Steady-State Diffusion Under Conditions of Generalized Source and Incident Current Distributions

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The method of Laplace transformations is extended to the more general case of neutron diffusion in half-infinite media containing source distributions, and an exact expression for the emergent current distribution is obtained. This expression depends on the total m.f.p., on the ratio N of the capture to scattering m.f.p., and on the type of source distribution. For a uniform source distribution and with N sufficiently large, the values of the normalized current as a function of angle agree closely with those given by the Fermi law. More significant discrepancies between the results of elementary theories and the exact theory are to be found in the ratio R of detector activities corresponding to positions deep within the medium and at the surface. The exact theory gives $R=(N+1)^{\frac{1}{2}}+1$ for uniform source distributions. A recent quotation of $R=14$ for the case of thermal neutron diffusion in paraffin then implies $N=168$ instead of 196 as given by the elementary theory.

INTRODUCTION

THE diffusion of thermal neutrons in hydrogenous materials lends renewed significance to a class of problems in the theory of gases which have not hitherto had adequate treatment. In order to make application to the experimental situation of interest, one requires exact solutions of the Boltzmann conservation equation in closed form subject to boundary conditions which determine the behavior of the gas as a whole. The point of view which one adopts for the contemplation of the theoretical problem is necessarily very different from that adopted in the usual theory of gases. According to the latter one seeks solutions of the Boltzmann equation which describes the state of a gas within a differential volume element subject to arbitrarily assigned values of the gas parameters and their derivatives within this element. Such solutions are completely general, and they are in principle sufficient to determine the steady-state condition prevailing in the experimental problem of neutron diffusion. But there are two difficulties connected with such applications of the general solutions. On the one hand these solutions are obtained by perturbation methods and the resulting series converges rapidly to the exact solution only in the limiting case of small deviations from spherical symmetry, a condition which may not be even approximately fulfilled when an arbitrary assignment is made on the boundary conditions in the steady-state problem. On the other hand, one would require not only the solutions of the Boltzmann equation giving the distribution function in terms of the gas parameters and their derivatives, but also the solution of the steady-state hydrodynamical equations giving the spatial dependence of the gas parameters subject to the imposed boundary conditions.

A more direct approach would appear to be the search for solutions of the Boltzmann equation itself subject to the boundary conditions imposed on the entire gas. This point of view has been used already in the case of no external forces to give solutions of well-known problems in astrophysics, namely, the transmission of radiant energy through stellar[1,2] and planetary atmospheres.[3] The treatment has not been based explicitly on the Boltzmann equation, but this can be done by considering radiation as a gas[4] of photons.

In the problem of neutron diffusion one is interested in such questions as the distribution emergent from the surfaces of a medium when neutron currents of a particular character are incident on the surfaces and source distributions

[1] E. A. Milne, "Thermodynamics of Stars," *Handbuch der Astrophysik*, Vol. III/1 (Springer, 1930), pp. 96–183; V. A. Kostitzen, Mem. Sc. Math. **69** (1935).
[2] E. Hopf, *Problems of Radiative Equilibrium*, Cambridge Tract No. 31 (1934).
[3] L. V. King, Phil. Trans. Roy. Soc. **A212**, 375–433 (1913).
[4] G. Jaffe, Ann. d. Physik [5] **6**, 195–252 (1930).

exist within the medium. The assumption of source distributions and the property of finite absorption probabilities possessed by neutrons introduce new features into the physical situation insofar as particle diffusion is concerned; and the requirement of exact solutions in closed form demanded by the importance of the boundary effect and the slow convergence of usual series solutions introduces new features into the mathematical treatment.

The method of attack on this problem used by Halpern, Lueneburg and Clark[5] in connection with the study of the albedo coefficient is most satisfactory. According to this method the integral equation associated with the Laplace transform of the density function is studied, and solutions of this equation subject to the imposed boundary conditions are obtained before attempting to find the general solution of the Boltzmann equation. It is the object of the present paper to extend this method to the more general case of diffusion in a half-infinite medium within which there exist source distributions, to establish a certain equivalence between the problem of source distributions and that of finite incident currents, to consider questions of uniqueness of solutions, and to state more fully than has been done in the past the analyticity considerations fundamental to the rigorous and complete solution of the problem. Numerical applications are then made to the problem of thermal neutron diffusion in paraffin.

I. The Fundamental Integral Equations

§1. Boltzmann equation

In the class of problems under consideration mutual interactions are regarded as negligible and all interactions with centers of force within the medium are regarded as occurring isotropically without the exchange of energy. Such conditions represent a convenient mathematical idealization satisfactorily describing many physical situations involving the scattering, absorption and penetration of particles of one kind through a fixed uniform distribution of particles of another kind, and, in particular, the diffusion of thermal neutrons in certain hydrogenous materials. A further idealization according to which all spatial variations are one dimensional with azimuthal symmetry of all functions about this direction may be introduced for purposes of simplification without restricting essentially the validity of applications of the theory to many experimental situations. Isotropy of scattering without energy exchange has two important consequences. The first of these is that the absolute value of the velocity enters only parametrically in the equations. The second is that the total scattering into, as well as out of, a particular velocity class (defined by the differential solid angle containing the velocity vector) is proportional to a total scattering probability. Under these conditions the fundamental Boltzmann equation describing the steady-state condition in the absence of external forces is

$$v\alpha\frac{\partial f(x,\alpha)}{\partial x}=q(x,\alpha)-(P_s+P_c)f(x,\alpha)+\frac{P_s}{2}\int_{-1}^{1}f(x,\alpha)d\alpha, \quad (1)$$

where $f(x, \alpha)$ represents the number of particles per unit volume per unit solid angle at the point x having a velocity of magnitude v lying in $d\alpha$ where $\alpha=\cos\theta$ and θ is the angle between the velocity vector and the x-direction. Furthermore, P_s and P_c represent the total scattering and capture probabilities, and $q(x, \alpha)$ is the source strength representing the rate of production of v-speed particles per unit volume per unit solid angle.

§2. Integral equations in $f(x, \alpha)$

Equation (1) together with certain boundary conditions defines $f(x, \alpha)$ everywhere within a single homogeneous isotropic medium. If there are several media possessing different scattering properties, an analogous equation exists for each. The properties of these equations are such that the boundary conditions may be specified in terms of arbitrary functions of α only in a particular way. The physical equivalent of such a specification is the arbitrary assignment of incident current distributions on the two outermost faces of the several media together with the requirement of continuity in the distribution function, but not its derivative, in passing from one medium to the next.

[5] O. Halpern, R. Lueneburg and O. Clark, Phys. Rev. **53**, 173–83 (1938).

For a single homogeneous isotropic medium lying between the planes $x=0$ and $x=a$, one may specify consequently

$$f(0, \alpha)=f_0(\alpha), \quad \alpha>0; \qquad f(a, \alpha)=f_a(\alpha), \quad \alpha<0, \tag{2a}$$

where $f_0(\alpha)$ and $f_a(\alpha)$ are arbitrary functions defined in the restricted range of α and piece continuously in x onto the distributions within the medium. Equation (1) may now be integrated to give two integral equations in $f(x, \alpha)$ instead of a single integro-differential equation. Introducing the concept of mean free paths defined by

$$l_s=v/P_s, \quad l_c=v/P_c, \quad 1/l=1/l_s+1/l_c$$

and using the integrating factor $e^{x/l\alpha}/\alpha v$, one obtains

$$f(x, \alpha)=f_0(\alpha)e^{-x/l\alpha}+\int_0^x \frac{dx'}{\alpha}\left[\frac{F(x')}{4\pi l_s}+\frac{q(x')}{v}\right]e^{(x'-x)/l\alpha}, \quad \alpha>0, \quad 0\leqslant x\leqslant a; \tag{2b}$$

$$f(x, \alpha)=f_a(\alpha)e^{(a-x)/l\alpha}-\int_x^a \frac{dx'}{\alpha}\left[\frac{F(x')}{4\pi l_s}+\frac{q(x')}{v}\right]e^{(x'-x)/l\alpha}, \quad \alpha<0, \quad 0\leqslant x\leqslant a \tag{2c}$$

where

$$F(x)=2\pi\int_{-1}^{1} d\alpha f(x, \alpha) \tag{2d}$$

is the total density of particles at x.

These equations have a simple physical interpretation in terms of kinetic theory concepts. Thus, for example, the first part of the integral represents the density contribution to the distribution function at a specified x plane due to free flight trajectories (straight lines for no external forces) from a plane at x' where the density is $F(x')$. The contribution from a layer of thickness dx' is made up of the product of four factors: (a) the number of collisions per second $vF(x')dx'/l\alpha$ in the "ray volume element" dx'/α; (b) the probability $1/4\pi$ of scattering into unit solid angle about α under isotropic conditions; (c) the probability l/l_s that the collision is a scattering act and not one of capture; and (d) the probability $\exp[-|(x'-x)/l\alpha|]$ that the particle will travel the ray distance $|(x'-x)/\alpha|$ without further collision. The product of these factors actually yields the current density per unit solid angle about α at x, and division by v must be made to obtain the corresponding distribution function. A similar interpretation in terms of the contribution due to incident currents and source distributions can be made for the other terms of the equations.

§3. Integral equation in F(x)

A general method of solving the simultaneous integral equations in $f(x, \alpha)$ is to solve first the single integral equation in $F(x)$ obtained by performing the integration of Eq. (2d) on the functions f defined by Eqs. (2b) and (2c). This integral equation may be put into the form

$$F(x)=g(x)+\frac{l}{2l_s}\int_0^a dx' K(|x'-x|)F(x'), \quad 0\leqslant x\leqslant a, \quad 0\leqslant\frac{l}{l_s}\leqslant 1, \tag{3a}$$

where all distances are measured in units of the total m.f.p. l. The inhomogeneous term is a continuous function for all problems of physical interest, and is given in terms of incident current distributions and isotropic source distributions by the equation

$$g(x)=2\pi\left[\int_0^1 d\alpha f_0(\alpha)e^{-x/\alpha}+\int_{-1}^0 d\alpha f_a(\alpha)e^{(a-x)/\alpha}+\frac{l}{v}\int_0^a dx' K(|x'-x|)q(x')\right]. \tag{3b}$$

The symmetric kernel of the integral equation is the logarithmic integral function defined by

$$K(s)=\int_0^1 d\alpha e^{-s/\alpha}\alpha^{-1}, \quad s>0. \tag{3c}$$

Because of the properties of K, the general solution of Eq. (3a) may be obtained in the form of an infinite series (the Neumann iteration solution)[6] by the method of successive approximations. One can show that no other bounded continuous solution exists.[7] However, for half-infinite medium problems ($a \gg 1$, a measured in units of l) and a small capture probability ($l/l_s \sim 1$), which is the case of interest for example in the diffusion of thermal neutrons in extended hydrogenous materials, convergence of the series is slow and a solution in closed form is desirable. Such a solution may be obtained by the method of Laplace transformations.

§4. Integral equation in f(0, z)

The form of the emission current expression for the case of isotropic sources ($q(x, \alpha) = q(x)$) and half-infinite medium conditions ($a = \infty$), which is given by Eq. (2c) when x is set equal to zero, suggests the use of the Laplace transformation of the density function. The following Laplace transformation is obtained by analytic continuation of (2c) into the complex plane

$$-zf(0, z) = \int_0^\infty dx' e^{x'/lz} \left[\frac{F(x')}{4\pi l_s} + \frac{q(x')}{v} \right], \quad R(z) < 0. \tag{4a}$$

The integral defines a regular function for $R(z) < 0$ as long as $F(x)$ and $q(x)$ are absolutely integrable over any finite interval and the above integral exists.[8] If, furthermore, $F(x)$ and $q(x)$ are continuous, the transformation (4a) has a unique inversion of the form[9]

$$\frac{F(x)}{4\pi l_s} + \frac{q(x)}{v} = \lim_{t_2 \to \infty} \frac{1}{2\pi i} \int_{t_1 - it_2}^{t_1 + it_2} dt \frac{e^{xt}}{lt} f(0, -1/lt), \quad t_1 > 0, \quad 0 < x < \infty. \tag{4b}$$

It is now evident that the problem can be solved in closed form if a closed analytic expression for $f(0, z)$ with $R(z) < 0$ can be found. The integral equation which $f(0, z)$ must satisfy in the general case of isotropic sources and axially symmetric incident currents on the face of the half-infinite medium can be obtained in a manner analogous to that for the case of incident current distributions alone.[5] One obtains

$$p(z, \sigma) f(0, z) - I(z) = \sigma \int_{-1}^{0} d\alpha \frac{\alpha f(0, \alpha)}{\alpha - z}, \quad R(z) < 0 \quad \text{and} \quad z \neq (-1, 0), \tag{4c}$$

where

$$p(z, \sigma) = 1 + \sigma z \int_{-1}^{1} d\alpha / (\alpha - z), \quad \sigma = l/2l_s, \tag{4d}$$

$$I(z) = \sigma \int_0^1 d\alpha \frac{\alpha f_0(\alpha)}{\alpha - z} - \int_0^\infty dx e^{x/lz} q(x)/vz, \quad R(z) < 0. \tag{4e}$$

II. Function Theory Treatment for Half-Infinite Medium Problems

§5. Regularity conditions

The integral equation (4c) and its analytic continuation have the particular significance of giving the analytic behavior of $f(0, z)$ everywhere in the complex z plane where it is not already known, i.e., in the right half-plane. For, assuming $f(0, \alpha)$, $-1 \leq \alpha \leq 0$, piecewise continuous, the right-hand side of (4c) or

$$\sigma \int_{-1}^{0} d\alpha \frac{\alpha f(0, \alpha)}{\alpha - z} = \sigma M(z) \tag{5a}$$

represents a regular function of z off the path

[6] E. Whittaker and G. Watson, *Modern Analysis* (Cambridge University Press, 3rd edition, 1920), § 11.4. The logarithmic infinity at the zero argument of the kernel function causes no difficulty since $\frac{1}{2}\int_0^a dx' K(|x'-x|) \leq 1$, $a \leq \infty$, and therefore the convergence of the iterative series may be demonstrated easily for either $l/l_s < 1$ or $a < \infty$. The limit case of $l/l_s = 1$, $a = \infty$ has been settled by E. Hopf, reference 2, Theo. VII, pp. 38–40.

[7] Cf. ahead in §8 and references 20, 21.

[8] G. Doetsch, *Theorie und Anwendung der Laplace Transformation* (Springer, 1937), pp. 40–3.

[9] Reference 8, p. 105, Theo. 2.

of integration.[10] Hence, the behavior of $p(z,\sigma)$ and of the given inhomogeneous term $I(z)$ (assumed capable of analytic continuation into the right half-plane) will impose certain analyticity requirements on $f(0,z)$ in order that the left-hand side, known regular for $R(z)<0$ and $z\neq(-1,0)$ by the validity of Eq. (4c), continue to be regular for $R(z)>0$ by the unique analytic continuation called for by $M(z)$. More important, it will appear that satisfaction by a function of these regularity conditions is not only necessary but *sufficient* that this function be a solution of the integral Eq. (4c). Briefly, the sufficiency establishment may be carried out along the following lines,[11] with the essential regularity conditions appearing in the process.

The regularity and first order vanishing at infinity of $M(z)$ gives zero as the value of the contour integral[12]

$$\frac{1}{2\pi i}\oint_L ds\frac{\sigma M(s)}{s-z}=0, \qquad \begin{array}{l}L \text{ containing line}\\ (-1,0) \text{ and point}\\ z\neq(-1,0),\end{array} \tag{5b}$$

where L is a closed ordinary[13] contour which may touch the line $(-1,0)$ at the origin.[14] From Eq. (4c) and its analytic continuation one may restate (5b) as

$$\frac{1}{2\pi i}\oint_L ds\frac{p(s,\sigma)f(0,s)-I(s)}{s-z}=0 \tag{5c}$$

with the ordinary contour L arbitrary except for the restrictions mentioned in connection with (5b). Therefore, by taking the point z to be anywhere in the complex plane except the interval $(-1,0)$, one sees from Eq. (5c) that a proposed solution $f_1(0,z)$ must be such that the expression

> $p(z,\sigma)f_1(0,z)-I(z)$ is made regular for any $z\neq(-1,0)$ and is continuous into the origin for all approaches not including the negative axis. (A)

Under such conditions one may write by Cauchy's theorems:

$$\frac{1}{2\pi i}\oint_L ds\frac{p(s,\sigma)f_1(0,s)-I(s)}{s-z}=p(z,\sigma)f_1(0,z)-I(z)+\frac{1}{2\pi i}\oint_{L'} ds\frac{p(s,\sigma)f_1(0,s)-I(s)}{s-z}, \tag{5d}$$

where L' is an ordinary contour enclosing only the line segment $(-1,0)$ which it touches at the origin while otherwise remaining in the left half-plane (the integration sense is the same as on L). Now if

> $f_1(0,z)$ and $I(z)$ are taken regular in the left half-plane and continuous into the origin from $R(z)<0$ approaches, (B)

the introduction of p's definition (4d) into the L' integral gives

$$\frac{1}{2\pi i}\oint_{L'} ds\left(1+\sigma s\int_{-1}^{1}\frac{d\alpha}{\alpha-s}\right)\frac{f_1(0,s)}{s-z}=\frac{\sigma}{2\pi i}\oint_{L'} ds\frac{sf_1(0,s)}{s-z}\int_{-1}^{1}\frac{d\alpha}{\alpha-s}=-\sigma\int_{-1}^{0}d\alpha\frac{\alpha f_1(0,\alpha)}{\alpha-z}, \tag{5e}$$

where the last expression follows formally by interchanging the order of integration and applying the residue rule, and can be justified in detail for the ordinary contours employed here by usual limit arguments. Finally, the left-hand side of Eq. (5d) will have zero as its value if

> $z[p(z,\sigma)f_1(0,z)-I(z)]$ is made regular at the infinite point. (C)

Thus the regularity conditions (A, B, C) insure through Eqs. (5d, e) that $f_1(0,z)$ actually is a solution of the integral equation (4c).

§6. Analyticity properties

In order to use the regularity conditions in constructing a solution, the analytical nature and continuation of the inhomogeneous term $I(z)$ must be given over the entire complex plane. For zero incident currents, inspection of a table of Laplace transformations shows that the simplest form for $I(z)$ is obtained by special-

[10] E. J. Townsend, *Functions of a Complex Variable* (Holt, 1915), p. 80, Theo. III.
[11] This treatment was suggested by the related one of reference 5 for the special case of incident currents of δ-function type.
[12] Reference 10, p. 273, Theo. X.
[13] Reference 10, p. 47.
[14] Reference 10, p. 71, Theo. II; only continuity with the surrounding regular values is required on the integration path for the validity of Cauchy's theorem.

izing the source emission rate to be a constant or an exponentially decaying function, namely:

$$q(x)=qe^{-mx},\quad q=\text{const.}\geqslant 0,\quad m\geqslant 0,\quad x\geqslant 0,\tag{6a}$$

when, with $R(z)<0$,

$$I(z)=-\int_0^\infty dxe^{x/lz}q(x)/vz=\frac{q/vm}{1/ml-z}.\tag{6b}$$

From the integrated form it is apparent that such a source term has a unique analytic continuation over the entire complex z plane with a simple pole on the positive real axis at $1/ml$.

The function $p(z,\sigma)$ must be examined now in detail. From its definition given by (4d), it may be expressed as

$$p(z,\sigma)=1+\sigma z\ln\frac{z-1}{z+1}\begin{cases}\text{principal value of ln,}\\ \text{branch cut }(-1,1).\end{cases}\tag{6c}$$

The definition (4d) shows that p is a regular function, $z\neq(-1,1)$, with the limit value $1-2\sigma$ at infinity; also, that $p(z,\sigma)=p(-z,\sigma)$. Equation (6c) shows that p has the limit value 1 at the origin, has branch points which are logarithmic singularities at $z=\pm 1$, and has just two simple zeros[15] at $z=\pm s_0$ where s_0 is the positive real root of the condition for which p vanishes:

$$1=\sigma s_0\ln\frac{s_0+1}{s_0-1}=2\sigma\left(1+\frac{1}{3s_0^2}+\frac{1}{5s_0^4}+\cdots\right),\quad 0<\sigma\leqslant\tfrac{1}{2}.\tag{6d}$$

In order that $p(z,\sigma)f_1(0,z)$ be regular on (0,1), in particular single-valued, as required by the regularity condition (A), one is forced to conclude that p can be expressed as a product of two functions one of which is regular for z on the strip (0,1) so that the other function may be taken as a reciprocal factor in f_1 and hence disappear in the product $p(z,\sigma)f_1(0,z)$. At the same time, condition (B) makes the reciprocal factor in f_1 have no zero in the left half-plane. This regularity and nonzero value required of one factor for $R(z)<0$ is met by the following product decomposition of p through the aid of Cauchy's integral formula on $\ln p$:

$$\ln p(z,\sigma)=\frac{-2iz}{2\pi i}\oint_{N'+N''}\frac{\ln p(it,\sigma)}{t-iz}\frac{dt}{t-(-iz)},\quad -iz\text{ inside contour }N'+N''\tag{6e}$$
$$\equiv\ln p_1(z,\sigma)+\ln p_2(z,\sigma),$$

where the closed contour $N'+N''$ of integration encloses no singularities of $\ln[p(it,\sigma)]/(t-iz)$ and is considered to be composed of two parts N' and N'' uniting only at the origin (positive-real side) and $+\infty$ with, say, N' being the part that starts from the origin and passes to the infinite point beneath the point $-iz$ in the complex t-plane, and where the N' portion of the integral is to be associated with $\ln p_1(z,\sigma)$, the N'' with $\ln p_2(z,\sigma)$. By taking $R(z)<0$, the N' path may be deformed continuously to coincide with the positive-real axis giving for $\ln p_1$:

$$\ln p_1(z,\sigma)=\frac{-z}{\pi}\int_0^\infty dt\frac{\ln p(it,\sigma)}{z^2+t^2},\quad \frac{\pi}{2}<\arg z<\frac{3\pi}{2}\quad (R(z)<0),\quad 0\leqslant\sigma\leqslant\tfrac{1}{2}.^{16}\tag{6f}$$

With $R(z)<0$, analogously

$$\ln p_2(z,\sigma)=\frac{z}{\pi}\int_0^\infty dt\frac{\ln p(it,\sigma)}{z^2+t^2}=\ln p_1(-z,\sigma)=\frac{1}{\pi}\int_0^\infty du\frac{\ln p(izu,\sigma)}{1+u^2},\quad -\frac{\pi}{2}<\arg z<\frac{\pi}{2},\quad 0\leqslant\sigma\leqslant\tfrac{1}{2}.\tag{6g}$$

The last integral form in (6g) is uniformly convergent, $0\leq z\leq\infty$, for $\sigma<\frac{1}{2}$ by the dominance in absolute value of $-\ln(1-2\sigma)\int_0^\infty du/(1+u^2)\pi=-\ln(1-2\sigma)^{\frac{1}{2}}$, so that $z=0$ and $z=\infty$ may be inserted inside the integral sign[17] to evaluate $\ln p_2(0,\sigma)=0$ and $\ln p_2(\infty,\sigma)=\ln(1-2\sigma)^{\frac{1}{2}}$. The uniform con-

[15] Cf. E. T. Copson, *Theory of Functions of a Complex Variable* (Oxford University Press, 1935), p. 119, for the "principle of the argument" by which it can be shown that no other zeros of p exist.

[16] This is essentially the same result for $p_1(z,\sigma)$ as obtained in reference 5 where, however, the validity for $\sigma=\frac{1}{2}$ was not demonstrated.

[17] E. Wilson, *Advanced Calculus* (Ginn, 1912), p. 369.

tinuity continues to exist for $\sigma=\frac{1}{2}$ and z finite, $R(z)>0$, since the integrand vanishes sufficiently strongly at infinity. In fact, the integral represents an analytic function in z whose derivatives of all orders may be formed by differentiation under the integral sign,[18] $R(z)>0$. This last property is useful in establishing the rate of p_2's decrease along the positive real-axis evident in Eq. (6g), in particular, that p_2 has a vertical tangent of logarithmic order at the origin. By the definitions of $p_1(z, \sigma)$ and $p_2(z, \sigma)$ through the N' and N'' contour integrals of (6e) and all analytical continuations possible by continuous deformations of the paths of integration N' and N'', it follows that p_1 and p_2 are regular functions of z over the whole plane except that p_1 contains p's branch cut singularities (0,1) and p's zero in the right half-plane while p_2 shows these same properties of p in the left half-plane ($p_1(z, \sigma) = p_2(-z, \sigma)$ and $p_1(z, \sigma)p_2(z, \sigma) = p(z, \sigma)$).

§7. Solution for exponentially-distributed sources

The complete satisfaction of the regularity conditions is now readily verified for the inhomogeneous term (6b) by the following function:

$$f(0, z) = \frac{q/vm}{1/ml - z} \frac{1}{p_1(z, \sigma)p_2(1/ml, \sigma)}, \quad \sigma<\tfrac{1}{2} \quad \text{or} \quad m>0, \tag{7a}$$

which therefore constitutes the solution of the integral equation (4c) for exponentially-distributed sources. When z is negative-real, Eq. (7a) gives a positive definite function, as is, indeed, necessary for physical meaning. When $\sigma=\frac{1}{2}$ and $m=0$, one can show that $ql\sqrt{3z}/vp_1(z, \frac{1}{2})$ satisfies the regularity conditions because $p_2(z, \frac{1}{2})$ behaves like $1/\sqrt{3z}$ around the infinite point;[19] however, this is a non-physical solution, being negative when z is in $(-1,0)$ with $q>0$, and is a consequence of the fact that the homogeneous integral equation arising from (4c) has solutions when $\sigma=\frac{1}{2}$. The latter are found from the corresponding regularity conditions obtained by setting $I(z)=0$ in statements (A, C). A one-signed solution for $\sigma=\frac{1}{2}$ is evidently:

$$f_h(0, z) = C/p_1(z, \tfrac{1}{2}), \quad C=\text{const.} \tag{7b}$$

§8. Uniqueness considerations

The uniqueness of the solution (7a) and the significance of (7b) must now be determined. A related point is the verification that these solutions of the complex-variable integral are actually the ones coming from the Boltzmann equation with given boundary conditions, or from the integral equations (2b and c) for the half-infinite medium case. These questions can be answered through the uniqueness of the Laplace transformation and its inversion, and the uniqueness of the solutions for the number density integral equation (3a). The existence of a solution for the number density equation has already been indicated. Its uniqueness among continuous bounded solutions when either $l/l_s<1$ or $a<\infty$ follows out of the zero solution of the corresponding homogeneous integral equation arising from (3a) for $g(x)\equiv 0$.[20] The same uniqueness exists in the limit case, too, unless unbounded solutions at infinity are allowed. Then a one-signed solution exists which is unique except for a multiplicative constant and which represents the limit solution of a problem (free of sources) in which the medium slab thickness becomes indefinitely large while the sole incident current at the far face becomes likewise indefinitely large. These last two statements are inferred from the work of E. Hopf[21] on the equivalent problem (pure scattering, half-infinite medium) in astrophysical radiation theory. Such number density uniqueness means that there is essentially only one associated $f(0, z)$ coming from the integral equations in f and therefore satisfying the relations (4a, b). The question is then whether the solutions (7a, b) are

[18] Reference 15, pp. 110–1, the simultaneous integrand continuity in z and u is assured by that of the product zu and by the regularity of $\ln p(izu)$ here.
[19] Reference 5, Eqs. (34), (55) and (58).

[20] Cf. K. Schwarzschild, Berlin Sitz., 1914, Part 2, pp. 1191–2, who gives a simple method applicable here.
[21] E. Hopf, Zeits. f. Physik **46**, 374 (1927–8); **49**, 155 (1928); also, reference 2, p. 37 (Lemma 2).

capable of representation in the form (4a), i.e., that the $F(x)$ function they define by (4b) is inversely related to them by (4a). This representation is indeed possible for functions of the type called for by the regularity conditions and the inhomogeneous terms involved.[22] Hence the uniqueness of the solutions of the complex-variable integral equation (4c) which are regular in the left half-plane with a finite limit value at the origin and positive-realness on the negative-real axis is just that of the corresponding number density solutions obtained from the integral Eq. (3a). Furthermore, these $f(0, z)$ functions necessarily satisfy the Boltzmann equation and boundary conditions since they have the required representation character. The significance and essential uniqueness of the solution (7b) is evident from the corresponding one-signed number density solution for $\sigma=\frac{1}{2}$, $a=\infty$ and its interpretation as the limit of a finite medium problem (by which the arbitrary constant C can be considered determined). When z lies in the interval $(-1,0)$, a suitable transformation of form displays the equivalence of the solution (7b) and that obtained by E. Hopf for starlight emission[23] by means of somewhat different analytical arguments.

§9. Incident current equivalence

For the case of no interior sources within the medium but with incident currents on the finite face, considered in detail by Halpern, Lueneburg and Clark,[5] the same form of inhomogeneous term as (6b) arises when the incident distribution (axially-symmetric) is restricted to a single direction, i.e.,

$$f_0(\alpha)=f\delta(\alpha-\alpha_0), \quad f=\text{const.}\geqslant 0, \quad 0<\alpha\leqslant 1, \quad 0<\alpha_0=\text{const.}\leqslant 1. \tag{9a}$$

Then

$$I(z)=\sigma\int_0^1 d\alpha\frac{\alpha f_0(\alpha)}{\alpha-z}=\frac{f\sigma\alpha_0}{\alpha_0-z} \tag{9b}$$

for which the solution can be obtained from Eq. (7a) by the parameter association:

$$1/ml=\alpha_0 \quad \text{and} \quad q=f\sigma v/l. \tag{9c}$$

The restriction of $\alpha_0\leqq 1$ is interesting as it is one evidence of the lack of complete equivalence of an incident current "beam" to an exponentially-decaying interior source distribution of arbitrary decay constant m. A beam of particles incident on a half-infinite medium decays in primary strength as $e^{-r/l}$ where $r=x/\alpha_0$ is the "ray" distance into the medium from the finite face and l is the total m.f.p. in the medium. Likewise, the number of particles left behind in any element dr is proportional to $e^{-x/l\alpha_0}$. Because of the isotropic no-energy-exchange character of the scattering assumed here, this decay of the direct beam as it enters the medium acts like a source distribution of exponentially decaying form e^{-mx} with $m\geqq 1/l$ ($m=1/l\alpha_0$, $\alpha_0\leqq 1$). For such a type of source distribution the equivalence is complete insofar as the emission current from the medium and the interior distribution $f(x,\alpha)$ with $\alpha\neq\alpha_0$. The form and limitations of the equivalence are evidently determined by the type of medium scattering-capture character.

§10. Analytic continuation for general distributions

The solution (7a) for $f(0, z)$ is an analytic function in $1/ml$ (considered as a complex variable) when $1/ml$ is excluded from the point z, the points of the line $(-1,0)$, and the zero of p in the left half-plane. As far as the emission distribution (4a) or the inversion of the Laplace transformation in (4b) is concerned, it is only necessary to have $R(z)<0$ in $f(0, z)$. Hence, the analyticity in $1/ml$ in (7a) exists for the whole right half-plane including the imaginary axis (but only continuity at the origin); and therefore, a unique analytic continuation of the solution from positive-real $1/ml$ values to those with pure imaginary ones can be made. Since the complex-variable integral equation (4c) is linear,

[22] Reference 8, p. 126–8, Theo. 2–3.
[23] Reference 2, pp. 31, 77, 105.

TABLE I. *Numerical values for* $\ln 1/p_2(z, N)=\ln 1/p_1(-z, N)$.

N	$z=0$	1/8	1/4	1/2	3/4	7/8	98/100	1	∞
∞	0	0.2629	0.4358	0.6980	0.901	0.9887	1.0558	1.0674	∞
150	0		0.4004	0.6297	0.7998	0.8715		0.9395	$\ln(151)^{\frac{1}{2}}$

there exists a superposition principle of individual solutions corresponding to particular inhomogeneous terms, which superposition is unique provided the homogeneous integral equation has no solutions—as certainly is the case for $\sigma < \frac{1}{2}$ or bounded solutions. Thus, by adding solutions corresponding to $m = in$ and $m = -in$ (n = positive integer), the solution corresponding to a cosine source distribution will be obtained; and analogously, for sine distributions. Therefore, the solution for any source distribution representable by a finite Fourier series can be written down in principle. Superposition of the incident current solutions then takes care of the more general case.

III. Applications to Thermal Neutron Diffusion

§11. Emission from paraffin

Thermal neutrons from natural radioactive sources are normally obtained from the Fermi standard paraffin cylinder.[24] A Rn-Be capsule inside the cylinder gives off high speed neutrons which are quickly degraded by collisions with the many protons in the paraffin molecule into the thermal energy region. Once the neutron's energy is below the lowest vibrational level of the C−H bond (0.1 ev[25]), collision effects are with the CH_2 group as a whole insofar as conservation laws are concerned. Lower energies make still more of the paraffin molecule ($\sim C_{22}H_{46}$) the effective mass group with which the neutrons interact. Because of the short range of interaction, only the s type of scattering is of importance for slow neutrons. The mutual

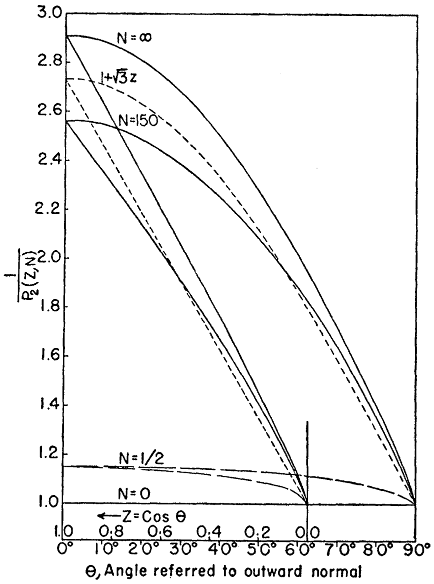

Fig. 1. Emergent number distribution-in-angle for uniform sources.

interactions of the neutrons can be neglected for the low concentrations available in the laboratory; and the diffraction effects from the space lattice appear to be small. Hence one is led to consider the scattering of thermal neutrons in paraffin to be approximately of the type considered in the preceding theory for which exact solutions are now available when the mean free paths and source distribution are given. The experimental m.f.p. for thermal neutrons in paraffin is 0.3 cm, with scattering occurring about 150 times as often as complete capture. In first approximation it is usual to assume that the production of thermal neutrons is uniform over the hydrogenous material in the final steady state.

[24] E. Amaldi and E. Fermi, Phys. Rev. **50**, 901 (1936).
[25] H. A. Bethe, Rev. Mod. Phys. **9**, 126 (1937).

Table II. *Normalized current density emission values.*

θ or $\cos^{-1}\alpha$	$\frac{\alpha p_2(1,150)}{p_2(\alpha,150)}$	$\frac{\alpha p_2(1,\infty)}{p_2(\alpha,\infty)}$	$\frac{\alpha+\sqrt{3}\alpha^2}{1+\sqrt{3}}$	$\frac{\alpha+3\alpha^2/2}{1+3/2}$	$\frac{\alpha p_2(1,150)}{0.896 p_2(\alpha,150)}\left(1-\frac{0.209}{1+\alpha}\right)$
0°	1.	1.	1.	1.	1.
10°	0.978	0.976	0.976	0.976	0.977
20°	0.911	0.905	0.904	0.906	0.907
30°	0.805	0.794	0.793	0.797	0.798
40°	0.671	0.656	0.653	0.659	0.661
50°	0.522	0.501	0.498	0.506	0.509
60°	0.367	0.345	0.342	0.350	0.353
70°	0.219	0.202	0.200	0.207	0.207
80°	0.0925	0.0832	0.0827	0.0876	0.0847
90°	0	0	0	0	0

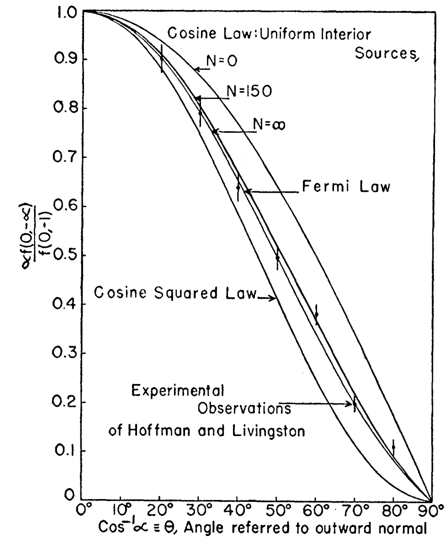

FIG. 2. Normalized emission current distribution-in-angle for thermal neutrons in paraffin.

Concerning the nature of the thermal neutron emission from the paraffin block, Fermi proposed the current density distribution (per unit zonal angle) $\cos\theta(1+\sqrt{3}\cos\theta)$, where θ is the zonal angle referred to the *outward* normal of the emitting face. Fermi[26] and Bethe[27] in deriving this relation neglect the small capture effects, introduce the classical diffusion coefficient, assume isotropic scattering, and consider no distributed interior sources in the block (idealized into a half-infinite medium) but merely assume the number density to vary linearly near the boundary. These many assumptions are not completely compatible. Therefore, it is of interest to obtain the emergent number distribution-in-angle on the basis of the present theory which incorporates source and capture effects directly, makes no use of a diffusion coefficient, and has an exact solution.

The emission distribution is given by Eq. (7a) when z lies in the interval $(-1,0)$ or $z=-\cos\theta$. Only the case of a uniform source distribution $(m=0)$ will be considered in detail. Since $p_1(z,\sigma)$, or $p_2(-z,\sigma)$, appears to constitute a new transcendental function, numerical evaluation of the definite integral definition was necessary to obtain the results given in Table I for various values of the ratio N of capture to scattering mean free path, or the inverse ratio of the cross sections σ_c, σ_s; i.e.,

$$N=l_c/l_s=\sigma_s/\sigma_c=2\sigma/(1-2\sigma). \qquad (10)$$

The notation $p_2(z,N)$ has been introduced for $p_2(z,\sigma)$, where σ and N are related as in (10). To effect the evaluation, the infinite integral of Eq. (6g) was transformed to the following proper one by introducing the new variable arctan $(1/zu)$ and then integrating partially the fractional part of the resulting integrand:

$$\ln\frac{p_2(z,N)}{(1+z)^{\frac{1}{2}}}=\frac{-1}{\pi}\int_0^{\pi/2}d\theta\frac{\cot\theta+N\theta}{N+1-N\theta\cot\theta}\arctan(z\tan\theta),\quad R(z)<0. \qquad (11)$$

The curves of $1/p_2(z,N)$ as a function of z and of θ are shown in Fig. 1.

These curves can be interpreted as being proportional to the emission distribution from the half-infinite medium. The pure scattering case of $N=\infty$ has meaning only in a limiting sense, since the factor of proportionality $1/p_2(\infty,N)=(N+1)^{\frac{1}{2}}$ becomes infinite, unless the interpretation is made on the basis of the homogeneous equation solution (7b) due to an "infinitely-strong" current incident on the "infinitely-far off" face. The exact solutions have a marked dependence on N; and the Fermi relation $1+\sqrt{(3)}z$ is but a rough approximation to the case $N=150$, having a slope more nearly corresponding to $N=\infty$. If the distributions had been reduced to the same value at normal emission, $z=1$, then the Fermi relation and the $N=\infty$ case would agree for not too large angles—which is not surprising considering the homogeneous equation interpretation of the $N=\infty$ case and the generally-corresponding character of the assumptions underlying the Fermi relation.

Passing to the quantity which can be observed directly in experiments, namely, the current density distribution normalized with respect to the normal emission value, one obtains the curves of Fig. 2 together with the

[26] E. Fermi, Ricerca Scient. 7, 13 (1936).
[27] Reference 25, pp. 132–3.

experimental observations of Livingston and Hoffmann[28] on thermal neutron emission from the standard paraffin cylinder. Any value of N from infinity down to 150 in the exact solution appears compatible with the experimental results. The simple cosine law corresponding to pure capture is definitely outside of the experimental data, as also is the cosine-squared law drawn in for comparison. To the scale used, the Fermi law and the $N=\infty$ case are indistinguishable, while the discrepancy with the $N=150$ case has been reduced over that present in the number distribution-in-angle because of the normalization process and the addition of the $\cos\theta$ factor. It may be noted that the relation $\cos\theta(1+\frac{3}{2}\cos\theta)/(1+\frac{3}{2})$ gives a curve intermediate between $N=\infty$ and $N=150$ with, therefore, a better approximation to the exact solution for $N=150$ than the Fermi law.[29] However, the differences for N greater than 150 are hardly large enough to be emphasized in comparison with the indicated statistical uncertainty of the experimental observations. The numerical data are summarized in Table II (the last column will be referred to in the next section).

§12. Ratio of capture to scattering mean free path

The comparison with experiments just made indicates that the lower limit to be expected for N is approximately 150. However, it is to be observed that the deduction is based on the hypothesis of a uniform production of thermal neutrons, whereas it is probably more reasonable to expect the production rate to fall off near the emitting face. To obtain some idea of the effect of such a falling-off, the solution for a source distribution like $1-e^{-mx}$, $m=1/l$, was obtained by superposition (the smallest m value possible with the extent of the calculations given in Table I was used). The resulting current distribution lies roughly halfway between the $N=150$ and the $N=\infty$ uniform source cases, as shown by the last column of Table II. Thus N can possibly take smaller values than 150. The source distribution or the emission current must be known more precisely to fix N in this way.

Another method for the determination of N appears in the cross section expression. Combining two different experimental values for the scattering cross section[30, 31] with the capture cross section measured for water[32] (a direct measurement for paraffin is not known), one obtains for N the two values, 169 ± 8 percent and 195 ± 8 percent, indicating that N actually lies between 150 and 200.

The Fermi "albedo" formula[33] gives an expression for evaluating N as

$$N=R^2, \tag{12a}$$

where R is the ratio of the activity of a thin detector placed deep in the interior of the paraffin to that at the surface of the paraffin. The exact solution for uniform sources furnishes a somewhat modified formula:

$$N=R^2-2R. \tag{12b}$$

If one uses a recent quotation[34] of $R=14$ he

[28] J. G. Hoffman and M. S. Livingston, Phys. Rev. **53**, 1020L (1938).

[29] The current law $\cos\theta[1+(3/2)\cos\theta]$ can be obtained by a modified treatment based on the assumptions underlying the Fermi law.

[30] J. G. Hoffman and M. S. Livingston, Phys. Rev. **53**, 929A (1938).

[31] H. Carroll and J. R. Dunning, Phys. Rev. **54**, 541L (1938); H. B. Hanstein, Phys. Rev. **57**, 1045L (1940).

[32] A. H. Spees, W. F. Colby and S. Goudsmit, Phys. Rev. **53**, 326A (1938).

[33] Reference 24, p. 910, Eqs. (10) and (11).

[34] J. H. Manley, H. H. Goldsmith and J. Schwinger, Phys. Rev. **55**, 44 (1939).

TABLE III. *Transmission ratios for absorption coefficient measurements.*

Kx	Eq. (14) with Exact Solution $N=150$	$\frac{K_1(Kx)+(3/2)K_2(Kx)}{1+3/4}$	$\frac{K_1(Kx)+\sqrt{3}K_2(Kx)}{1+\sqrt{3}/2}$	Eq. (14) with Exact Solution $N=\infty$
0	1.	1.	1.	1.
0.2	0.620	0.629	0.634	0.634
0.4	0.435	0.443	0.448	0.447
0.6	0.316	0.322	0.326	0.325
0.8	0.234	0.238	0.242	0.241
1.0	0.175	0.179	0.181	0.181
1.4	0.101	0.103	0.105	0.105
2.0	0.046	0.047	0.048	0.048

obtains $N=168$. The expression (12b) involves the usual assumption that a thin detector's activity is proportional to the number density at its location and that N is constant over the thermal speed band. The expression comes from

$$R=\frac{F(\infty)}{F(0)}=\frac{f(0,-\infty)-ql/v}{f(0,0)-ql/v}=(N+1)^{\frac{1}{2}}+1, \quad (12c)$$

where the second equality comes from the Laplace transformation (4a), and the third, from the use of the solution (7a) for the f's with $m=0$.[35]

§13. Thermal neutron absorption coefficients

With the character of the thermal neutron emission from the standard paraffin block known, the experiments with plane parallel layers of absorbers and detectors exposed to the emission can be interpreted. Scatterers, however, may introduce complications due to the disturbances of the boundary condition by the back scattering. A class of experiments for which the emergent number density distribution (and not current density) enters more directly into the interpretation of the results is represented by absorption coefficient measurements for different materials with thermal neutrons. Varying thicknesses of the absorber material are placed between the paraffin face and the detector; and the resulting reduction in the activity of the latter depends essentially upon the particle numbers allotted by the paraffin block to the different oblique directions through the absorber, as the following analysis shows:

Let K and x denote the absorption coefficient and thickness of the pure absorber, and k and t the same quantities for the detector. Then the number absorbed per second per unit area by the detector in the v-speed class will be:

$$a(v, Kx)=\int_0^1 d\alpha v\alpha f(0,\alpha)e^{-Kx/\alpha}(1-e^{-kt/\alpha})$$
$$\doteq vkt\int_0^1 d\alpha f(0,\alpha)e^{-Kx/\alpha}, \quad (13a)$$

where $v\alpha f(0,\alpha)$ represents the emission current density (α = direction cosine referred to outward normal from the paraffin block), $e^{-Kx/\alpha}$ represents the fraction emerging from the absorber plate, and $1-e^{-kt/\alpha}$ represents the chance of capture by the detector plate, all edge effects due to finite area of extent being neglected. The last integral in (13a), valid for an infinitely-thin detector, is just the number density $F(x)$ of v-speed neutrons at the location of the detector. From this one sees that proportionality between absorption rate and number density will always exist for a very thin detector producing no appreciable disturbance at the place of insertion. The activity of the detector for a given speed class should bear a definite relation to the absorption rate, and in the final steady state should be proportional to it; i.e.,

$$A(v, Kx)=C(v)\alpha(v, Kx)\doteq C(v)vktF(x), \quad (13b)$$

where $C(v)$ is a proportionality coefficient.

The transmission ratios become

$$A(v, Kx)/A(v, 0)$$
$$\doteq\int_0^1 d\alpha f(0,\alpha)e^{-Kx/\alpha}\Big/\int_0^1 d\alpha f(0,\alpha), \quad (14)$$

giving the reduction in detector activity due to the interposition of an absorber of thickness x in terms of its absorption coefficient K and the source's characteristic number density distribution $f(0,\alpha)$. Amaldi and Fermi[36] have given plots of Eq. (14) for the cases: (a) f constant, giving the first associated logarithmic integral $K_1(Kx)$,[37] and (b) the Fermi relation $1+\sqrt{3}\alpha$, giving $[K_1(Kx)+\sqrt{3}K_2(Kx)]/(1+\sqrt{3}/2)$.

For the exact solution obtained in this paper

[35] In connection with the albedo it should be mentioned that the approximation $\sigma=\frac{1}{2}$ used by Halpern, Lueneburg and Clark in their numerical formula for albedo with incident currents on a half-infinite medium is quite serious if N is actually of the order of 150. This is easily seen in the case of a normally-incident current when the numerical factor $2.91(N=\infty)$ in Eq. (66a) should be changed to approximately $2.56(N=150)$, and the corresponding N value inferred from Amaldi's and Fermi's measurements lowered from 261 to approximately 203—the exact value requiring further calculations of the $p_2(1, N)$ function. The case of uniform incident currents has been discussed explicitly by W. de Groot and K. F. Niessen, Physica **7**, 199 (1940).

[36] Reference 24, p. 902, Fig. 2.

[37] The associated logarithmic integrals are defined as: $K_n(s)=\int_0^1 d\alpha e^{-s/\alpha}\alpha^{n-1}$, $n=0, 1, 2, \cdots$ with the logarithmic integral being the zeroth member. V. A. Kostitzen, Institut des Math. de Moscous (1926), p. 15, gives a tabulation of the first five associated logarithmic integrals.

in the case of $N=150$ and uniform sources, the second column of Table III gives the results of an approximate calculation for the transmission ratios. The calculation has been made by averaging an upper and lower linear approximation to the curve shown in Fig. 1 (z-scale) for $N=150$. It is evident from Table III that $f=1+(\frac{3}{2})\alpha$ is again a better approximation to the $N=150$ case than the Fermi relation, and that the $N=\infty$ case follows closely the Fermi-Amaldi transmission ratios. Further possible applications of the exact solutions, such as in nuclear level widths,[34] will not be considered at this time.

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Temperature Dependence of the Work Function of Tungsten from Measurement of Contact Potentials by the Kelvin Method*

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The Kelvin method for the determination of contact potentials is adapted for measurements between filaments in vacuum. Measurements between a heated and a cool tungsten filament indicate an increase in the work function of tungsten, with temperature, of 6.3×10^{-5} volt per degree centigrade. This effect is distinguished from changes in work function arising from variations, with temperature, in contaminating layers on the tungsten, which tend to obscure the true temperature coefficient even under the best vacuum conditions. The observed true temperature coefficient appears to be comparatively independent of the existing degree of surface contamination. This suggests that the temperature effect must reside primarily in the thermodynamic potential of the electrons inside the metal and not in the potential barrier at the surface. The coefficient obtained resolves the discrepancy between the experimental value of A in the Richardson equation and the theoretical factor of 120 without the introduction of a reflection coefficient.

An increase in the electronic work function of tungsten, with temperature, of approximately 6×10^{-5} volt per degree centigrade was estimated by Waterman and Potter[1] in 1936 from data they obtained from measurements of contact potentials employing the Kelvin method. This positive temperature coefficient of the work function was in definite disagreement with a negative coefficient of approximately 4×10^{-4} volt per degree reported for pure tungsten, earlier in the same year by D. B. Langmuir.[2] His calculations were made from shifts of thermionic curves attributed to changes in contact potential between a collector and an emitter. Although Langmuir expressed the opinion that the negative coefficient might be an effect resulting from imperfect vacuum conditions, this coefficient did agree well with some deductions of Nottingham.[3]

As pointed out by Becker and Brattain,[4] if the work function, w, is of the form, $w=w_0+\alpha T$, α being the temperature coefficient and T the absolute temperature in °K, then when w appears in the exponent of the Richardson equation, the effect of the second term in w is to multiply the theoretical thermionic current by the factor, $e^{-\alpha/k}$, where k is Boltzmann's constant. After the value of 6×10^{-5} volt per degree had been reported for α in the case of tungsten,[1] it was noticed that this value made the factor, $e^{-\alpha/k}$,

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[1] A. T. Waterman and J. G. Potter, Phys. Rev. **51**, A63 (1937).
[2] D. B. Langmuir, Phys. Rev. **49**, 428 (1936).
[3] W. B. Nottingham, Phys. Rev. **49**, 78 (1936).
[4] J. A. Becker and W. H. Brattain, Phys. Rev. **45**, 694 (1934).