The integration is elementary, and gives

$$S_2 = Ce^4/8r_sD^3. \quad (80)$$

Returning to Eq. (66), we find that

$$W_B = -(r^2)_{Av}Ce^4/24r_s\Delta_A D^3 = -(Ce^2/2r_s\Delta_A)W_A, \quad (81)$$

where W_A is the image value as given by (58). For the case of a general molecule, $(r^2)_{Av}$ should be replaced by $(R^2)_{Av}$ as defined by Eq. (50). In terms of the polarizability of the molecule, α_A, W_B is

$$W_B = -C\alpha_A e^2/16r_sD^3. \quad (82)$$

The approximation (44) gives finally

$$W \sim W_AW_B/(W_A+W_B) = -(e^2(r^2)_{Av}/12D^3)\cdot\frac{Ce^2/2r_s\Delta_A}{1+Ce^2/2r_s\Delta_A}. \quad (83)$$

W will be smaller than the image value if Δ_A is large or if r_s is large.

For ordinary electron densities in the metal, and with Δ_A equal to about one Rydberg unit (13.5 ev), the factor $Ce^2/2r_s\Delta_A$ is about unity, so that W is about one-half of the image value W_A.

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Theory of the Approach to Magnetic Saturation

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The standard theoretical treatment of magnetization at high fields leads to a formula $J=J_s-b/H^2$; experimentally, a term $-a/H$ is also found. In the standard treatment it is assumed that the internal forces responsible for the incompleteness of saturation are approximately uniform over distances containing a very large number of atoms. If these forces vary rapidly over shorter distances, the interatomic coupling forces prevent the direction of microscopic magnetization from varying with equal rapidity, and the $1/H^2$ law no longer follows. It is shown here that for $H\gg 4\pi J_s$, point, line, and plane concentrations of force lead to laws of the form $J=J_s-a/H^{n/2}$, with $n=1$, 2, and 3, respectively. For $H\ll 4\pi J_s$ the law is of the same form, but with different apparent values of J_s and a. For $H\cong 4\pi J_s$ the behavior is more complicated. The conclusion that the observed a/H term is due to line concentrations of force is supported by the dependence of a on degree of plastic strain, since the mechanism of plastic flow is believed to be a propagation of dislocation lines through the lattice.

§1. INTRODUCTION

AT fields much larger than the coercive force, a ferromagnetic material behaves almost reversibly. The theoretical calculation of this part of the magnetization curve is therefore comparatively simple. The conventional treatment leads, at sufficiently high fields, to a formula[1]

$$J=J_s-b/H^2, \quad (1)$$

where H is the magnetic field, J is the magnetization, J_s is the "spontaneous" magnetization predicted by the Weiss-Heisenberg theory at the temperature in question, and b is a constant. In deriving this formula it is assumed that the material is magnetized to a value J_s at each point but that the *direction* of the magnetization varies from one region to another. This is illustrated very crudely in the upper part of Fig. 1. A soft specimen is assumed to consist of unstrained crystals, a hard specimen of regions each under approximately uniform internal stress. Forces due to crystalline anisotropy or to the distortion of the lattice tend to pull the magnetization vector into certain directions, represented by the dotted arrows; these directions are related to the principal axes of the crystal or of the system of internal stresses. As the field is increased, it overcomes these forces and pulls the magnetization vector toward the field direction.

[1] F. Bitter, *Introduction to Ferromagnetism* (McGraw-Hill, 1937), p. 222.

Experimentally it is found that Eq. (1) does not describe the results accurately. They can, however, be fitted to an empirical formula[2]

$$J=J_s-a/H-b/H^2+cH. \qquad (2)$$

The value required for b is in satisfactory agreement with the theoretical value in Eq. (1). The origin of the term in H is at least partially understood, although the theoretical c is too small. The present investigation was undertaken in the hope of discovering a mechanism that could explain the $1/H$ term.

The derivations of this term offered previously have usually been based on a statistical theory of one kind or another.[3] Near saturation the statistical theory may be put into a very simple form. Because of internal forces that interfere with perfect saturation, the magnetization direction deviates from the field direction by a small angle α that varies irregularly from point to point. This increases the energy of the specimen in the field, $-HJ_s\cos\alpha$ for unit volume, by an amount whose mean value per unit volume is $-HJ_s[(\cos\alpha)_{Av}-1]=\frac{1}{2}HJ_s(\alpha^2)_{Av}$. If it is now assumed that the internal forces are able to increase this mean energy density by an amount ϵ independent of the field, then the resulting magnetization is $J=J_s(\cos\alpha)_{Av}=J_s[1-\frac{1}{2}(\alpha^2)_{Av}]=J_s(1-\epsilon/HJ_s)$. This is of the required form; but the assumption made is not easy to defend. The complete version of the theory is equally unsatisfactory in this respect; yet it leads to a relation between ϵ and the initial susceptibility which is in fair agreement with experiment, and which has still not been derived by any other method.

In the derivation of Eq. (1) two facts have been ignored, both of which will be taken into account in the analysis to follow. First, if the magnetization direction varies from point to point, the magnetization vector will in general have a nonvanishing divergence; that is, there will be a Poisson magnetic charge density, and this will produce a non-uniform magnetic field. In the theory, H is assumed uniform. Second and more important, the internal forces, instead of being approximately uniform over fairly large distances, may conceivably undergo rapid variations over distances of a few lattice spacings. Then it becomes necessary to take account of the coupling forces between the magnetic moments of neighboring atoms—the "exchange" forces responsible for the spontaneous magnetization itself. These tend to aline successive atoms with their moments as nearly as possible parallel. Thus suppose, as an extreme case, that a force acts on a single atom and produces a deviation of its magnetic moment from the field direction. Because of the coupling forces, other atoms as well will have their moments pulled out of the field direction, as illustrated in the lower left part of Fig. 1. The disturbance dies down to $1/e$ of its original value in some distance d. In the simplest cases this distance turns out to be inversely proportional to $\sqrt{H}$; for nickel it is about $10^{-4}/\sqrt{H}$ cm, which at 100 oersteds is about 500 lattice spacings. Stress (or other) variations over distances much shorter than this will be too fast for the magnetization to follow.

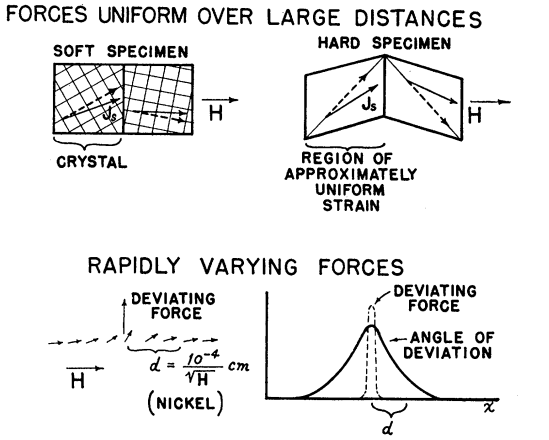

FIG. 1. Contrast between two extremes in the mode of spatial variation of the internal forces that prevent perfect saturation. Above, forces that behave in accordance with the conventional theory: whether due to crystalline anisotropy or to distortion of the lattice, they are assumed to be approximately uniform over fairly large regions, so that large groups of atoms behave as units. Below, force acting on a single atom (left) or concentrated within a small region (right): interatomic coupling forces spread the disturbance out over a distance of order of magnitude d.

[2] H. Polley, Ann. d. Physik **36**, 625 (1939); A. R. Kaufmann, Phys. Rev. **57**, 1089(A) (1940).

[3] For a summary see W. F. Brown, Phys. Rev. **54**, 279 (1938) and **55**, 568 (1939). The theory has been extended by M. Takagi, Sci. Rep. Tohoku Imp. Univ. [1] **28**, 20 and 85 (1939), and R. Kimura, Proc. Phys.-Math. Soc. Japan **22**, 219 and 233 (1940). F. Bitter, Phys. Rev. **37**, 91 (1931), attributed the $1/H$ term to temperature agitation of blocks of atoms. The insensitivity of a to temperature change (Polley, reference 2) is difficult to reconcile with this view. See also P. Weiss, Ann. de physique **12**, 279 (1929), especially pp. 286–287.

Thus a lattice distortion concentrated in a small region will produce a deviation from perfect saturation over a much larger region, as shown at the lower right in Fig. 1.

§2. Illustrative Case

The nature of the problem and the methods available for solving it may be illustrated by considering first a simple one-dimensional case, pictured at the left in Fig. 2. The field acts along the z axis, the deviating forces tend to rotate the magnetization vector toward the x axis, and these forces are supposed to vary only in the y direction. Therefore the small angle of deviation α also varies only with y. The divergence of the magnetization vanishes and H is uniform, so that there are no long distance magnetic effects to complicate the problem; but the effect of the interatomic coupling forces must still be considered.

In the conventional theory these coupling forces are neglected, and the magnetization vector at each point is assumed to be in equilibrium under the joint action of the field H and of the deviating force. α may be found as a function of y by minimizing the free energy W of a portion of the specimen of unit cross section, extending along the y axis from one bounding surface at y_1 to the other bounding surface at y_2. W consists of the energy in the field,

$$W_H=-\int_{y_1}^{y_2} HJ_s\cos\alpha dy$$
$$=\tfrac{1}{2}HJ_s\int_{y_1}^{y_2}\alpha^2dy+\text{const.}, \quad (3)$$

and the free energy W_a due to anisotropy, stresses, or other internal forces. For small α, the volume density of W_a may be assumed to vary linearly with α and may be written $g\alpha+$const., where g is a function of y; thus

$$W_a=\int_{y_1}^{y_2} g\alpha dy+\text{const.} \quad (4)$$

If W_H+W_a is to be a minimum, then for $\delta\alpha$ an arbitrary function of y

$$\int_{y_1}^{y_2}[HJ_s\alpha+g]\delta\alpha dy=0, \quad (5)$$

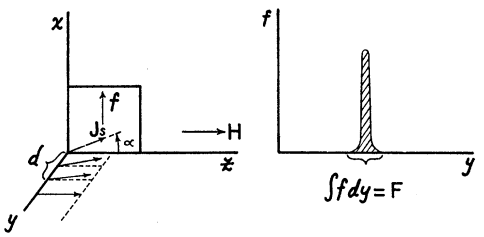

Fig. 2. Simple one-dimensional case illustrating the effects of uniform and of concentrated forces. Forces f uniform in xz planes tend to rotate the magnetization vector from the field direction z toward the direction x. The forces and the resulting deviation angle α are functions of y. In the conventional theory the forces f are assumed to be approximately uniform over distances containing many atoms, and the coupling forces between adjacent atoms are neglected. If the forces f are concentrated in small y intervals as shown at the right, the interatomic forces must not be neglected.

whence $HJ_s\alpha+g=0$, or $\alpha=-g/HJ_s$, $\alpha^2=g^2/H^2J_s^2$, and

$$J=J_s[1-\tfrac{1}{2}(\alpha^2)_{Av}]=J_s[1-\tfrac{1}{2}(g^2)_{Av}/H^2J_s^2], \quad (6)$$

which is of the form (1).

This treatment must now be modified by taking account of the interatomic coupling forces. The coupling energy of a pair of neighboring atoms, whose magnetic moments are at an angle ϕ to each other, may be written $-\frac{1}{2}I\cos\phi$, where I is a constant whose order of magnitude may be estimated by multiplying Boltzmann's constant by the Curie temperature; this classical approximation has proved adequate in the treatment of similar problems at low fields.[4] If for simplicity the lattice is assumed to be simple cubic with interatomic distance a_0 and to have a cube edge along the y axis, then $\phi=a_0d\alpha/dy$, $\cos\phi=1-\frac{1}{2}\phi^2=1-\frac{1}{2}a_0^2(d\alpha/dy)^2$, and the effect of a varying α is an increase of the mutual energy of successive atoms along the y axis by an amount $\frac{1}{4}Ia_0^2(d\alpha/dy)^2$. The total addition to the free energy of the portion of the specimen under consideration is

$$W_C=\tfrac{1}{2}C\int_{y_1}^{y_2}(d\alpha/dy)^2dy, \quad (7)$$

where $C=I/2a_0$. To the left member of Eq. (5)

[4] For an elementary treatment and for references see W. F. Brown, J. App. Phys. **11**, 160 (1940).

must now be added a term

$$C\int_{y_1}^{y_2}(d\alpha/dy)(d\delta\alpha/dy)dy$$
$$=C(d\alpha/dy)\delta\alpha\Big|_{y_1}^{y_2}-C\int_{y_1}^{y_2}(d^2\alpha/dy^2)\delta\alpha dy. \quad (5a)$$

The left member of (5) with (5a) added must vanish for $\delta\alpha$ an arbitrary function of y. This leads to the differential equation

$$-Cd^2\alpha/dy^2+HJ_s\alpha+g=0 \quad (8)$$

and the boundary conditions

$$d\alpha/dy=0 \quad \text{at} \quad y=y_1 \quad \text{and at} \quad y=y_2. \quad (9)$$

Equation (8) may be written

$$d^2\alpha/dy^2-\eta\alpha=f \quad (10)$$

with

$$\eta=HJ_s/C, \quad f=g/C. \quad (11)$$

This reduces to the previous result if f is assumed to vary only slowly; for then α also varies only slowly, and the term $d^2\alpha/dy^2$ in (10) is negligible. The other extreme is the *plane concentration* of force sketched at the right in Fig. 2. Here f vanishes everywhere except in a very thin section parallel to the zx plane, where it attains large values so that its integral across this y interval is a finite quantity F. If this section is located at $y=0$, integration of (10) across it gives for the discontinuity at $y=0$

$$(d\alpha/dy)\big|_-^+=F; \quad (12)$$

for $y\neq 0$, f may be set equal to zero in (10).

For an infinite specimen the solution is

$$\alpha=-(F/2\lambda)e^{-\lambda|y|}, \quad (13)$$

where

$$\lambda=\sqrt{\eta}=(HJ_s/C)^{\frac{1}{2}}. \quad (14)$$

The decay distance $d=1/\lambda$ is of the order of magnitude already stated and is very small compared with the dimensions of ordinary specimens or their constituent crystals. These may therefore be considered infinite without introducing appreciable error, as may be verified by solving the problem rigorously with proper attention to the boundary conditions. If the actual length of the specimen in the y direction is l, the mean value of α^2 is

$$(\alpha^2)_{Av}=(1/l)\int_{-\infty}^{\infty}\alpha^2dy$$
$$=(F^2/4\lambda^2l)\int_{-\infty}^{\infty}e^{-2\lambda|y|}dy=F^2/4\lambda^3l. \quad (15)$$

When there are a number of such plane sources of disturbance, N per cm along the y axis, with F-values F_i, the corresponding α-values α_i must be superposed. Then

$$(\alpha^2)_{Av}=[(\sum_i\alpha_i)^2]_{Av}=(\sum_i\sum_j\alpha_i\alpha_j)_{Av}$$
$$=\sum_i(\alpha_i^2)_{Av}+\sum_i\sum_{j\neq i}(\alpha_i\alpha_j)_{Av}. \quad (16)$$

If the F_i's are uncorrelated or if the mutual distances are $\gg d$, the second term may be neglected. Then

$$J=J_s[1-\tfrac{1}{2}(\alpha^2)_{Av}]=J_s[1-N(F^2)_{Av}/8\lambda^3]. \quad (17)$$

This gives an approach to saturation according to a $1/H^{\frac{1}{2}}$ law, which has not been observed. Other types of force concentration must therefore be investigated; but first the simple one-dimensional case will be used to illustrate one further point. The explicit evaluation of α or its equivalent in terms of familiar functions, as in Eq. (13), is difficult or impossible in some of the more complicated cases. The final formula for $(\alpha^2)_{Av}$ or its equivalent can be obtained in these cases by expressing the solution as a Fourier integral.[5] Thus for a plane concentration the right member of (10) may be written $F\delta(y)$, where $\delta(y)$ is the "delta function," defined by $\delta(y)=0$ for $y\neq 0$ and $\int\delta(y)dy=1$ for any interval including $y=0$. The delta function may be expressed as a Fourier integral,

$$\delta(y)=(1/2\pi)\int_{-\infty}^{\infty}\cos pydp; \quad (18)$$

and the solution of (10) for a single sinusoidal component is easily obtained. Superposition of the results gives

$$\alpha(y)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\psi(p)\cos pydp, \quad (19)$$

with

$$\psi(p)=-F\cdot(2\pi)^{-\frac{1}{2}}(p^2+\lambda^2)^{-1}. \quad (20)$$

[5] Cf. J. M. Burgers, Akad. v. wet. (Amsterdam) Proc. 42, 378 (1939).

Although in the present case this result can be transformed into the simpler form (13), such a transformation is in general not possible and is fortunately not necessary for the evaluation of $(\alpha^2)_{Av}$. For by the reciprocity property of Fourier integrals,[6]

$$\psi(p)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\alpha(y)\cos pydy, \tag{21}$$

hence

$$\int_{-\infty}^{\infty}\alpha^2dy=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha(y)\psi(p)\cos pydpdy$$
$$=\int_{-\infty}^{\infty}\psi^2dp, \tag{22}$$

and the last integral is easily evaluated. This leads to the final result in (15) without use of (13).

§3. General Theory

In the general case, let the applied field $\mathbf{H}_0$ be along the z axis, and let the direction cosines of the magnetization $\mathbf{J}$ be (α, β, γ). The magnetic energy consists of two parts. The first is the energy in the applied field,

$$W_H=-\int\mathbf{H}_0\cdot\mathbf{J}d\tau; \tag{23}$$

the region of integration is the volume τ of the specimen. The second part is the mutual magnetic energy of the atoms; the variation of this energy in a small change $\delta\mathbf{J}$ is

$$\delta W_M=\int(\nabla V)\cdot\delta\mathbf{J}d\tau, \tag{24}$$

where V is the scalar potential of the field of the specimen. In τ, V satisfies Poisson's equation

$$\nabla^2V=4\pi\nabla\cdot\mathbf{J}; \tag{25}$$

outside it is harmonic and regular at infinity; and on the boundary S, with outward normal $\mathbf{n}(\partial/\partial n=\mathbf{n}\cdot\nabla,\ J_n=\mathbf{n}\cdot\mathbf{J})$,

$$V_{\text{in}}=V_{\text{out}},$$
$$-(\partial V/\partial n)_{\text{in}}+4\pi J_n=-(\partial V/\partial n)_{\text{out}}. \tag{26}$$

The energy due to anisotropy and stresses is the volume integral of some function $w_1(x, y, z, \alpha, \beta, \gamma)$ which may be supposed re-expressed in the form $w(x, y, z, \alpha, \beta)$ by means of the relation $\alpha^2+\beta^2+\gamma^2=1$. Thus

$$W_a=\int w(x, y, z, \alpha, \beta)d\tau. \tag{27}$$

Finally, the interatomic coupling energy is[7]

$$W_C=\tfrac{1}{2}C\int[(\nabla\alpha)^2+(\nabla\beta)^2+(\nabla\gamma)^2]d\tau+\text{const.}, \tag{28}$$

with

$$C\cong I/a_0.$$

The variation of $W_H+W_M+W_a+W_C$ must be zero for arbitrary $\delta\alpha$ and $\delta\beta$, with

$$\delta\gamma=-(\alpha\delta\alpha+\beta\delta\beta)/\gamma.$$

After the variation δW_C has been freed of such factors as $\nabla\delta\alpha$ by an integration by parts (by use of the divergence theorem), the coefficients of $\delta\alpha$ and $\delta\beta$ may be equated to zero in τ and on S. This gives in τ

$$-C[\nabla^2\alpha-(\alpha/\gamma)\nabla^2\gamma]+J_s[\partial V/\partial x-(\alpha/\gamma)\partial V/\partial z]+H_0J_s\alpha/\gamma+\partial w/\partial\alpha=0, \tag{29}$$

together with a similar equation in which β and y replace α and x; and on S,

$$\partial\alpha/\partial n-(\alpha/\gamma)\partial\gamma/\partial n=\partial\beta/\partial n-(\beta/\gamma)\partial\gamma/\partial n=0. \tag{30}$$

If α and β are now assumed to be small, and only first-order terms are retained, these equations become

$$-C\nabla^2\alpha+J_s(\partial V/\partial x-\alpha\partial V/\partial z)+H_0J_s\alpha+g_1=0, \tag{31}$$
$$\partial\alpha/\partial n=\partial\beta/\partial n=0.$$

Here $g_1=(\partial w/\partial\alpha)_{\alpha=\beta=0}$; it has been assumed that $\partial^2w/\partial\alpha^2$ and $\partial^2w/\partial\alpha\partial\beta$ are $\ll H_0J_s$. When $H_0=\infty$, $\alpha=\beta=0$, and inside a specimen in the form of an ellipsoid $V=N'J_sz$, where N' is the demagnetizing factor. The ellipsoidal shape will henceforth be assumed; then for α and β small, $V=N'J_sz+U$, where U is small. Insertion of this expression in (31) and (25) and subtraction from (26) of its form for $H_0=\infty$ lead to the following

[6] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* I, second edition (Springer, 1931), p. 68.

[7] L. Landau and E. Lifshitz, Physik. Zeits. Sowjetunion **8**, 153 (1935); W. C. Elmore, Phys. Rev. **53**, 757 (1938).

set of equations, determining the functions α, β, and U and valid to the first order in these quantities:

In τ,

$$\begin{aligned} -C\nabla^2\alpha+J_s\partial U/\partial x+HJ_s\alpha+g_1=0,\\ -C\nabla^2\beta+J_s\partial U/\partial y+HJ_s\beta+g_2=0,\\ \nabla^2U=4\pi J_s(\partial\alpha/\partial x+\partial\beta/\partial y). \end{aligned} \tag{32}$$

Outside τ, $\nabla^2U=0$. At infinity, U is regular. On S,

$$\begin{aligned} \partial\alpha/\partial n=\partial\beta/\partial n=0, \quad U_{\text{in}}=U_{\text{out}},\\ -(\partial U/\partial n)_{\text{in}}+4\pi J_s(n_x\alpha+n_y\beta)=-(\partial U/\partial n)_{\text{out}}. \end{aligned}$$

Here $H=H_0-N'J_s$ is (except for terms of order α^2) the actual field, corrected for the demagnetizing effect; (n_x, n_y, n_z) are the direction cosines of the outward normal **n**. It may be shown by standard methods that the solution of these equations is unique and that, in particular, for $g_1=g_2=0$ everywhere the solution is $\alpha=\beta=0$, $U=0$.

As before, the specimen will be assumed infinite in calculating the effect of a single concentration of force; the order of magnitude of the resulting decay distances will justify this simplification. Then Eqs. (32) may be replaced by

$$\nabla^2\alpha-\partial u/\partial x-\eta\alpha=f_1, \tag{33}$$

$$\nabla^2\beta-\partial u/\partial y-\eta\beta=f_2, \tag{34}$$

$$\nabla^2u=h(\partial\alpha/\partial x+\partial\beta/\partial y), \tag{35}$$

together with suitable conditions at infinity. Here

$$\begin{aligned} u=UJ_s/C, \quad \eta=HJ_s/C,\\ f_i=g_i/C, \quad h=4\pi J_s^2/C. \end{aligned} \tag{36}$$

Given f_1 and f_2 as functions of (x, y, z), the problem is to find α and β; then

$$J=J_s[1-\tfrac{1}{2}(\alpha^2)_{\text{Av}}-\tfrac{1}{2}(\beta^2)_{\text{Av}}]. \tag{37}$$

The solution will now be carried out for forces concentrated in planes, along lines, and at isolated points.

(a) **Plane concentrations of force: f_1, f_2 constant in planes**

Case 1. Planes parallel to the field direction.—Take the zx plane as the constancy plane; then $\partial/\partial x=\partial/\partial z=0$. Eqs. (33)–(35) become

$$d^2\alpha/dy^2-\eta\alpha=f_1, \tag{38}$$

$$d^2\beta/dy^2-du/dy-\eta\beta=f_2, \tag{39}$$

$$d^2u/dy^2=hd\beta/dy. \tag{40}$$

α is determined by f_1, and β independently by f_2. Eq. (38) has been solved in §2. Integration of (40) gives $du/dy=h\beta+$const., and for finiteness of u at infinity the constant must vanish; thus (39) becomes identical in form with (38) except that η is replaced by $\eta+h$, and in the final result $\lambda=\sqrt{\eta}$ must be replaced by

$$k=(\eta+h)^{\frac{1}{2}}=[(H+4\pi J_s)J_s/C]^{\frac{1}{2}}. \tag{41}$$

Combining the two results gives, for an incoherent distribution of such plane sources with density N per cm along y,

$$J/J_s=1-(N/8)[(F_1^2)_{\text{Av}}/\lambda^3+(F_2^2)_{\text{Av}}/k^3]. \tag{42}$$

This contains a term in $1/H^{\frac{1}{2}}$ and another term in $1/(H+4\pi J_s)^{\frac{1}{2}}$. The quantity $H+4\pi J_s$ may be identified with the induction $B=H+4\pi J$, since the difference between J and J_s is of no significance in a term that is itself of order α^2.

Case 2. Planes perpendicular to the field direction.—In this case z replaces y as independent variable, and α and β are each determined by an equation of the form (38); the result is (42) with k replaced by λ.

Case 3. Planes with normal at an angle θ_1 to the field direction.—Let the normal lie in the yz plane: then $\theta_1=\pi/2$ is Case 1, and $\theta_1=0$ is Case 2. If s is distance measured along the normal, then $\partial/\partial x=0$, $\partial/\partial y=\sin\theta_1 d/ds$, $\partial/\partial z=\cos\theta_1 d/ds$. The substitutions $u=u'/\sin\theta_1$, $h=h'/\sin^2\theta_1$ reduce the equations to the form already solved in Case 1. The result is the same as (42) except that k is replaced by

$$k'=(\lambda^2\cos^2\theta_1+k^2\sin^2\theta_1)^{\frac{1}{2}}. \tag{43}$$

(b) **Line concentrations of force: f_1, f_2 constant along lines**

Case 1. Lines parallel to the field direction.—The equations to be solved are Eqs. (33)–(35) with $\partial/\partial z=0$. For a single line source, $f_i=0$ except at, say, $x=y=0$, while $\int f_i dS_z$ ($dS_z=dxdy$) has a finite value F_i for any region including the origin. Integration of (33) and (34) over such a region leads to conditions of the form

$$\int_0^{2\pi}(\partial\alpha/\partial r)rd\theta\to F_1 \quad \text{as} \quad r\to 0; \tag{44}$$

in addition u, $r\alpha$, and $r\beta$ must be finite at the origin. Here r and θ are polar coordinates in the xy plane.

For $f_i=0$, the two-dimensional vector $\mathbf{v}=\alpha\mathbf{i}+\beta\mathbf{j}$ is easily shown to satisfy the equations

$$(\nabla^2-k^2)\nabla\cdot\mathbf{v}=0, \tag{45}$$

$$(\nabla^2-\lambda^2)\nabla\times\mathbf{v}=0. \tag{46}$$

Thus the magnetic charge density $-\nabla\cdot\mathbf{J}$ falls to $1/e$ of its original value in a distance $1/k$ determined by the induction B; the equivalent current density $\nabla\times\mathbf{J}$ has a decay distance $1/\lambda$ determined by the field H.

The solution of Eqs. (33)–(35) and (44), derivation of which must be omitted but which may be verified by direct substitution, is

$$\begin{aligned} u=u_1+u_2,\\ \mathbf{v}=-(1/\eta)\nabla u_1+(1/h)\nabla u_2+\nabla\times\mathbf{A}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} u_1=-(h/2\pi k^2 r)(F_1\cos\theta+F_2\sin\theta),\\ u_2=[hK_1(kr)/2\pi k](F_1\cos\theta+F_2\sin\theta),\\ A_z=[K_1(\lambda r)/2\pi\lambda](F_1\sin\theta-F_2\cos\theta),\\ A_x=A_y=0. \end{aligned} \tag{48}$$

The evaluation of the integral $S_z(\alpha^2+\beta^2)_{Av}=\int v^2 dS_z$ may be carried out by making use of the properties of the modified Bessel functions K_n.[8] The result for an incoherent aggregate of line sources, N of them per cm² of the xy plane, is

$$J/J_s=1-(N/16\pi)[(F_1^2)_{Av}+(F_2^2)_{Av}](1/\lambda^2+1/k^2). \quad (49)$$

This contains a term in $1/H$ and an equal term in $1/B$. For $H\gg 4\pi J_s$, both terms vary essentially as $1/H$. For $H\ll 4\pi J_s$ the term in $1/B$ is practically constant, and the curve again follows a $1/H$ law, but with a coefficient only half as great and with a slightly smaller apparent saturation value.

Case 2. Lines perpendicular to the field direction.—Take the x axis as the direction of the lines. Then

$$\begin{aligned}&\nabla^2\alpha-\eta\alpha=f_1,\\&\nabla^2\beta-\partial u/\partial y-\eta\beta=f_2,\\&\nabla^2 u=h\partial\beta/\partial y,\end{aligned} \quad (50)$$

with $\nabla^2=\partial^2/\partial y^2+\partial^2/\partial z^2$. The α and β equations separate. For a concentrated source α is given by

$$\alpha=-(F_1/2\pi)K_0(\lambda r), \quad (51)$$

and

$$S_x(\alpha^2)_{Av}=\int\alpha^2 dS_x=F_1^2/4\pi\lambda^2. \quad (52)$$

The value of $(\beta^2)_{Av}$ may be found by the Fourier integral method. In this case

$$f_2=F_2\delta(y)\delta(z)=(F_2/4\pi^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos py\cos qz\,dpdq, \quad (53)$$

and the solution for $f_2=(F_2/4\pi^2)\cos py\cos qz$ may be found without difficulty. Superposition of the results gives

$$\beta(y,z)=(1/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(p,q)\cos py\cos qz\,dpdq, \quad (54)$$

where

$$\psi(p,q)=-(F_2/2\pi)(p^2+q^2)/[(p^2+q^2)^2+(\lambda^2p^2+k^2q^2)], \quad (55)$$

and

$$S_x(\beta^2)_{Av}=\int\int\beta^2 dydz=\int\int\psi^2 dpdq=F_2^2/4\pi\lambda k; \quad (56)$$

the last integral is easily evaluated by transforming to polar coordinates in the pq plane.

Combining the α and β contributions gives

$$J/J_s=1-(N/8\pi)[(F_1^2)_{Av}/\lambda^2+(F_2^2)_{Av}/\lambda k], \quad (57)$$

with a term in $1/H$ and a term in $1/\sqrt{(HB)}$. The second term is equal to the first for $H\gg 4\pi J_s$ and small for $H\ll 4\pi J_s$.

Case 3. Lines at an angle θ_2 to the field direction.—This case is still more complicated than the last; the Fourier integral method leads to the result

$$\frac{J}{J_s}=1-\frac{N(k''-k\cos^2\theta_2)}{8\pi\lambda^2\sin^2\theta_2}\left\{\frac{(F_1^2)_{Av}}{k''}+\frac{(F_2^2)_{Av}}{k}\right\}, \quad (58)$$

where

$$k''=(\lambda^2\sin^2\theta_2+k^2\cos^2\theta_2)^{\frac{1}{2}}. \quad (59)$$

This reduces to (49) for $\theta_2=0$ and to (57) for $\theta_2=\pi/2$. The lines are perpendicular to the y axis. In this case and in the next, the delta-function is most conveniently written in the form

$$\delta(x)=(1/2\pi)\int_{-\infty}^{\infty}e^{ipx}dp. \quad (60)$$

(c) **Point concentrations of force**

Here F_i is a volume integral over a region containing the point source. The general equations (33)–(35) must be used. For $H\gg 4\pi J_s$, u may be neglected, and

$$\alpha/F_1=\beta/F_2=-e^{-\lambda r}/4\pi r, \quad (61)$$

where r is the distance from the source to the point at which α and β are being evaluated. At lower fields the curl of $\mathbf{v}$ about the field direction has a decay distance $1/\lambda$, but the equations determining u and $\nabla\cdot\mathbf{v}$ contain both decay distances. The Fourier integral method leads to the following result for an incoherent aggregate of point sources of number N per cm³:

$$\frac{J}{J_s}=1-\frac{N}{32\pi}[(F_1^2)_{Av}+(F_2^2)_{Av}]\left\{\frac{1}{\lambda}+\frac{1}{h^{\frac{1}{2}}}\sin^{-1}\frac{h^{\frac{1}{2}}}{k}\right\}. \quad (62)$$

The quantity in braces reduces to $2/\lambda$ for $H\gg 4\pi J_s$ and to $1/\lambda+$const. for $H\ll 4\pi J_s$, and thus gives a $1/H^{\frac{1}{2}}$ law of approach to saturation in either case.

Summary of theoretical conclusions

At fields high enough to justify the neglect of magnetic interactions ($H\gg 4\pi J_s$), point, line, and plane concentrations of force lead to laws of the form $J/J_s=1-a/H^{n/2}$, with $n=1$, 2, 3, respectively; to these may be added the result of the conventional theory, $n=4$ for forces uniform throughout an extended volume. At much lower fields the power of H is the same as at high fields, but in most cases its coefficient is smaller and a constant term must be added. In the range $H\leqq 4\pi J_s$ there occurs a transition from one behavior to the other.

When two or more of these types of force distribution are present and there is no correlation between them, their contributions to J_s-J may be added, for the cross-product term in Eq. (16) is negligible.

The solution for arbitrary one-, two-, or three-dimensional functions f_i may be expressed formally as an integral of the plane, line, or point solution over the sources, but in most cases the resulting expressions will be too complicated to be of much value.

§4. Interpretation of Experiments

The actual occurrence of a $1/H$ term suggests the presence of line concentrations of force. The precision of any data at present available is probably not sufficient to determine with cer-

[8] Gray, Mathews, and MacRobert, *Treatise on Bessel Functions* (Macmillan, 1922), Chapters 3 and 6.

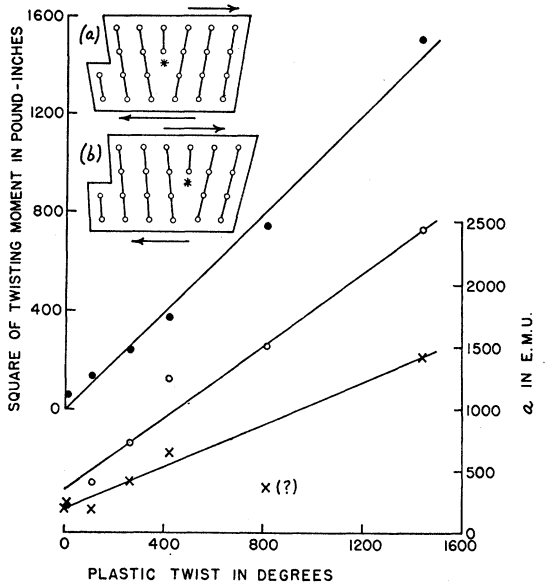

FIG. 3. Test of the hypothesis that the $1/H$ term in the magnetization curve is due to dislocation lines. Data plotted were taken by A. R. Kaufmann on a 42" length of annealed nickel wire $\frac{1}{4}$" in diameter. Top curve, test of Taylor's theory of hardening, according to which the plastic strain and the square of the stress producing it are proportional to the density N of dislocation lines and therefore to each other. Lower curves, test of present theory of approach to saturation, according to which the coefficient a of the $1/H$ term should be proportional to N and therefore to the plastic strain. The circles represent values with the torque still acting, the crosses after its release. Two successive stages in the propagation of a dislocation line through the lattice are shown schematically in (a) and (b), which represent cross sections perpendicular to the line. The atoms are represented by circles and the "dislocation" is indicated by an asterisk. The shearing stress acts as shown by the arrows; the lines joining the atoms are intended merely as an aid to the eye.

tainty whether $1/H^{\frac{1}{2}}$ and $1/H^{\frac{3}{2}}$ terms are also present.

The explanation of the $1/H$ term as due to line concentrations is borne out by Kaufmann's measurements on nickel subjected to various degrees of plastic twist.[9] He finds that the coefficient a in Eq. (2) increases with the plastic twist. Plastic flow is now explained as a propagation of "dislocation lines" through the lattice; (a) and (b) in Fig. 3 represent a cross section of such a dislocation line in two successive stages of the propagation.[10] According to Taylor's theory of hardening, the application of a shearing stress greater than the initial yield value produces new dislocations, each of which is propagated a certain distance and then stopped by a flaw; this continues until the stopped dislocations have become so numerous that their stress field, which opposes the applied stress, becomes large enough to counteract it completely and thus to create a stable condition with a new yield value. The plastic shear γ is proportional to the number N of dislocations per cm^2 that have been produced, propagated, and stopped, and N is proportional to the square of the applied shearing stress. To what extent Kaufmann's specimen follows this law is shown by the upper graph in Fig. 3; from the slope of the curve and from other data on the specimen it follows that $N/\gamma \doteq 1.4\times10^{12}$ dislocations per cm^2 per unit of plastic shear. The data used so far are purely mechanical. But if the dislocation lines are identical with the line concentrations of force responsible for the term $-a/H$ in the magnetization curve, then a should be proportional to N and therefore to the plastic shear. The lower curves in Fig. 3 show that this is at least approximately true. From the slope and from magnetic data, the order of magnitude of $N(F_1^2)_{Av}/\gamma$ is found to be 2×10^{11} c.g.s. units per unit of plastic shear. Combining the mechanical and magnetic results gives for the root mean square value of CF_1, 2×10^{-6} erg/cm. This is equal to $\int g_1 dS_x$ and may be set equal to $g_1 a_0^2$ in order of magnitude, whence $g_1 \cong 5\times10^9$ erg/cm^3. But $g_1 = \partial w/\partial\alpha \cong w$; the figure 5×10^9 erg/cm^3 therefore gives the order of magnitude of the free energy density associated with the forces that oppose saturation. It is interesting to compare it with the volume densities of anisotropy energy, $K_1 = 5\times10^4$; dipole energy, $\frac{2}{3}\pi J_s^2 = 5\times10^5$; magnetostrictive energy for a strain $\cong 1$, $\lambda_\infty E = 8\times10^7$; and exchange energy, $I/a_0^3 \cong 10^{10}$.[11] The last alone is of a comparable order of magnitude. Is it the breaking up of exchange forces at a dislocation that causes the deviation from saturation?

[9] A. R. Kaufmann, reference 2. The writer is indebted to Dr. Kaufmann for providing him with the data analyzed here.

[10] G. I. Taylor, Proc. Roy. Soc. **A145**, 362 (1934); J. M. Burgers, Proc. Phys. Soc. **52**, 23 (1940).

[11] K_1 = anisotropy constant, λ_∞ = saturation magnetostriction, E = Young's modulus; values of these and the other quantities may be found in the books listed in the bibliography of reference 4.