Their general results, however, become simple and useful, without a detailed knowledge of the system, only if one assumes further that the compound state has a long life.

We want to indicate a simple derivation of the dispersion formula where this assumption is introduced from the beginning, thus allowing the usual methods of perturbation theory, but where we start from the very existence of the compound state without any further assumption as do Kapur and Peierls. Particularly we want to abandon the unnecessary restriction that all nuclear processes which do not show resonance are to be described as potential scattering, i.e., are caused by forces acting solely on the incident particle. It is indeed noteworthy that the distinction between resonance and nonresonance processes is merely an approximate one, to be made only so long as the compound state has a comparatively long lifetime.* While for certain states of one and the same dynamical system this condition may be fulfilled, it will generally turn out that for others, particularly the states with higher energy, this distinction becomes entirely impracticable. In the study of resonance processes one is only interested, however, in the anomalies occurring within a certain small range of energies and it is quite sufficient to examine specially only those compound states which lie close to or within that range. We can and shall choose the number of these states to be finite, since, as we shall see below, it is by no means necessary that the eigenfunctions describing them form a complete set.

Let us consider a dynamical system consisting of certain nuclear particles. These particles may be any heavy nuclear fragments or even light quanta, electrons, neutrinos or mesotrons. If one could find the rigorous stationary solutions of the wave-mechanical problem they could obviously be classified into those which are energetically stable and those which are not. The stable solutions would be described by wave functions which vanish if any one of the particles

[1] See the preceding paper of H. Staub and H. Tatel; also W. P. Heydenburg and N. F. Ramsey, Phys. Rev. **57**, 106 (1940).

[2] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936); H. A. Bethe and G. Placzek, *ibid.* **51**, 450 (1937); H. A. Bethe, Rev. Mod. Phys. **9**, 69 (1937); F. Kalckar, J. R. Oppenheimer and R. Serber, Phys. Rev. **52**, 273 (1937); P. L. Kapur and R. Peierls, Proc. Roy. Soc. **A166**, 277 (1938); A. Siegert, Phys. Rev. **56**, 750 (1939).

* *Note added in proof.*—This point is also emphasized in a recent paper by G. Breit, Phys. Rev. **58**, 506 (1940).

is moved to infinite distance from any other, while the unstable ones could be written asymptotically (i.e., for large distances of certain particles from a remaining residual nucleus) as a product of a free solution of that particle times a wave function, describing the residual nucleus. In the idealized limit of infinitely long lifetimes of certain compound states there will now be a degeneracy between some of the unstable states which, by proper linear combinations, will yield new states of the same energy whose wave functions have the same asymptotic behavior as those of the energetically stable states, i.e., to vanish, as any one of the particles is infinitely removed from any other. A finite number of such metastable states, obtained in the way just described, we shall denote by a subscript $r(r=1, 2\cdots n)$ and their wave functions by φ_r. Besides these states r there will of course still be an infinite set of states, to be described by the subscript s, whose wave functions φ_s have the asymptotic behavior generally characteristic of unstable solutions.

Omitting the energetically stable solutions, we can then write the wave function describing our system in this limit in the form

$$\psi=\sum_r c_r e^{-iE_r t/\hbar}+\sum_s c_s e^{-iE_s t/\hbar}, \tag{1}$$

where the quantities c_r and c_s are arbitrary constants.

The rigorous time-dependent Schrödinger equation which ψ has to satisfy, will now imply that these quantities are generally no longer constants, but depend on the time in a way which, without further restrictions but merely by an appropriate choice of the states r and s is described by the equations

$$-(\hbar/i)\dot{c}_r=\sum_s V_{rs}e^{i(E_r-E_s)t/\hbar}, \tag{2a}$$

$$-(\hbar/i)\dot{c}_s=\sum_r V_{sr}e^{i(E_s-E_r)t/\hbar} \tag{2b}$$

with $V_{rs}=V_{sr}^*$. It is not necessary and generally not even possible to know the quantities V_{rs} nor any "small perturbation" V of which they may be considered to be the matrix elements.

As in the theory of spontaneous emission of radiation, one can show that there exist n characteristic solutions of Eqs. (2) which after a sufficient time $t\gg T$ take the form

$$c^+{}_{r\rho}=e^{-i(E_\rho-E_r-\frac{1}{2}i\Gamma_\rho)t/\hbar}d^+{}_{r\rho}, \tag{3a}$$

$$c^+{}_{s\rho}=\sum_r V_{sr}\frac{e^{-i(E_\rho-E_s-\frac{1}{2}i\Gamma_\rho)t/\hbar}-1}{E_\rho-E_s-\frac{1}{2}i\Gamma_\rho}d^+{}_{r\rho}; \tag{3b}$$

$$\rho=1, 2\cdots n,$$

where the quantities $d^+{}_{r\rho}$ are constants, satisfying the system of homogeneous equations

$$(E_\rho-\tfrac{1}{2}i\Gamma_\rho)d^+{}_{r\rho}=\sum_{r'}(H_{rr'}-\tfrac{1}{2}i\Gamma_{rr'})d^+{}_{r'\rho}; \tag{4}$$

$$r=1, 2\cdots n$$

with the n complex eigenvalues

$$E_\rho-\tfrac{1}{2}i\Gamma_\rho. \tag{5}$$

The index ρ of the quantities c_r and c_s indicates that they refer to a special solution of (4) representing the compound state ρ and the + sign indicates that this solution is valid for sufficiently large positive times.

Equation (3a) represents the exponential decay of the compound state ρ with lifetime $\hbar/\Gamma_\rho$ and Eq. (3b) the corresponding building up of the "free particle states" s.

Both the hermitian matrices $H_{rr'}$ and $\Gamma_{rr'}$ in (4) are to be derived from the matrix function

$$F_{rr'}(E_s)=2\pi\sum_\sigma V_{r\sigma}(E_s)V^*{}_{r'\sigma}(E_s)P_\sigma(E_s), \tag{6}$$

where instead of the subscript s we have used the energy E_s of the state s and another subscript σ which may be necessary for its complete characterization; the number of states lying between E_s and E_s+dE_s is given by $P_\sigma(E_s)dE_s$.

The limits of validity of Eqs. (3) are also determined by (6); let E_0 be the amount by which E_s has to vary in order that $F_{rr'}$ varies in the neighborhood of the values E_ρ by its own order of magnitude. E_0 will largely depend on the kinds of processes under consideration. If the outgoing particles are neutrons of not too high energies so that the main variation in $F_{rr'}$ arises from $P_\sigma(E_s)$ one will have to choose E_0 of the order of magnitude of the kinetic energy of the neutrons. In the presence of a Coulomb barrier E_0 may have to be chosen considerably smaller and for energies comparable to the nuclear binding energy E_0 will be of the order of these binding forces since in both these latter cases the quantity $V_{r\sigma}V^*_{r'\sigma}$ will be the determining factor.

Equations (3) can now claim validity if the two conditions

$$t \gg T = \hbar/E_0; \quad \Gamma_\rho \ll E_0 \quad (7) \text{ and } (8)$$

are satisfied. Relation (8) expresses the main assumption which underlies the perturbation treatment, namely that the lifetime $\hbar/\Gamma_\rho$ of the compound state ρ be large compared to the characteristic time $T = \hbar/E_0$.

From the solutions (3) we can obtain a somewhat more general solution of Eqs. (2) in the form

$$c^+_r = \sum_\rho K^+_\rho c^+_{r\rho}, \tag{9a}$$

$$c^+_s = \sum_\rho K^+_\rho c^+_{s\rho} \tag{9b}$$

with arbitrary constants K_ρ^+. Although (9) is valid only for $t \gg T$ we can investigate the form it will take if we integrate Eqs. (2) backward to negative times t. First we remark that the quantities c_r and c_s satisfy the same Eqs. (2) as c_r^* and c_s^* if V_{rs} is replaced by $V^*_{rs} = V_{sr}$ and t by $-t$. We thus obtain n further solutions of Eqs. (2), valid for $-t \gg T$ which, in analogy to (3), take the form

$$c^-_{r\rho} = e^{-i(E_\rho - E_r + \frac{1}{2}i\Gamma_\rho)t/\hbar} d^-_{r\rho}, \tag{10a}$$

$$c^-_{s\rho} = \sum_r V_{sr} \frac{e^{-i(E_\rho - E_s + \frac{1}{2}i\Gamma_\rho)t/\hbar} - 1}{E_\rho - E_s + \frac{1}{2}i\Gamma_\rho} d^-_{r\rho} \tag{10b}$$

with

$$(E_\rho + \tfrac{1}{2}i\Gamma_\rho) d^-_{r\rho} = \sum_{r'} (H_{rr'} + \tfrac{1}{2}i\Gamma_{rr'}) d^-_{r'\rho}. \tag{11}$$

Again condition (8) has to be satisfied and the $-$ sign indicates that (10) claims validity for sufficiently large negative times. Equations (4) and (11) lead to the same eigenvalues E_ρ and Γ_ρ since $H_{rr'} = H^*_{r'r}$ and $\Gamma_{rr'} = \Gamma^*_{r'r}$ so that the conjugate complex of (11) differs from (4) merely by the interchange of rows and columns of the matrix $H_{rr'} - \frac{1}{2}i\Gamma_{rr'}$ and thus leaves its complex eigenvalues $E_\rho - \frac{1}{2}i\Gamma_\rho$ unaltered. We further not that with proper normalization of the constants $d_{r\rho}$ (4) and the conjugate complex of (11) lead immediately to the relation

$$\sum_r d^+_{r\rho} d^*_{r\rho'} = \delta_{\rho\rho'}. \tag{12}$$

Finally we can write in analogy to (9)

$$c^-_r = \sum_\rho K^-_\rho c^-_{r\rho}, \tag{13a}$$

$$c^-_s = \sum_\rho K^-_\rho c^-_{s\rho} \tag{13b}$$

with n more constants K_ρ^-.

In order that (9) and (13) join on to the same solution at $t=0$ it can be shown under the condition (8) that there must exist the n relations

$$\sum_r K^+_\rho d^+_{r\rho} = \sum_r K^-_\rho d^-_{r\rho}; \quad r = 1, 2 \cdots n, \tag{14}$$

between the constants K_ρ^+ and K_ρ^- so that through (12) one set of constants can be expressed in terms of the other. Equations (10) evidently describe the manner in which a compound state ρ is built up for negative times which for positive times decays according to Eqs. (3).

Substituting (9) and (13) into (1) we now obtain two wave-functions ψ^+ and ψ^- which, except for higher order terms in Γ_ρ/E_0 express correctly one and the same solution ψ of the time-dependent Schrödinger equation for positive and negative times, respectively. For the dispersion problem we are interested in a stationary solution corresponding to a given energy E of the total system which we obtain from ψ by Fourier analysis in the form

$$\chi(E) = \int_{-\infty}^{+\infty} e^{iEt/\hbar} \psi(t) dt. \tag{15}$$

Furthermore we only need the asymptotic form $\chi^\tau(E)$ which (15) will assume as any particle Q is far removed from the residual nucleus. We shall denote by R, θ, Φ the polar coordinates of the particle Q relative to the center of gravity of the system and by τ a system of quantum numbers, characterizing the internal state of Q and the remaining nucleus as R approaches infinity. Before we can write down the definite expression for $\chi^\tau(E)$ we have to define the corresponding asymptotic forms of the functions φ_s, used in (1) since these only will contribute to χ.

Including the case where both particle Q and remaining nucleus are charged we then can write asymptotically

$$\varphi_s \rightarrow \varphi^\tau{}_\sigma(E_s) = \frac{1}{R} \sum_{l,m} [a^\tau{}_{\sigma,l,m}(E_s) e^{i(kR-\frac{1}{2}l\pi+\eta l-\alpha \log 2kR)} + b^\tau{}_{\sigma,l,m}(E_s) e^{-i(kR-\frac{1}{2}l\pi+\eta l-\alpha \log 2kR)}] Y_l{}^m(\theta, \Phi), \quad (16)$$

where the functions $Y_l{}^m$ are spherical harmonics, normalized to 4π on the sphere. Here again we have characterized the state s by its energy E_s and by an additional set of quantum numbers σ. The quantities $a^\tau{}_{\sigma,l,m}$ and $b^\tau{}_{\sigma,l,m}$ depend on all dynamical variables except R, θ and Φ. If $v = v(E_s)$ is the relative velocity of particle Q and remaining nucleus, corresponding to the total energy E_s of the system we have

$$k = mv/\hbar = k(E_s) \quad (17)$$

with m as the effective mass of particle Q. Further it is

$$\alpha = e_R e_Q / \hbar v = \alpha(E_s), \quad (18)$$

e_R and e_Q being the charges of remaining nucleus R and particle Q, respectively, and finally it is

$$\eta_l = \arg [(l+i\alpha)!] = \eta_l(E_s). \quad (19)$$

With (1), (9), (13) and (16) we then obtain from (15) asymptotically

$$\chi^\tau(E, R, \theta, \Phi) = -\frac{2\pi\hbar}{R} \sum_{r,\rho,\sigma,l,m} \left[K_\rho{}^+ d^+{}_{r\rho} a^\tau{}_{\sigma,l,m}(E) \frac{e^{i(kR-\frac{1}{2}l\pi+\eta l-\alpha \log 2kR)}}{E_\rho - E - \frac{1}{2}i\Gamma_\rho} + K_\rho{}^- d^-{}_{r\rho} b^\tau{}_{\sigma,l,m}(E) \frac{e^{-i(kR-\frac{1}{2}l\pi+\eta l-\alpha \log 2kR)}}{E_\rho - E + \frac{1}{2}i\Gamma_\rho} \right] V_{\sigma r}(E) P_\sigma(E) Y_l{}^m(\theta, \Phi). \quad (20)$$

This can be written in a more convenient form by using the relations (14). Writing as an abbreviation

$$S_\sigma{}^\pm(E) = -2\pi\hbar \sum_{r\rho} P_\sigma(E) V_{\sigma r}(E) d^+{}_{r\rho} \frac{K_\rho{}^\pm}{E_\rho - E \mp \frac{1}{2}i\Gamma_\rho}, \quad (21)$$

we have

$$\chi^\tau(E, R, \theta, \Phi) = \frac{1}{R} \sum_{\sigma,l,m} [S_\sigma{}^+(E) a^\tau{}_{\sigma,l,m}(E) e^{i(kR-\frac{1}{2}l\pi+\eta l-2 \log 2kR)} + S_\sigma{}^-(E) b^\tau{}_{\sigma,l,m}(E) e^{-i(kR-\frac{1}{2}l\pi+\eta l-2 \log 2kR)}] Y_l{}^m(\theta, \Phi), \quad (22)$$

where the quantities $S_\sigma{}^+$ and $S_\sigma{}^-$ are related by

$$S_\sigma{}^+(E) = S_\sigma{}^-(E) + i \sum_{\rho\sigma'} \frac{\Gamma_{\rho;\sigma\sigma'}(E)}{E_\rho - E - \frac{1}{2}i\Gamma_\rho} S^-{}_{\sigma'}(E) \quad (23)$$

with

$$\Gamma_{\rho;\sigma\sigma'}(E) = 2\pi \sum_{rr'} \overset{*}{\bar{d}}_{r'\rho} V_{r'\sigma'}(E) P_\sigma(E) V_{\sigma r}(E) d^+{}_{r\rho}. \quad (24)$$

Equation (22) may properly be called the general dispersion formula, since it contains all those features in the asymptotic form of the wave-function which are caused by long life compound states. Comparing (22) with the functions $\varphi_\sigma{}^\tau(E)$ of (16) we see that the compound states establish essentially a changed relation between the amplitudes of incoming and outgoing particle waves which is described by the relations (23) between the quantities $S_\sigma{}^+$ and $S_\sigma{}^-$. Besides the energies E_ρ and widths Γ_ρ of the compound states there enter in (23) the quantities $\Gamma_{\rho;\sigma\sigma'}(E)$ which over a range of the energy E, small compared to E_0 may be replaced by constants. These values are not quite independent of the matrices $H_{rr'}$ and $\Gamma_{rr'}$ from which E_ρ, Γ_ρ and $d^{\pm}{}_{r\rho}$ are derived through the equations (4) and (11). Indeed in this range, from (6)

$$\Gamma_{rr'} = F_{rr'}(E) = 2\pi \sum_\sigma V_{r\sigma}(E) V_{\sigma r'}(E) P_\sigma(E), \tag{25}$$

so that we have

$$\sum_\sigma \Gamma_{\rho;\sigma\sigma} = \sum_{rr'} \overset{*}{d^{-}}_{r'\rho} \Gamma_{rr'} d^{+}{}_{r\rho}. \tag{24a}$$

It seems to us that the closed form (22) forms a most convenient basis for the discussion of dispersion problems and we shall now proceed to specialize it to the case in which we are interested.

2. Elastic Scattering of a Particle with Spin $\frac{1}{2}$ by a Nucleus with Spin Zero

The main simplifying feature here lies in the conservation of angular momentum. We shall assume that the incident wave represents a particle with spin $\frac{1}{2}$ [3] in the direction of incidence (z direction). We then have to consider only states of the total system for which j_z, the z component of the total angular momentum, has the value $\frac{1}{2}$. Assuming that the spin orbit coupling is small compared to the spacing of levels with different orbital momentum l of the compound state, so that a given value l is still a good quantum number we can for $j_z = \frac{1}{2}$ characterize a state s besides its energy E_s by its orbital momentum l and its total angular momentum $j = l \pm \frac{1}{2}$. It will be sufficient to replace the index σ in (16) by $l \pm \frac{1}{2}$, having it understood that at the same time the orbital momentum shall assume the value l.

It is then easily seen that

$$\varphi^{\tau}{}_{l+\frac{1}{2}}(E_s) = \frac{1}{R} A^\tau \sin\left(kR - \frac{l\pi}{2} + \eta_l + \delta_{l+\frac{1}{2}} - \alpha \log 2kR\right) \frac{1}{(2l^2+l)^{\frac{1}{2}}} \times \left[[l(l+1)]^{\frac{1}{2}} Q_{\frac{1}{2}} Y_l{}^0(\theta, \Phi) + l Q_{-\frac{1}{2}} Y_l{}^1(\theta, \Phi) \right] \tag{26}$$

and

$$\varphi^{\tau}{}_{l-\frac{1}{2}}(E_s) = \frac{1}{R} A^\tau \sin\left(kR - \frac{l\pi}{2} + \eta_l + \delta_{l-\frac{1}{2}} - \alpha \log 2kR\right) \frac{1}{(2l^2+l)^{\frac{1}{2}}} \times \left[l Q_{\frac{1}{2}} Y_l{}^0(\theta, \Phi) - [l(l+1)]^{\frac{1}{2}} Q_{-\frac{1}{2}} Y_l{}^1(\theta, \Phi) \right], \tag{27}$$

where $Q_{\pm\frac{1}{2}}$ represents a spin eigenfunction of the particle Q with spin orientation in the positive and negative z direction, respectively, and where A^τ represents the wave function of the residual nucleus with spin zero. By comparison with (16) we obtain thus first from (26) and (27) the quantities a and b, entering in (22). They contain essentially the phases $\delta_{l\pm\frac{1}{2}}$ about which nothing can be said without knowing finer details of the interaction between particle Q and residual nucleus except that they will generally vary slowly with E.

It remains to determine the quantities $S_\sigma{}^{\pm}$ or $S^{\pm}{}_{l\pm\frac{1}{2}}$ in (22). If E lies close to the energy of a compound state with angular momentum l^* while all other compound states lie far away, we can for all values $l \neq l^*$ write

$$S^{+}{}_{l\pm\frac{1}{2}} = S^{-}{}_{l\pm\frac{1}{2}} = C_{l\pm\frac{1}{2}} e^{i(\eta_l + \delta_{l\pm\frac{1}{2}})}. \tag{28}$$

Since only states with equal angular momentum combine we obtain from (23) two separate equations for $S^{\pm}{}_{l+\frac{1}{2}}$ and $S^{\pm}{}_{l-\frac{1}{2}}$ which, using (24a) for $l = l^*$, can be written in the form

$$S^{+}{}_{l^*\pm\frac{1}{2}} = S^{-}{}_{l^*\pm\frac{1}{2}} \left(1 + i \frac{\Gamma_{l^*\pm\frac{1}{2}}}{E_{l^*\pm\frac{1}{2}} - E - \frac{1}{2} i \Gamma_{l^*\pm\frac{1}{2}}}\right) = S^{-}{}_{l^*\pm\frac{1}{2}} \frac{E_{l^*\pm\frac{1}{2}} - E + \frac{1}{2} i \Gamma_{l^*\pm\frac{1}{2}}}{E_{l^*\pm\frac{1}{2}} - E - \frac{1}{2} i \Gamma_{l^*\pm\frac{1}{2}}}. \tag{29}$$

[3] All angular momenta are measured in units $\hbar = h/2\pi$.

Or with

$$\gamma_{l^*\pm\frac{1}{2}}=\arctan\frac{1}{2}\frac{\Gamma_{l^*\pm\frac{1}{2}}}{E_{l^*\pm\frac{1}{2}}-E-\frac{1}{2}i\Gamma_{l^*\pm\frac{1}{2}}}, \tag{30}$$

$$e^{-i\gamma_{l^*\pm\frac{1}{2}}}S^{+}{}_{l^*\pm\frac{1}{2}}=e^{i\gamma_{l^*\pm\frac{1}{2}}}S^{-}{}_{l^*\pm\frac{1}{2}}=C_{l^*\pm\frac{1}{2}}e^{i(\eta_{l^*}+\delta_{l^*\pm\frac{1}{2}})}. \tag{31}$$

Using (26), (27), (28), and (31) we can now write (22) in the form

$$\chi^{\tau}(E,R,\theta,\Phi)=\frac{A^{\tau}}{R}\sum_{l}\bigg\{C_{l+\frac{1}{2}}e^{i\beta_{l+\frac{1}{2}}}\sin\left(kR-\frac{l\pi}{2}+\beta_{l+\frac{1}{2}}-\alpha\log 2kR\right)\frac{1}{(2l^2+l)^{\frac{1}{2}}}[[l(l+1)]^{\frac{1}{2}}Q_{\frac{1}{2}}Y_l^0(\theta,\Phi)$$
$$+lQ_{-\frac{1}{2}}Y_l^1(\theta,\Phi)]+C_{l-\frac{1}{2}}e^{i\beta_{l-\frac{1}{2}}}\sin\left(kR-\frac{l\pi}{2}+\beta_{l-\frac{1}{2}}-\alpha\log 2kR\right)\frac{1}{(2l^2+l)^{\frac{1}{2}}}$$
$$\times[lQ_{\frac{1}{2}}Y_l^0(\theta,\Phi)-[l(l+1)]^{\frac{1}{2}}Q_{-\frac{1}{2}}Y_l^1(\theta,\Phi)]\bigg\}, \tag{32}$$

where we have introduced

$$\beta_{l\pm\frac{1}{2}}=\delta_{l\pm\frac{1}{2}}+\eta_l \quad \text{for} \quad l\neq l^* \tag{33a}$$

and

$$\beta_{l^*\pm\frac{1}{2}}=\delta_{l^*\pm\frac{1}{2}}+\gamma_{l^*\pm\frac{1}{2}}+\eta_{l^*} \tag{33b}$$

and where the energy dependent quantities η_l, $\delta_{l\pm\frac{1}{2}}$ and k are to be taken with those values which they assume for the energy E. We now have to determine the constants $C_{l\pm\frac{1}{2}}$ in (32) in such a way that the incoming spherical waves are those belonging asymptotically to a plane wave of the form

$$A^{\tau}e^{i[kZ+i\alpha\log k(R-Z)]}=\frac{1}{kR}A^{\tau}\sum_{l=0}^{\infty}(2l+1)^{\frac{1}{2}}i^l\sin\left(kR-\frac{l\pi}{2}-\alpha\log 2kR\right)Q_{\frac{1}{2}}Y_l^0(\theta,\Phi). \tag{34}$$

This means that

$$lC_{l+\frac{1}{2}}-[l(l+1)]^{\frac{1}{2}}C_{l-\frac{1}{2}}=0,\quad [l(l+1)]^{\frac{1}{2}}C_{l+\frac{1}{2}}+lC_{l-\frac{1}{2}}=(2l^2+l)^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}i^l/k$$

or

$$C_{l+\frac{1}{2}}=(l+1)^{\frac{1}{2}}i^l/k;\quad C_{l-\frac{1}{2}}=l^{\frac{1}{2}}i^l/k. \tag{35a) and (35b}$$

Substituting these values into (32) we find

$$\chi^{\tau}=\frac{A^{\tau}}{kR}\sum_{l}\frac{i^l}{(2l+1)^{\frac{1}{2}}}\{e^{i\beta_{l+\frac{1}{2}}}\sin(kR-\tfrac{1}{2}l\pi+\beta_{l+\frac{1}{2}}-\alpha\log 2kR)[(l+1)Q_{\frac{1}{2}}Y_l^0+[l(l+1)]^{\frac{1}{2}}Q_{-\frac{1}{2}}Y_l^1]$$
$$+e^{i\beta_{l-\frac{1}{2}}}\sin(kR-\tfrac{1}{2}l\pi+\beta_{l-\frac{1}{2}}-\alpha\log 2kR)[lQ_{\frac{1}{2}}Y_l^0-[l(l+1)]^{\frac{1}{2}}Q_{-\frac{1}{2}}Y_l^1]\}. \tag{36}$$

Subtracting (34) from (36) we obtain for the scattered wave

$$\chi_s=\frac{A^{\tau}}{kR}e^{i(kR-\alpha\log 2kR)}\bigg\{Q_{\frac{1}{2}}\sum_l\frac{1}{(2l+1)^{\frac{1}{2}}}[(l+1)e^{i\beta_{l+\frac{1}{2}}}\sin\beta_{l+\frac{1}{2}}+le^{i\beta_{l-\frac{1}{2}}}\sin\beta_{l-\frac{1}{2}}]Y_l^0$$
$$+Q_{-\frac{1}{2}}\sum_l\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}[e^{i\beta_{l+\frac{1}{2}}}\sin\beta_{l+\frac{1}{2}}-e^{i\beta_{l-\frac{1}{2}}}\sin\beta_{l-\frac{1}{2}}]Y_l^1\bigg\}. \tag{37}$$

The first part in the bracket of (37) represents those particles which have not altered the orientation of their spin in the scattering process, the second those which have reversed it. In the total scattering cross section these two parts contribute incoherently and we thus find for the cross section per unit solid angle

$$\frac{dS}{d\omega}=\frac{1}{k^2}\left\{\left|\sum_l \frac{1}{(2l+1)^{\frac{1}{2}}}[(l+1)e^{i\beta_{l+\frac{1}{2}}}\sin\beta_{l+\frac{1}{2}}+le^{i\beta_{l-\frac{1}{2}}}\sin\beta_{l-\frac{1}{2}}]Y_l{}^0\right|^2 + \left|\sum_l\left(\frac{l(l+1)}{2l+1}\right)^{\frac{1}{2}}[e^{i\beta_{l+\frac{1}{2}}}\sin\beta_{l+\frac{1}{2}}-e^{i\beta_{l-\frac{1}{2}}}\sin\beta_{l-\frac{1}{2}}]Y_l{}'\right|^2\right\}. \quad (38)$$

It remains to further specialize this formula to the case where the compound state is a P state, i.e., where $l^*=1$ and to introduce those simplifications which seem likely to be valid in the case of the scattering of protons and neutrons on He^4. First of all, because of the smallness of spin-orbit forces we may assume that

$$\delta_{l+\frac{1}{2}}=\delta_{l-\frac{1}{2}}=\delta_l. \quad (39)$$

This means with Eq. (33a) that

$$\beta_{l+\frac{1}{2}}=\beta_{l-\frac{1}{2}}=\beta_l=\eta_l+\delta_l \quad \text{for} \quad l\neq l^*. \quad (40)$$

For $l=l^*$ this equation cannot be maintained to be valid since $\gamma_{l^*+\frac{1}{2}}$ and $\gamma_{l^*-\frac{1}{2}}$ will vary irregularly in the neighborhood of the two different values $E_{l^*+\frac{1}{2}}$ and $E_{l^*-\frac{1}{2}}$ of the energy E, respectively. However, the smallness of spin-orbit forces will allow one further simplification, namely, that we may set

$$\Gamma_{l^*+\frac{1}{2}}=\Gamma_{l^*-\frac{1}{2}}=\Gamma \quad (41)$$

In Eq. (30). Formula (38) may thus be simplified to

$$\frac{dS}{d\omega}=\left\{\left|f(\theta)+\frac{1}{2k}\frac{e^{2i(\eta_{l^*}+\delta_{l^*})}}{(2l^*+1)^{\frac{1}{2}}}\left[(l^*+1)\frac{\Gamma}{E_{l^*+\frac{1}{2}}-E-\frac{1}{2}i\Gamma}+l^*\frac{\Gamma}{E_{l^*-\frac{1}{2}}-E-\frac{1}{2}i\Gamma}\right]Y^0{}_{l^*}\right|^2 + \frac{1}{4k^2}\frac{l^*(l^*+1)}{2l^*+1}\left|\frac{\Gamma}{E_{l^*+\frac{1}{2}}-E-\frac{1}{2}i\Gamma}-\frac{\Gamma}{E_{l^*-\frac{1}{2}}-E-\frac{1}{2}i\Gamma}\right|^2|Y^1{}_{l^*}|^2\right\} \quad (42)$$

with

$$f(\theta)=\frac{1}{k}\sum_l (2l+1)^{\frac{1}{2}}e^{i(\eta_l+\delta_l)}\sin(\eta_l+\delta_l)Y_l{}^0(\theta) \quad (43)$$

and using (30) and (41)

$$e^{i(\eta_{l^*}+\delta_{l^*}+\gamma_{l^*\pm\frac{1}{2}})}\sin(\eta_{l^*}+\delta_{l^*}+\gamma_{l^*\pm\frac{1}{2}})-e^{i(\eta_{l^*}+\delta_{l^*})}\sin(\eta_{l^*}+\delta_{l^*})=e^{2i(\eta_{l^*}+\delta_{l^*})}\frac{\Gamma}{E_{l^*\pm\frac{1}{2}}-E-\frac{1}{2}i\Gamma}.$$

Specializing further to $l^*=1$ and using

$$Y_1{}^0=3^{\frac{1}{2}}\cos\theta, \quad Y_1{}^1=(3/2)^{\frac{1}{2}}e^{i\varphi}\sin\theta,$$

we obtain

$$\frac{dS}{d\omega}=\left|f(\theta)+\frac{1}{k}e^{2i(\eta_1+\delta_1)}\left(\frac{\Gamma}{E_{3/2}-E-\frac{1}{2}i\Gamma}+\frac{1}{2}\frac{\Gamma}{E_{1/2}-E-\frac{1}{2}i\Gamma}\right)\cos\theta\right|^2 + \frac{1}{4k^2}\left|\frac{\Gamma}{E_{3/2}-E-\frac{1}{2}i\Gamma}-\frac{\Gamma}{E_{1/2}-E-\frac{1}{2}i\Gamma}\right|^2\sin^2\theta. \quad (44)$$

This formula cannot be simplified any further without special assumptions about the function $f(\theta)$, or, according to (43) about the phases δ_l.

3. Scattering of Protons on Helium

We will here for simplicity assume, that for all values of l we may take $\delta_l=0$ so that all the non-resonant scattering is due to the Coulomb field; there may be course easily be a noticeable nuclear "potential" scattering, particularly for $l=0$ and which, if necessary, could be taken account by taking in (43) $\delta_0 \neq 0$. Omitting this correction we obtain for $f(\theta)$ the well-known form which leads to the Rutherford scattering:

$$f(\theta) = -\frac{\alpha}{2k \sin^2 \theta/2} e^{i\alpha \log 2(1-\cos\theta)+2i\eta_0}. \tag{45}$$

Using with (19)

$$\eta_1 - \eta_0 = \arg \frac{(1+i\alpha)!}{(i\alpha)!} = \arg(1+i\alpha) = \arctan \alpha,$$

we obtain from (44)

$$\frac{dS}{d\omega} = \lambda\!\!\!^{-2} \left[\left| \frac{\alpha}{2\sin^2\theta/2} - e^{i[2\arctan\alpha + \alpha\log 2(1-\cos\theta)]} \cos\theta \left(\frac{\Gamma}{E_{3/2}-E-\frac{1}{2}i\Gamma} + \frac{1}{2}\frac{\Gamma}{E_{1/2}-E-\frac{1}{2}i\Gamma} \right) \right|^2 + \frac{\sin^2\theta}{4} \left| \frac{\Gamma}{E_{3/2}-E-\frac{1}{2}i\Gamma} - \frac{\Gamma}{E_{1/2}-E-\frac{1}{2}i\Gamma} \right|^2 \right\}. \tag{46}$$

Here we have written $1/k = \lambda\!\!\!^-$ and α stands according to (18) for $2e^2/\hbar v$, v being the velocity of the incident proton and $2\pi\lambda\!\!\!^-$ its wave-length in the system in which the center of gravity is at rest. It is evident from (46) that after subtraction of the pure Rutherford term $\alpha^2\lambda\!\!\!^{-2}/4\sin^4\theta/2$ there still remains even for appreciable scattering angles θ and for protons of about 3 Mev ($\alpha=0.18$) a noticeable influence of the Coulomb scattering, due to its interference with the resonance term. Particularly for $\theta<\pi/2$ this fact must be taken into account for any conclusion drawn from the scattering of protons on He^4 upon the existence of a compound P state of Li^5.

4. Scattering of Neutrons on Helium

Since in this case $\alpha=0$ it follows from (19) that $\eta_l=0$ for all values of l. Evidently we may not assume here that also all the phases δ_l are zero since otherwise according to (43) $f(\theta)=0$ which would mean, that the scattering of neutrons on He^4 is caused entirely by resonance. It will, however, for the energies actually used be a good approximation to assume that only $\delta_0 \neq 0$. In this case we obtain from (43), since $Y_0^0(\theta)=1$

$$f(\theta) = (1/k)e^{i\delta_0}\sin\delta_0 \tag{47}$$

and from (44)

$$\frac{dS}{dw} = \lambda\!\!\!^{-2} \left\{ \left| \sin\delta_0 + e^{i\delta_0} \left(\frac{\Gamma}{E_{3/2}-E-\frac{1}{2}i\Gamma} + \frac{1}{2}\frac{\Gamma}{E_{1/2}-E-\frac{1}{2}i\Gamma} \right) \cos\theta \right|^2 + \frac{\sin^2\theta}{4} \left| \frac{\Gamma}{E_{3/2}-E-\frac{1}{2}i\Gamma} - \frac{\Gamma}{E_{1/2}-E-\frac{1}{2}i\Gamma} \right|^2 \right\}, \tag{48}$$

where we have again written $2\pi\lambda\!\!\!^- = 2\pi/k$ for the neutron wave-length in the system in which the center of gravity is at rest. The discussion of this formula is given in the preceding paper by H. Staub and H. Tatel.