NOVEMBER 1, 1940 PHYSICAL REVIEW VOLUME 58

The Directional Dependence of Electrical Conductivity*

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(Received July 23, 1940)

An attempt is made to apply the theory of conductivity to the question of directional dependence. The first section reviews the usual theory, developing the basic equations in the somewhat generalized form necessary for the problem. The next section discusses the nature of the electron-lattice interaction in some detail. This seems desirable since a number of very significant points—notably the polarization of the lattice waves, and the role of strictly transverse waves when these actually occur—seem to have received much too causal attention in the past. In a third section the directional dependence is calculated for several simple models. No very serious attempt is made to relate the theoretical results with experimental data.

INTRODUCTION

THE conductivity of a general metallic crystal is adequately specified when conductivities along three principal conductivity axes are given. This means that the conductivity is completely described by a second-order symmetric tensor, and that by proper choice of axes this tensor can be diagonalized. The effect of an arbitrary electric field may be determined by considering the components along the principal axes. Each component of the current density is obtained by multiplying the field component by the appropriate conductivity. In general, of course, the resultant current is not in the direction of the field.

What has been said indicates that the conductivity of a cubic crystal is isotropic, an anisotropy first entering for tetragonal, hexagonal and trigonal crystals. These have two principal conductivities—parallel and perpendicular to the axis of maximum symmetry. The directional dependence of the conductivity can be indicated by the ratio of these two. For some common metals at room temperature:

$$\sigma_{\parallel}/\sigma_{\perp}=1.21\ (\mathrm{Mg});\quad 0.93\ (\mathrm{Zn});\quad 0.83\ (\mathrm{Cd});\quad 0.69\ (\mathrm{Sn}).$$

The first three are hexagonal close-packed; the last is tetragonal.

The general equations of the theory of conductivity include the factors from which the directional dependence arises, but, since in most applications this dependence is not of interest, simplifying assumptions are introduced which lead to an isotropic result. In the present investigation it is proposed to return to the general equations and consider the way in which anisotropy enters for certain models. Some attention will be given to application to real metals, but it will be seen that any serious attempt to make such application is premature.

(1) GENERAL THEORY[1]

A calculation of the conductivity requires the determination of the slight modification produced in the distribution function of the electrons by an external field. The modified distribution may be assumed to be:

$$f(\mathbf{K})=f_0(E)+g(\mathbf{K}). \tag{1}$$

This gives the probability that the state $\mathbf{K}$ in wave vector space is occupied. In the absence of the field it reduces to the Fermi function, $f_0(E)$. The assumed distribution is more definitely determined by substitution into the stationary state equation:

$$\left(\frac{\partial f}{\partial t}\right)_{\text{lattice}}=+\frac{e}{\hbar}\frac{df_0}{dE}\operatorname{grad}_K E\cdot\mathbf{F}, \tag{2}$$

which states that the distribution is unchanging, the effect of the external field, $\mathbf{F}$ being just balanced by transitions arising from the interaction of the electrons with the lattice.

The effect of the electron lattice interaction is

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[1] This section is based on the treatment of Sommerfeld and Bethe (*Handbuch der Physik*, Vol. 24/2). Very slight modifications have been made to fit the treatment to the present purpose.

calculated by the usual perturbation methods. The probability that $\mathbf{K}'$ is occupied at some finite time as a result of the transition $\mathbf{K}\rightarrow\mathbf{K}'$, $\mathbf{K}$ having been occupied and $\mathbf{K}'$ empty at zero time is:

$$\frac{|U(\mathbf{K},\mathbf{K}')|^2}{\hbar^2}\Omega(x) \equiv \frac{|U(\mathbf{K},\mathbf{K}')|^2}{\hbar^2}\,\frac{2[1-\cos(xt/\hbar)]}{(x/\hbar)^2}. \qquad (3)$$

Here: $x=E_{K'}-E_K\pm\hbar\omega$ ($\hbar\omega$ is the energy of the lattice quantum involved), and $U(\mathbf{K},\mathbf{K}')$ is the matrix element of the perturbation. (Departure of the lattice potential, $V(\mathbf{r})$ from periodicity.)

The probable transitions are those in which energy is conserved and for which:

$$\mathbf{K}'=\mathbf{K}\pm\mathbf{q}+2\pi\mathbf{g}. \qquad (4)$$

$\mathbf{q}$ is the propagation vector of the lattice wave; $\mathbf{g}$ a vector in the reciprocal lattice. The plus sign corresponds to absorption of a lattice quantum; the negative to emission. The matrix element to be associated with a given transition is:

$$U(\mathbf{K},\mathbf{K}')=\left(\frac{N_{qj}+1}{2M\omega_{qj}}\right)^{\frac{1}{2}}K_j \quad \text{(emission)}$$
$$\text{or } U(\mathbf{K},\mathbf{K}')=\left(\frac{N_{qj}}{2M\omega_{qj}}\right)^{\frac{1}{2}}K_j \quad \text{(absorption).} \qquad (5)$$

(j indicates the polarization of the lattice wave.)

$$N_{qj}=(\exp[\hbar\omega_{qj}/kT]-1)^{-1} \qquad (6)$$

gives the probability that the wave is excited.

$$K_j=\int_{\text{unit cell}} u_K(\mathbf{r})(\partial V/\partial s_j)u_{K'}{}^*(\mathbf{r})d\tau, \qquad (7)$$

$\partial V/\partial S_j$ being the component of grad V in the direction of polarization of the wave, and $u_K(\mathbf{r})$ the periodic part of the electron wave function.

The contribution to $(\partial f(\mathbf{K})/\partial t)_{\text{lattice}}$ arising from interaction with states of higher energy is:

$$\sum_{K',j}\frac{K_j{}^2\Omega(E_K-E_{K'}+\hbar\omega_{qj})}{2\hbar^2M\omega_{qj}}\{(N_{qj}+1)f(\mathbf{K}')(1-f(\mathbf{K}))-N_{qj}f(\mathbf{K})(1-f(\mathbf{K}'))\}, \qquad (8)$$
$$\mathbf{K}-\mathbf{K}'=\pm\mathbf{q}.$$

The $\mathbf{K}'$ summation may be replaced by an integral if we introduce:

$$\text{Density of states}=1/8\pi^3, \qquad (9)$$
$$\text{Volume element}=(dE_{K'}/|\text{grad}_{K'}\,E|)dS',$$

where dS' is an element of area lying in the constant energy surface. As is always the case in this type of problem the energy integration can be carried out. One finally has for $(\partial f/\partial t)_{\text{lattice}}$ due to exchange of electrons with states near $\mathbf{K}'$ of higher energy:

$$\frac{K_j{}^2dS'}{8\pi^2\hbar M\omega_{qj}|\text{grad}_{K'}\,E|}\{(N_{qj}+1)f(\mathbf{K}')(1-f(\mathbf{K}))-N_{qj}f(\mathbf{K})(1-f(\mathbf{K}'))\}, \qquad (10)$$
$$E_{K'}=E_K+\hbar\omega_{qj}.$$

The interaction with states of lower energy leads to a similar expression:

$$\frac{K_j{}^2dS'}{8\pi^2\hbar M\omega_{qj}|\text{grad}_{K'}\,E|}\{N_{qj}f(\mathbf{K}')(1-f(\mathbf{K}))-(N_{qj}+1)f(\mathbf{K})(1-f(\mathbf{K}'))\}, \qquad (11)$$
$$E_{K'}=E_K-\hbar\omega_{qj}.$$

At *high temperatures* ($T\gg\theta_D$), $\hbar\omega_{qj}/kT\ll1$ for all lattice waves, so that:

$$N_{qj}\doteq N_{qj}+1\doteq kT/\hbar\omega_{qj} \text{ and } f(\mathbf{K}'), \text{ at } E_K+\hbar\omega_{qj}\doteq f(\mathbf{K}'), \text{ at } E_K-\hbar\omega_{qj};\ f_0(E_K)\doteq f_0(E_K\pm\hbar\omega_{qj}). \qquad (12)$$

Both emission and absorption terms simplify and when combined lead to the following condition for the stationary state:

$$\frac{kT}{4\pi^2M\hbar^2}\int_{S'}\sum_j\left(\frac{K_j}{\omega_{qj}}\right)^2\{g(\mathbf{K}')-g(\mathbf{K})\}\frac{dS'}{|\text{grad}_{K'}\,E|}=\frac{e}{\hbar}\frac{df_0}{dE}\text{grad}_K\,E\cdot\mathbf{F}. \qquad (13)$$

The substitution:

$$g(\mathbf{K}) = -(df_0/dE)g(\vartheta, \varphi) \tag{14}$$

is convenient. Here $g(\vartheta, \varphi)$ is independent of the energy, ϑ and φ being coordinates which fix the point on a surface of constant energy in K-space. For present purposes the high temperature equation can be written:

$$\int_{S'} \sum_j \left(\frac{K_j}{\omega_{q_j}}\right)^2 \{g(\vartheta', \varphi') - g(\vartheta, \varphi)\} \frac{dS'}{|\mathrm{grad}_{K'} E|} = \mathrm{constant} \frac{\partial E}{\partial K_{\mathrm{field}}}. \tag{15}$$

At *low temperatures* ($T \ll \theta_D$), $\hbar\omega_{q_j}/kT \gg 1$ for most lattice waves. The energy difference between initial and final states is important and specific values of N_{q_j} and $N_{q_j}+1$ must be considered. $(\partial f/\partial t)_{\mathrm{lattice}}$ is found to be proportional to:

$$\frac{1}{|\mathrm{grad}_K E|} \sum_j \int_0^{q_{\max}} dq \int_0^{2\pi} d\beta \frac{N_{q_j} C_j^2 q^3}{\omega_{q_j}} \{g(\vartheta', \varphi') - g(\vartheta, \varphi)\} \{f_0(E+\hbar\omega)e^{\hbar\omega/kT} + f_0(E-\hbar\omega)\}. \tag{16}$$

Since only small q are significant, K_j has been taken proportional to q ($K_j = C_j q$) (justification later) and $dS = q\,dq\,d\beta$ where β is the azimuthal angle of a polar coordinate system with axis along $\mathrm{grad}_K E$.

It is desirable to expand in powers of q the difference entering the stationary state equation. The linear term is found to give zero on integration over β, so that the approximation is:

$$g(\vartheta', \varphi') - g(\vartheta, \varphi) \doteq q^2 \delta(\vartheta, \varphi, \beta). \tag{17}$$

It is convenient in the low temperature calculations to treat ω as the independent variable rather than q. Since for low frequencies q is proportional to ω, this can be done by setting:

$$q_j(\omega, \Theta, \Phi) = F_j(\Theta, \Phi)\omega/\bar{u}_j, \tag{18}$$

where $\bar{u}_j$ is an average velocity of the lattice wave of polarization j, and $F_j(\Theta, \Phi)$ takes into account the dependence of the velocity on the direction of propagation. Further, the range of the ω (or q) integral may be extended indefinitely. Finally the double integral becomes:

$$\int_0^\infty \frac{f_0(E+\hbar\omega)e^{\hbar\omega/kT} + f_0(E-\hbar\omega)}{e^{\hbar\omega/kT} - 1} \omega^4 d\omega \int_0^{2\pi} \sum_j \left(\frac{C_j F_j^3}{\bar{u}_j^3}\right)^2 \delta(\vartheta, \varphi, \beta) d\beta. \tag{19}$$

In the approximation used here the second integral does not depend on the energy. Its value is to be determined for $E = \zeta$. With $E = \zeta$, the first integral may be easily evaluated. This brings in the T^5 variation of the resistance, but is of no particular interest here. For present purposes the stationary state equation can be written:

$$\int_0^{2\pi} \sum_j \left(\frac{C_j F_j^3}{\bar{u}_j^3}\right)^2 \delta(\vartheta, \varphi, \beta) d\beta = \mathrm{constant} \left(|\mathrm{grad}_K E| \frac{\partial E}{\partial K_{\mathrm{field}}}\right)_{E=\zeta}. \tag{20}$$

Solutions of the equations

If one assumes (1) that the surfaces of constant energy are spheres, and (2) that the "interaction function" ($\sum_j (K_j/\omega_{q_j})^2$ or $\sum_j (C_j F_j^3/\bar{u}_j^3)^2$) does not depend on the direction of propagation of the lattice wave the stationarity equations are readily solved. It is found that the modification in the distribution function is proportional to the cosine of the angle between $\mathbf{K}$ and the electric field, and that the coefficient does not depend on the direction of the field. This means that the distribution is shifted bodily without distortion, and that the amount of the shift is independent of the field direction. If either assumption fails, the solution becomes more complicated. The center of gravity of the distribution is shifted and the shape slightly altered, both the magnitude of the shift and the nature of the distortion depending on the direction of the field. In Section (3) the equations will be solved for several simple models.

Calculation of the current from the distribution function

The current density in the zth direction is:

$$j_z=\frac{2e}{8\pi^3\hbar}\int_0^\infty -\frac{df_0}{dE}dE\int_S \frac{g(\vartheta,\,\varphi)(\partial E/\partial K_z)}{|\mathrm{grad}_K\, E|}dS. \tag{21}$$

The functions in the second integral are all taken independent of E and written for $E=\zeta$, since the integrand in the first integral has a sharp maximum at this value. Evaluating the first integral:

$$j_z=\frac{e}{4\pi^3\hbar}\int_S \frac{g(\vartheta,\,\varphi)(\partial E/\partial K_z)}{|\mathrm{grad}_K\, E|}dS. \tag{22}$$

For a metal with general axial symmetry and a spherical Fermi surface of radius K this gives:

$$j_{||}=\frac{eK^2}{2\pi^2\hbar}\int_0^\pi g(\vartheta)\cos\vartheta\sin\vartheta d\vartheta$$

(field parallel to axis) (23)

$$j_\perp=\frac{eK^2}{4\pi^3\hbar}\int_0^{2\pi} d\varphi\int_0^\pi d\vartheta g(\vartheta,\,\varphi)\sin^2\vartheta\cos\varphi.$$

(field perpendicular to axis).

If the energy depends on the wave vector as:

$$E(\mathbf{K})=\zeta\left(\frac{K_x^2+K_y^2}{a^2}+\frac{K_z^2}{c^2}\right), \tag{24}$$

we get an ellipsoidal Fermi surface:

$$(K_x^2+K_y^2)/a^2+K_z^2/c^2=1, \tag{25}$$

which may be represented parametrically:

$$K_x=a\sin\vartheta\cos\varphi,\quad K_y=a\sin\vartheta\sin\varphi,\; K_z=c\cos\vartheta.$$

(ϑ is the usual parametric angle—not the polar angle. In the limiting case of the sphere the distinction disappears.) For this ellipsoid:

$$dS=a\sin\vartheta(a^2\cos^2\vartheta+c^2\sin^2\vartheta)^{\frac{1}{2}}d\vartheta d\varphi \tag{26}$$

$$\partial E/\partial K_x=(2\zeta/a)\sin\vartheta\cos\varphi,\quad \partial E/\partial K_y=(2\zeta/a)\sin\vartheta\sin\varphi,\quad \partial E/\partial K_z=(2\zeta/c)\cos\vartheta, \tag{27}$$

so that for field along z:

$$j_{||}=\frac{ea^2}{2\pi^2\hbar}\int_0^\pi g(\vartheta)\cos\vartheta\sin\vartheta d\vartheta, \tag{28}$$

while for field along x:

$$j_\perp=\frac{eac}{4\pi^3\hbar}\int_0^{2\pi} d\varphi\int_0^\pi d\vartheta g(\vartheta,\,\varphi)\sin^2\vartheta\cos\varphi. \tag{28'}$$

In addition to altering $g(\vartheta,\,\varphi)$, the change in the form of $E(\mathbf{K})$ causes the electron velocity to vary with position on the Fermi surface, multiplying $\sigma_{||}/\sigma_\perp$ by the factor a/c.

(2) Electron Lattice Interaction

This section considers in some detail the nature of the interaction functions:

$$\begin{array}{ll}\sum_j(K_j/\omega_{q_j})^2 & \text{(high temperatures),}\\ \sum_j(C_jF_j^3/\bar{u}_j^3)^2 & \text{(low temperatures).}\end{array} \tag{29}$$

This seems advisable since the published work in explicit calculations treats only spherical distributions and spherically symmetric wave functions, and assumes that the lattice waves can be treated as longitudinal and transverse. If this assumption is false the sum over three lattice waves must be retained in any case. If, further, general wave functions are admitted, even strictly transverse waves have importance. It is found in what follows that for the hexagonal metals at low temperatures, it is this last point which is significant. That is, the elastic waves are found to be essentially longitudinal and transverse, but both types contribute to the resistance. It seems likely that the transverse waves are the more important for most metals, particularly at low temperatures.

If by use of the wave equation the potential energy is eliminated from K_j [Eq. (7)], the following is obtained:[2]

$$K_j=\frac{\hbar^2}{2m}\mathbf{q}\cdot\int\{2i\,\mathrm{grad}\,u_K+(\mathbf{K}+\mathbf{K}')u_K\}\frac{\partial u_{K'}^*}{\partial s_j}d\tau. \tag{30}$$

This simplifies if E is a function of $|\mathbf{K}|$, $(|\mathbf{K}|=|\mathbf{K}'|)$, or if u has certain symmetry properties (see below).

$$K_j=\frac{\hbar^2 i}{m}\mathbf{q}\cdot\int\mathrm{grad}\,u_K\left(\frac{\partial u_{K'}}{\partial s_j}\right)d\tau. \tag{30'}$$

In general at low temperatures ($\mathbf{q}$ small, $\mathbf{K}\approx\mathbf{K}'$) it has the form:

$$K_j=\frac{\hbar^2}{m}\mathbf{q}\cdot\int(i\,\mathrm{grad}\,u_K+\mathbf{K}u_K)\frac{\partial u_K}{\partial s_j}d\tau. \tag{30''}$$

"Hexagonal metal"

We here assume that the scattering depends upon:

$$K_j=\frac{\hbar^2 i}{m}qC_j\equiv\frac{\hbar^2 i}{m}q\int\frac{\partial u}{\partial q}\frac{\partial u}{\partial s_j}d\tau, \tag{31}$$

where $\partial u/\partial q$ denotes the component of grad u along $\mathbf{q}$. The assumption implies that u is real and independent of $\mathbf{K}$, and that the second term in the integrand of the matrix element [Eq. (30)] can be neglected.

Let $\mathbf{q}$ make an angle Θ with the hexagonal axis and lie in the $\Phi=0$ plane. The unit polarization vectors are essentially:

$S_1\sim\sin\Theta,\quad 0,\quad \cos\Theta\qquad$ (longitudinal)
$S_2\sim 0,\quad 1,\quad 0\qquad$ (transverse $\perp$ plane)
$S_3\sim -\cos\Theta,\quad 0,\quad \sin\Theta.\qquad$ (transverse in plane)

If we then take $u(\mathbf{r})=u(r,\vartheta)$ we have:

$$\begin{aligned}C_1&=\int\left\{\cos^2\Theta\left(\frac{\partial u}{\partial z}\right)^2+\sin^2\Theta\left(\frac{\partial u}{\partial x}\right)^2\right\}d\tau=C_z\cos^2\Theta+C_x\sin^2\Theta,\\ C_2&=0,\\ C_3&=\int\sin\Theta\cos\Theta\left\{\left(\frac{\partial u}{\partial z}\right)^2-\left(\frac{\partial u}{\partial x}\right)^2\right\}d\tau=(C_z-C_x)\sin\Theta\cos\Theta.\end{aligned} \tag{32}$$

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[2] See A. Sommerfeld and H. A. Bethe, reference 2, p. 512.

Integrals for p state symmetric about z axis

Let us now calculate the C_j for the wave function:

$$u(r,\vartheta)=v(r)\cos\vartheta. \tag{33}$$

For this function:

$$\frac{\partial u}{\partial x}=\sin\vartheta\cos\vartheta\left\{\frac{\partial v}{\partial r}-\frac{v(r)}{r}\right\}\cos\varphi,\quad \frac{\partial u}{\partial z}=\frac{\partial v}{\partial r}\cos^2\vartheta+\frac{v(r)}{r}\sin^2\vartheta.$$

For a spherical atomic volume the integration over angles can be carried out, giving:

$$\begin{aligned} C_x&=\frac{4\pi}{15}\int\left\{\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{v}{r}\right)^2-2\frac{\partial v}{\partial r}\frac{v(r)}{r}\right\}r^2dr,\\ C_z&=\frac{4\pi}{15}\int\left\{3\left(\frac{\partial v}{\partial r}\right)^2+8\left(\frac{v}{r}\right)^2+4\frac{\partial v}{\partial r}\frac{v(r)}{r}\right\}r^2dr. \end{aligned} \tag{34}$$

It is to be noted that for this wave function the neglected term in the matrix element vanishes. Also, unless the integral of the cross-term, $(\partial v/\partial r)(v/r)$ is large and negative, $C_z \gg C_x$. Actually this integral should be small since the radial function and its derivative would be expected to change sign alternately. The assumed form of K_j [Eq. (31)] is reasonable then, and the C_j become [cf. Eq. (32)]:

$$C_z\cos^2\Theta,\quad 0,\quad C_z\sin\Theta\cos\Theta. \tag{35}$$

As was to be anticipated the angular dependence of the wave function has given importance to the transverse waves. (If u were a function of φ, C_2 would also be finite.) At low temperatures the scattering depends upon:

$$C_z^2\cos^2\Theta\{\cos^2\Theta(F_1/\bar{u}_1)^6+\sin^2\Theta(F_3/\bar{u}_3)^6\}. \tag{36}$$

Now $\bar{u}_3\approx\bar{u}_1/2$ and $F_1\approx F_3\approx 1$. This means that for a spherical electron distribution the second (transverse) term is more important than the first. At high temperatures the low velocity of the transverse wave exerts less influence since the second rather than the sixth-power enters. It is to be noted that the nature of the approximate matrix element to be used in a calculation would depend very markedly on the shape of the electron distribution since this determines which Θ's are of importance.

Propagation of elastic waves in hexagonal crystals[3]

An investigation of the propagation of plane waves in an elastic medium leads to the following secular equation for the allowed velocities:

$$\begin{vmatrix}(a-\lambda) & h & g\\ h & (b-\lambda) & f\\ g & f & (c-\lambda)\end{vmatrix}=0 \tag{37}$$

$\lambda=\rho u^2$, where ρ is the density of the medium, and $u(\alpha,\beta,\gamma)$, the velocity as a function of the direction cosines of the propagation vector. For tetragonal and hexagonal crystals:

$$\begin{aligned} a&=\alpha^2a_{11}+\beta^2a_{66}+\gamma^2a_{44}, & f&=(a_{13}+a_{44})\beta\gamma,\\ b&=\alpha^2a_{66}+\beta^2a_{11}+\gamma^2a_{44}, & g&=(a_{13}+a_{44})\gamma\alpha,\\ c&=\alpha^2a_{44}+\beta^2a_{44}+\gamma^2a_{33}, & h&=(a_{12}+a_{66})\alpha\beta. \end{aligned} \tag{38}$$

For hexagonal crystals the roots of the secular equation depend only on γ. For these crystals:

$$2a_{66}=a_{11}-a_{12} \tag{39}$$

and expansion of the determinant shows that α and β enter the coefficients of the cubic only in the combination $\alpha^2+\beta^2$. This conclusion can be reached without explicit reference to Eq. (39) by turning directly to symmetry considerations. The coefficients in the cubic show complete or tetragonal symmetry about the unique axis. Hexagonal symmetry is achieved only by *removal* of the tetragonal terms. Without loss of generality waves propagated in the $\Phi=0$ plane may be studied. With $\beta=0$, $\alpha=(1-\gamma^2)^{\frac{1}{2}}$ the

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[3] For a discussion of elastic waves in crystals, see K. Forsterling, Ann. d. Physik **61**, 549 (1920). For curves showing the directional dependence of wave velocities in Cd and Zn, see E. Grüneisen and E. Goens, Zeits. f. Physik **26**, 235 (1924). This paper does not give analytical expressions for solutions of the secular equation, however, and arbitrarily labels the waves longitudinal and transverse.

secular equation becomes:

$$\begin{vmatrix} (a_0-\lambda) & 0 & g_0 \\ 0 & (b_0-\lambda) & 0 \\ g_0 & 0 & (c_0-\lambda) \end{vmatrix} = 0 \qquad (37')$$

with the roots:

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(a_0+c_0)+\tfrac{1}{2}[(a_0-c_0)^2+4g_0{}^2]^{\frac{1}{2}}, \\ \lambda_2 &= b_0, \\ \lambda_3 &= \tfrac{1}{2}(a_0+c_0)-\tfrac{1}{2}[(a_0-c_0)^2+4g_0{}^2]^{\frac{1}{2}}. \end{aligned} \qquad (40)$$

The polarization of the waves can be determined from the amplitude equations. The first wave is nearly longitudinal, the second strictly transverse, the third nearly transverse.

The directional dependence of the velocity may be approximated by expressions of the form:

$$u_j(\gamma)=\bar{u}_j(1-\alpha_j P_2(\gamma)). \qquad (41)$$

The results shown in Table I are obtained for three common hexagonal close-packed metals.

The above discussion was intended to point out some important aspects of the electron-lattice interaction and to indicate some of the things to be borne in mind if a calculation of an interaction function is attempted. The discussion of elastic waves would have direct applicability only at low temperatures. In addition to playing a part in emphasizing the importance of transverse waves, the calculation can provide the basis of an estimate of directional dependence in the conductivity brought about by anisotropy in the lattice vibrations alone. This will be mentioned again in (3, b).

(3) Directional Dependence for Certain Models

(a) Ellipsoidal constant energy surfaces

We treat here a model characterized by an isotropic interaction function and the energy function of Eq. (24):

$$E(\mathbf{K})=\zeta\left(\frac{K_x{}^2+K_y{}^2}{a^2}+\frac{K_z{}^2}{c^2}\right).$$

TABLE I. *Values of* $\bar{u}$ *(in units of* 10^5 *cm/sec.) and* α *for* Mg, Zn, *and* Cd.

	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	α_1	α_2	α_3
Mg	5.7	3.1	3.2	0.00	0.00	0.00
Zn	4.34	2.80	2.12	0.19	0.14	0.14
Cd	3.17	1.80	1.36	0.23	0.18	0.00

From the relations (26) and (27) it is seen that:

$$\frac{dS}{|\mathrm{grad}_K\, E|}=\frac{a^2c}{2\zeta}\sin\vartheta d\vartheta d\varphi,$$

so that the *high temperature* equations are:

$$\int_{-1}^{+1}\{g(\eta')-g(\eta)\}d\eta'=G\frac{\eta}{c} \qquad (42)$$

(parallel field)
(η has been written for $\cos\vartheta$)

and

$$\int_0^{2\pi}d\varphi'\int_{-1}^{+1}d\eta'\{g(\eta',\varphi')-g(\eta,0)\} = 2\pi G\frac{(1-\eta^2)^{\frac{1}{2}}}{a}. \qquad (42')$$

(perpendicular field—along $\varphi=0$).

These have the very simple solutions:

$$-(G/2c)\eta, \quad -(G/2a)(1-\eta^2)^{\frac{1}{2}}\cos\varphi, \qquad (43)$$

corresponding to the conductivity ratio:

$$\sigma_{||}/\sigma_{\perp}=(a/c)^2. \qquad (44)$$

This is just the result implied in the discussion of the "effective number of free electrons" by Mott and Jones.[4] Its simplicity is associated with the fact that the problem can be treated without specific reference to the exclusion principle in terms of a surface density of states, $dS/8\pi^3|\mathrm{grad}_K\, E|$, giving the number of states of energy between E and $E+dE$ in the neighborhood of the element dS of the Fermi surface. This treatment is possible because the breadth of the Fermi surface ($\sim kT$) is large compared to the energy of all scattering quanta. At low temperatures the exclusion principle plays a more positive role. The directional dependence no longer arises only from variations in the velocity and effective mass of the electrons.

In treating the low temperature case it is convenient to write the condition for the stationary state as a differential equation. The function $\delta(\vartheta,\varphi,\beta)$ which was introduced in Eq. (17) is the second-degree coefficient in a Taylor expansion of $g(\vartheta',\varphi')$ in terms of q. When the interac-

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[4] N. F. Mott and H. Jones, *Theory of Metals and Alloys* (Oxford, 1936), p. 96.

tion function is assumed independent of the direction of **q**, integration over β leads to the Laplacian of $g(\vartheta, \varphi)$ for the ellipsoid. The differential equations are:

$$\frac{1}{a\sin\vartheta(a^2\cos^2\vartheta+c^2\sin^2\vartheta)^{\frac{1}{2}}}\frac{\partial}{\partial\vartheta}\left(\frac{a\sin\vartheta}{(a^2\cos^2\vartheta+c^2\sin^2\vartheta)^{\frac{1}{2}}}\frac{\partial g}{\partial\vartheta}\right)+\frac{1}{a^2\sin^2\vartheta}\frac{\partial^2 g}{\partial\varphi^2}$$
$$=\text{constant}\,\frac{(a^2\cos^2\vartheta+c^2\sin^2\vartheta)^{\frac{1}{2}}}{ac}\begin{matrix}\times\cos\vartheta/c\ (||)\\ \times\sin\vartheta/a\ (\perp).\end{matrix} \tag{45}$$

Nearly spherical distribution.—

$$a/c=1+\epsilon \quad |\epsilon|\ll 1. \tag{46}$$

The equations for the stationary state in terms of $\eta=\cos\vartheta$ to first order in ϵ are:

$$\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial g}{\partial\eta}\right)+\epsilon\left\{\frac{1-3\eta^2}{3}\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial g}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left((1-\eta^2)^2\frac{\partial g}{\partial\eta}\right)\right\}=-\frac{D}{M}\eta\left\{1+\epsilon\left(\frac{3\eta^2+1}{3}\right)\right\}$$

and

$$\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial g}{\partial\eta}\right)+\frac{1}{1-\eta^2}\frac{\partial^2 g}{\partial\varphi^2}+\epsilon\left\{\frac{1-3\eta^2}{3}\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial g}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left((1-\eta^2)^2\frac{\partial g}{\partial\eta}\right)-\frac{2}{3(1-\eta^2)}\frac{\partial^2 g}{\partial\varphi^2}\right\}$$
$$=-\frac{D}{M}(1-\eta^2)^{\frac{1}{2}}\left(1+\epsilon\frac{3\eta^2-2}{3}\right). \tag{47}$$

Solutions to first order in ϵ are:

$$\begin{gathered}D\eta(1+\tfrac{2}{3}\epsilon\eta^2)/2M,\\ D(1-\eta^2)^{\frac{1}{2}}(1+(\epsilon/3)(2\eta^2-1))\cos\varphi/2M.\end{gathered} \tag{48}$$

The ratio of conductivities is:

$$\frac{\sigma_{||}}{\sigma_{\perp}}=\frac{a}{c}\frac{1+2\epsilon/5}{1-\epsilon/5}=1+\frac{8\epsilon}{5}=\left(\frac{a}{c}\right)^{1.6} \tag{49}$$

(slightly closer to unity than the corresponding ratio at high temperatures).

Very flat ellipsoid.—

$$a/c\gg 1. \tag{50}$$

We are here concerned with the equation:

$$\Delta g=\text{const.}\ |\text{grad}_K E|\,\partial E/\partial K_{\text{field}}, \tag{51}$$

where Δ is the Laplacian for a very flat ellipsoid of revolution. For points away from the edges of the distribution ($\tan\vartheta\ll a/c$) the equation can be written:

$$\frac{\partial^2 g}{\partial K_x^2}+\frac{\partial^2 g}{\partial K_y^2}=\text{constant}\,\frac{\partial E}{\partial K_z}\frac{\partial E}{\partial K_{\text{field}}}. \tag{52}$$

For parallel field this becomes:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right)=\mp\frac{A}{a^2c^2}(a^2-\rho^2);\quad \rho^2=K_x^2+K_y^2=a^2\sin^2\vartheta. \tag{53}$$

(sign changes when ϑ goes through $\pi/2$)

The solution which is well-behaved at $\rho=0$ and vanishes at $\rho=a$ is:

$$g(\rho)=(\pm A/16a^2c^2)(a^2-\rho^2)(3a^2-\rho^2). \tag{54}$$

This is the significant solution of the stationary state equation for parallel field. Its range of validity covers most of the distribution, and includes the region where $g(\rho)$ and the contribution of a given electron to the current are large. It is incorrect only in the exact way in which it goes to zero near the edges of the ellipsoid.

For perpendicular field the ρ equation for $\tan\vartheta \ll a/c$ has the form:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right)-\frac{g(\rho)}{\rho^2}=\frac{A}{a^3c}(a^2-\rho^2)^{\frac{1}{2}}\rho. \tag{53'}$$

(complete solution $g(\rho)\cos\varphi$)

The function:

$$\frac{Aa}{15c}\,\frac{(1+x^2)-(1-x^2)^{5/2}}{x};\quad x=\rho/a \tag{54'}$$

is a solution and satisfies the required boundary conditions (behaves like x for small x, and has a derivative which vanishes at $x=1$).

Since for perpendicular field $g(\rho)$ and the contribution to the current from a given electron are greatest near the edge of the ellipsoid, it seems advisable to investigate the behavior of g for large $\tan\vartheta$. By introducing $y=\cos\delta$ (δ being the angle between the normal to the ellipsoid and the z axis) a simple equation for the range $\tan\vartheta\approx a/c$ to $\vartheta=\pi/2$ can be obtained:

$$\frac{\partial}{\partial y}\left((1-y^2)\frac{\partial g}{\partial y}\right)-\frac{g(y)}{1-y^2}\left(\frac{c}{a}\right)^4=A\left(\frac{c}{a}\right)^4\frac{1}{(1-y^2)^{\frac{3}{2}}}. \tag{55}$$

Keeping in mind that $c/a\ll 1$, we see that the solution is essentially constant in this range, that is, there is no abrupt change in the behavior of $g(\rho)$ near the edges of the distribution. The solution given above [Eq. (54′)] can, therefore, be used for the whole range of θ.

A computation of the conductivity ratio gives:

$$\frac{\sigma_{||}}{\sigma_{\perp}}=\frac{a^4}{24c^2}\bigg/\frac{a^2}{20}=\frac{5}{6}\left(\frac{a}{c}\right)^2, \tag{56}$$

slightly smaller than for high temperatures.

(b) Anisotropic interaction function

We now consider the anisotropy in the conductivity when the electron distribution is spherical and the interaction function of the form:

$$M(\Theta)=M_0(1+\mu\cos^2\Theta),\quad |\mu|\ll 1. \tag{57}$$

Low temperatures.—If as before η is written for $\cos\vartheta$, the differences entering the equation for the stationary state are:

Parallel field:

$$g(\eta')-g(\eta)=\frac{\partial g}{\partial\eta}(\delta\eta)+\frac{1}{2}\frac{\partial^2 g}{\partial\eta^2}(\delta\eta)^2. \tag{58}$$

Perpendicular field: (58)

$$g(\eta',\varphi')-g(\eta,0)\propto\frac{\partial g}{\partial\eta}(\delta\eta)+\frac{1}{2}\frac{\partial^2 g}{\partial\eta^2}(\delta\eta)^2+\frac{1}{2}\frac{\partial^2 g}{\partial\varphi^2}(\delta\varphi)^2,$$

where

$$\delta\eta=-\frac{q^2}{2K^2}\eta-\frac{q}{K}\cos\beta(1-\eta^2)^{\frac{1}{2}},\quad \delta\varphi=\frac{q}{K}\frac{\sin\beta}{(1-\eta^2)^{\frac{1}{2}}}.$$

(In the perpendicular equation the terms which would finally give zero on integration over β have been omitted.)

To sufficient approximation:

$$\cos\Theta = -(1-\eta^2)^{\frac{1}{2}}\cos\beta. \tag{59}$$

The differential equations obtained are:

$$\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial g}{\partial\eta}\right)+\mu(1-\eta^2)\left(\frac{3}{4}(1-\eta^2)\frac{\partial^2 g}{\partial\eta^2}-\eta\frac{\partial g}{\partial\eta}\right)=-\frac{D}{M_0}\eta$$

and

$$\left\{\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial g}{\partial\eta}\right)-\frac{g(\eta)}{(1-\eta^2)}\right\}+\mu(1-\eta^2)\left\{\frac{3}{4}(1-\eta^2)\frac{\partial^2 g}{\partial\eta^2}-\eta\frac{\partial g}{\partial\eta}-\frac{1}{4}\frac{g(\eta)}{(1-\eta^2)}\right\}=-\frac{D}{M_0}(1-\eta^2)^{\frac{1}{2}}. \tag{60}$$

(complete solution $g(\eta)\cos\varphi$)

For small μ these equations have the solutions:

$$\begin{aligned} g_{\parallel}(\eta) &= (D/2M_0)\eta[1+(\mu/12)(\eta^2-3)], \\ g_{\perp}(\eta) &= (D/2M_0)(1-\eta^2)^{\frac{1}{2}}[1+(\mu/12)(\eta^2-5)]. \end{aligned} \tag{61}$$

Introducing

$$\bar{M}=\int_0^1 M(\cos\Theta)d\cos\Theta = M_0(1+\mu/3), \tag{62}$$

the complete solutions for small μ are:

$$(D/2\bar{M})\eta[1+(\mu/12)(\eta^2+1)], \quad (D/2\bar{M})(1-\eta^2)^{\frac{1}{2}}[1+(\mu/12)(\eta^2-1)]\cos\varphi. \tag{63}$$

The ratio of conductivities is:

$$\frac{\sigma_{\parallel}}{\sigma_{\perp}}=\frac{1+2\mu/15}{1-\mu/15}=1+\frac{\mu}{5}. \tag{64}$$

High temperatures.—The following are stationary state equations for the model at high temperatures:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}d\varphi'\int_{-1}^{+1}d\eta'(1+\mu\cos^2\Theta)\{g(\eta')-g(\eta)\} &= -\frac{2D}{M_0}\eta, \\ \frac{1}{2\pi}\int_0^{2\pi}d\varphi'\int_{-1}^{+1}d\eta'(1+\mu\cos^2\Theta)\{g(\eta',\varphi')-g(\eta,0)\} &= -\frac{2D}{M_0}(1-\eta^2)^{\frac{1}{2}}, \end{aligned} \tag{65}$$

where

$$\cos^2\Theta=\tfrac{1}{2}(\eta'-\eta)^2/\{(1-\eta\eta')-(1-\eta^2)^{\frac{1}{2}}(1-\eta'^2)^{\frac{1}{2}}\cos\varphi'\}. \tag{66}$$

Solutions correct to first order in μ.—If the function:

$$B(\eta+bP_3(\eta))$$

is substituted into the parallel equation it becomes (to first order):

$$2\eta+2bP_3(\eta)-\frac{\mu}{2\pi}\int_{-1}^{+1}d\eta'\int_0^{2\pi}d\varphi'(\eta'-\eta)\cos^2\Theta = \frac{2D}{BM_0}\eta.$$

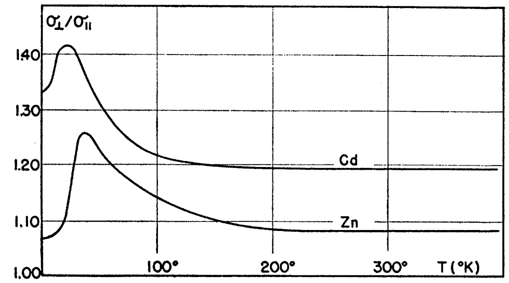

FIG. 1. Variation of $\sigma_{\perp}/\sigma_{\parallel}$ as a function of temperature (data of Grüneisen and Goens).

The φ' integral is:

$$\frac{1}{2}\int_0^{2\pi}\frac{d\varphi'}{(1-\eta\eta')-(1-\eta^2)^{\frac{1}{2}}(1-\eta'^2)^{\frac{1}{2}}\cos\varphi'}=\frac{\pi}{|\eta-\eta'|}.$$

This gives for the μ term:

$$-\frac{\mu}{2}\int_{-1}^{+1}d\eta'\frac{(\eta'-\eta)^3}{|\eta-\eta'|}=\mu\eta(\tfrac{1}{3}\eta^2+1)$$

and the equation becomes:

$$1+\tfrac{1}{2}b(5\eta^2-3)+\tfrac{1}{2}\mu(\tfrac{1}{3}\eta^2+1)=D/BM_0.$$

If we choose:

$$b=-\frac{\mu}{15},\quad B=\frac{D}{M_0}\left(1-\frac{3\mu}{5}\right)=\frac{D}{\bar{M}}\left(1-\frac{4\mu}{15}\right),$$

the equation is solved. The solution:

$$\frac{D}{\bar{M}}\left(1-\frac{4\mu}{15}\right)\left(\eta-\frac{\mu}{15}P_3(\eta)\right)$$

can be written:

$$D/\bar{M}\eta[1-(\mu/6)(\eta^2+1)]. \tag{67}$$

In a similar way a first-order solution of the perpendicular equation can be obtained:

$$(D/\bar{M})(1-\eta^2)^{\frac{1}{2}}[1-(\mu/6)(\eta^2-1)]\cos\varphi. \tag{67'}$$

These correspond to the conductivity ratio:

$$\frac{\sigma_{||}}{\sigma_{\perp}}=\frac{1-4\mu/15}{1+2\mu/15}=1-\frac{2\mu}{5}. \tag{68}$$

It is to be noticed that the ratios for low and high temperatures lie on different sides of unity. Something of this sort might well be expected since entirely different lattice waves may be of importance for the two cases. For example, consider the scattering of electrons near the pole of the distribution. At low temperatures electrons are exchanged directly only with neighboring points on the sphere. Waves traveling at right angles to the symmetry axis are responsible for the transitions. At high temperatures transitions to or from all points of the surface are possible, involving lattice waves propagated in all directions.

If the anisotropy of the interaction function is assumed to arise solely from anisotropy of the lattice waves one has:

$M(\Theta)=\bar{M}F_1{}^6(\Theta)$ (low temperatures)
$M(\Theta)=\bar{M}F_1{}^2(\Theta)$. (high temperatures)

If the anisotropy is assumed small these become:

$$M(\Theta)=\bar{M}(1+6\alpha_1P_2(\cos\Theta))$$
$$M(\Theta)=\bar{M}(1+2\alpha_1P_2(\cos\Theta)),$$

the corresponding μ's being:

$$9\alpha_1 \text{ and } 3\alpha_1$$

and the conductivity ratios:

$\sigma_{||}/\sigma_{\perp}=1+9\alpha_1/5$ (low temperatures)
$\sigma_{||}/\sigma_{\perp}=1-6\alpha_1/5$. (high temperatures)

The values of α_1 for cadmium and zinc (0.23 and 0.19) are not small enough to justify the use of these expressions except for extremely rough estimates. Results of their use are listed below. For comparison experimental values from data due to Grüneisen and Goens are written alongside in parenthesis.

	CADMIUM	ZINC	
$\sigma_{||}/\sigma_{\perp}$	1.41(0.71)	1.34(0.90)	(low)
	0.72(0.83)	0.77(0.92)	(high).

It is seen that although the calculated values are in accidental agreement with experiment in the high temperature case, the necessity for very serious attention to other sources of anisotropy is indicated by the low temperature data. In this connection we note that magnesium has a rather anisotropic conductivity, while the lattice waves seem to be isotropic at low temperatures.

The value 0.72 for cadmium at high temperatures agrees with that obtained by Houston[5] in an early paper on the theory of conductivity ($\sigma_{||}/\sigma_{\perp}=1/1.4$). Houston treated the scattering of the electrons by use of an expression derived by Debye for x-rays. This gives the probability of scattering in a given direction in terms of the binding constants of the lattice atoms, these in turn being computed from the elastic constants of the metal.

Further remarks

It is clear that the directional dependence is an essentially complex effect so that it would be futile to make estimates for serious comparison

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[5] W. V. Houston, Zeits. f. Physik **48**, 449 (1928).

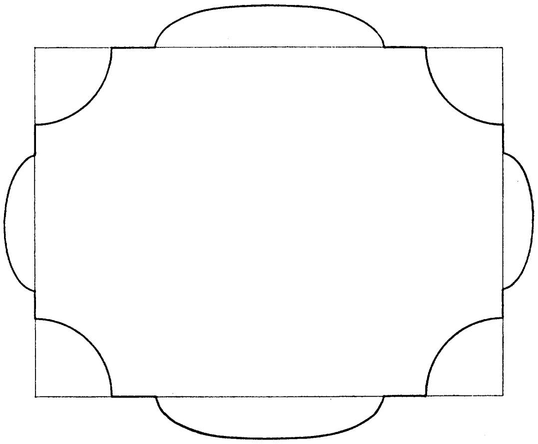

FIG. 2. Suggested distribution of electrons in K space for a metal with a unique axis.

with experiment without having available much more quantitative information than we do about electronic wave functions and energy surfaces for real metals. It is desirable, however, to make some further reference to measurements and to point out more or less how these fit with theoretical ideas. The following data plotted in Fig. 1 for cadmium and zinc are due to Grüneisen and Goens:[6]

	T(°K)	373	273	195	130	82	20
$\sigma_\perp/\sigma_\parallel$	Zinc	1.086	1.082	1.078	1.107	1.170	1.108
	Cadmium	1.189	1.191	1.194	1.200	1.230	1.406

As would be expected the ratio is independent of temperature at high temperatures—each conductivity following the $1/T$ law. (A similar behavior would be expected at low temperatures—each conductivity following the $1/T^5$ law.) As one moves from high to low temperatures a definite fluctuation seems to begin at a point well below the Debye temperature (Zn ≈ 235°K, Cd ≈ 168°K). This means that the temperature indicating the transition from high to low temperature conditions depends upon the field direction. For a complex distribution there are a variety of reasons for expecting this. For cadmium and zinc the transition evidently begins at a higher temperature for perpendicular field.

The shape of the K space electron distribution is of course very important for the directional effects. Unfortunately little is known at present about the distribution for multivalent metals. In general, however, it is to be expected that the Fermi surface will overlap the first Brillouin zone. A suggested picture for a metal with a unique axis is given in Fig. 2. The overlapping electrons fall into very flat ellipsoids of revolution. At high temperatures transitions between various parts of the distribution would be expected. If overlapping occurred at all places, as indicated, the anisotropy in the conductivity need not be large. At low temperatures the distribution would be equivalent to a number of independent parts, the observed current being the superposition of contributions from various parts. Again a comparatively small anisotropy would be understandable. On the other hand, it is entirely possible that for some metal the conducting electrons might lie principally in a single very flat ellipsoid, so that a large anisotropy would result.

In concluding the report, the author wishes to express his appreciation to Professor Bethe for suggesting the problem, and for many helpful consultations throughout the course of the work.

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[6] E. Grüneisen and E. Goens, Zeits. f. Physik **26**, 250 (1924).