The Interpretation of Resonances in Nuclear Reactions

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The theory of resonances in the one-body problem model is systematized. Energy shifts in the apparent position of levels are found. The apparent position of levels in scattering and in γ-ray emission is expected to be the same to within $\sim\Gamma^2/E$ in the usual notation if the only repulsive barrier is the centrifugal one. In the absence of strong repulsive barriers there may be additional shifts of the order Γ. Interference between levels is shown to occur for the scattering cross section in the numerator and the denominator of a fraction in accordance with Eqs. (3.3), (3.6). The artificiality of usual formulas with interference for cross sections is discussed in this connection. The construction of Green's function for two-dimensional separable problems is described. The construction works also for some many-dimensional problems. The reduction of the number of essential dimensions is outlined by a reformulation of the problem in one less dimension. Applications are made to the solution of scattering problems which represent schematically idealized nuclear problems. The single particle potential barriers enter the solutions through the regular and irregular solutions f, g of the radial equation. It is seen from the solutions that the yield may depend on the f and g for various excited states of the residual nucleus. A simplification occurs if the interaction is highly repulsive and is localized in the two-dimensional space of the distances of the particles from the center. In this case the regular functions f for the incident and final state enter linearly in the wave function as in Eq. (15.5). If the interaction region is made large and if the incident state happens to be especially important in the expansion of Green's function then the factor g of the incident wave occurs in the formula for the cross section as in Eq. (16.4). A resonance model is worked out by assuming the interaction to be highly attractive within a circle and repulsive in a ring surrounding the circle in the two-dimensional diagram. The damping constants in the resultant formula contain f^2 as in Eq. (18.8). Here also the single dependence on f^2 is a consequence of having localized the interaction to a narrowly defined radius for the incident particle. In the general case each damping constant involves the g of different states. The examples show that quantitative applications with simplified forms of damping constants and with interference types of dispersion formulas have only a limited validity.

INTRODUCTION

MANY nuclear reactions show excitation curves with pronounced peaks which suggest that there is a resonance of the nuclear system to certain energies. The older discussions of such resonances have been concerned mainly with "one-body" models in which the actual system was schematically represented as a system consisting of the incident particle moving in the field of the bombarded nucleus. The large number of resonances discovered in the bombardment of nuclei by slow neutrons as well as other reactions have shown the inapplicability of the "one-body" picture in the majority of cases and it has become clear that many nuclear particles are usually involved in a resonance state.

The general physical aspects of the problem are clear and well known.[1-5] The bombarded nucleus together with the incident particle can form a state which has an appreciable stability and a relatively long mean life. The width of the state is connected with the mean life by the indeterminacy relation so that a long life corresponds to a sharp level. The compound state is formed with a maximum intensity when the incident particle has a suitable energy and with smaller intensity when the energy of the bombarding particle is varied through the resonance width. The compound state disintegrates in several energetically possible ways producing either scattered incident particles or disintegration fragments.

This picture is good enough for a qualitative understanding of some reactions. It is not entirely satisfactory for quantitative applications. The compound state used in the above description is not defined by the description itself and it is understood in different ways by different physicists. Some think of it in strict analogy to pre-wave-mechanical pictures as being like an excited state of the electronic system of an atom which can emit photons of several different

[1] N. Bohr, Nature **137**, 344 (1936).
[2] G. Breit and E. Wigner, Phys. Rev. **49**, 519 (1936). Referred to as BW.
[3] H. A. Bethe and G. Placzek, Phys. Rev. **51**, 450 (1937).
[4] F. Kalckar, J. R. Oppenheimer and R. Serber, Phys. Rev. **52**, 273 (1937).
[5] H. A. Bethe, Rev. Mod. Phys. **9**, 71 (1937).

wave-lengths by making optical transitions. This picture is correct except for its pre-wave-mechanical features. Without wave mechanics one can describe qualitatively the decay of the atomic level by introducing Einstein's emission probabilities and saying that the atomic system jumps to one or another of the lower levels with the simultaneous emission of a photon. So far the picture is correct. It is then said that the level width is increased because it is not a real stationary level and that the emitted lines will be fuzzy. For a justification of the latter view an appeal is made to the indeterminacy principle. At this point there is a bad mix-up, the word "level" being used in two different ways. In the first part of the picture it was used in the sense of an atomic level, the atomic system being pictured as an idealized mechanical unit with its interaction with radiation neglected. This kind of "level" is a clear concept as long as it lies in the discrete part of the spectrum, as it is supposed to lie. It is not diffuse but absolutely sharp. When, toward the end of the explanation, one speaks of the "level" as diffuse, it is the level of the compound system consisting of atom+radiation field that one is thinking about. This kind of level is not well defined in the above explanation.

The actual situation in the optical case has been cleared up by Weisskopf and Wigner.[6] They showed by a perturbation calculation that if an atom is put into an excited state j, so that it is definitely in that excited state at the time $t=0$, then at later times there is (a) an exponentially decaying probability of the atom being in the excited state j, (b) a growing probability of lower atomic states i, k, l and of photons with corresponding energies. The energies of the photons emitted in the ji jump are approximately equal to the energy difference $E_j-E_i=E_{ji}$ between the upper and lower atomic states. The band of photons emitted in such a jump is found to have an intensity distribution of the resonance type

$$\frac{\text{const.}}{(E-E_{ji})^2+(\Gamma_i+\Gamma_k+\Gamma_l)^2} \qquad (1)$$

with a peak at the emission center E_{ji}. The half-width of the line $2(\Gamma_i+\Gamma_k+\Gamma_l)$ was shown by them to be the sum of half-widths due to separate emissions ji, jk, jl with emission probabilities $4\pi\Gamma_i/h$, $4\pi\Gamma_k/h$, $4\pi\Gamma_l/h$. They have also explained that for the compound system atom +radiation the energy of the initial state (no photons, atom in state j) is not sharp but has a width connected with

$$\Gamma=\Gamma_i+\Gamma_k+\Gamma_l \qquad (1')$$

by the indeterminacy relation. In their theory the "compound state" is the state of the system

atom+radiation field

in which there are no photons and the atom is in the state j. This compound state does not have a definite energy. The energies of the compound system out of which the state is composed have a probability distribution of the form (1).

In the optical problem the optical transition probabilities are small and it is natural as well as useful to define the compound state in the above manner in most cases. But even here this definition is not the most natural under all circumstances. If a group of atomic levels E_1, E_2, etc. have energy differences $|E_{12}|$, $|E_{13}|$, etc. smaller than the energy widths Γ_1, Γ_2, etc. the compound state obtained by exciting the atom to a single level, say E_2, and having no photons at $t=0$ does not correspond to a simple physical condition. The photons emitted from E_2 are reabsorbed so that the atom is partly in E_2 and partly also in other states E_1, $E_3\cdots$. It is here also possible to define[7] the compound state in a sensible manner, at least approximately. This corresponds to an initial state of the atom with a wave function which is a proper linear combination of the wave functions ψ_1, ψ_2, ψ_3 corresponding to E_1, E_2, $E_3\cdots$. If, at $t=0$ the atom is put into such a state, represented by $c_1\psi_1+c_2\psi_2+\cdots$, and if initially there are no photons, then at later times the probability of the atomic state $c_1\psi_1+c_2\psi_2+\cdots$ decreases exponentially and photons appear within a band having a width connected with the mean life of the atomic state by the indeterminacy relation. It is to be noted that the atomic states E_1, E_2, $\cdots$ have

[6] V. Weisskopf and E. Wigner, Zeits. f. Physik **63**, 54 (1930); P. A. M. Dirac, Zeits. f. Physik **44**, 594 (1927); V. Weisskopf, Ann. d. Physik **9**, 23 (1931).

[7] G. Breit, Rev. Mod. Phys. **5**, 91, 117 (1933); G. Breit and I. S. Lowen, Phys. Rev. **46**, 590 (1934).

different energies. One could, if one liked, speak of ψ_2 as the compound state. This would not be an especially useful procedure for physical applications, however, because (a) this compound state would not decay exponentially; (b) the scattering of light is expressed more simply in terms of the states $c_1\psi_1+c_2\psi_2+\cdots$.

It is seen that even in the optical case the definition of the compound state requires caution. The natural and useful definition is that of using linear combinations of atomic states which give an exponentially decaying probability of atomic excitation. This definition is slightly more complicated than that for the problem of Weisskopf and Wigner. It is definite as long as the interaction of the atom with the radiation is sufficiently weak. If the interaction were strong the exponential decay of the atomic excitation would not be obtained and the compound state could not be defined in the above manner.

Nuclear resonances differ from the optical problem in the following respects: (a) There are cases of nuclear resonances in which the peaks of excitation curves overlap. The coupling through emission which has just been discussed can be expected, therefore, to be of more frequent occurrence. (b) In the optical problem the direct change of one type of photon into another is of secondary importance. With Dirac's electron equation it even disappears altogether. The scattering of photons in this formulation can be explained entirely by a succession of emissions and absorptions affecting the wave function. In the general nuclear problem such a simplification cannot be made. Direct transitions between states in the continuum are possible. (c) The effect of a nuclear resonance is of interest also outside the main resonance peak. For slow neutrons this effect gives the $1/v$ law. There is evidence for it also in the γ-rays emitted by Li^7 under proton bombardment below the 440-kev resonance. (d) The interaction of the atom with radiation is always small. The level of the compound system responsible for the resonance is a state in which there are no photons present. The interactions between nuclear particles are not sufficiently small to make the analogous simplification possible.

It is desirable to be able to use experimental yield-energy curves and to draw from them conclusions regarding properties of the nuclear system. Data on widths of nuclear levels due to neutron emission should be comparable with similar data for widths due to proton emission. The comparison can be made only if proper allowance is made for the effect of the Coulomb barrier. A quantitative understanding of the problem is necessary for this purpose. Similarly it is desirable to know whether peaks in yield-energy curves indicate the position of levels of the compound nucleus with any precision and just what relation these energy levels have to ordinary nuclear levels. For example, the compound state $(C^{12})^*$ formed as $B^{11}+H^1$ shows resonances in gamma-emission at proton energies that correspond to a mass difference between $(C^{12})^*-B^{12}$ which would be expected if $(C^{12})^*$ as well as B^{12} were stable and if the binding energy of $(C^{12})^*$ differed from the binding energy of B^{12} only on account of the Coulomb energy. A question of direct physical interest is whether the hypothesis of a symmetric Hamiltonian for nuclear particles makes one expect a coincidence of this sort and to what accuracy. Another type of question arises in observations of the same level under different circumstances. The position of the level is not expected to be exactly the same and some calculations regarding the magnitude of the energy shift are made in the present paper.

For the discussion of such questions it is difficult to use too general and formal treatments such as have been attempted by Bethe and Placzek[3] and by Kapur and Peierls.[8] The point of view of these authors is that one must represent the collision cross section of a reaction as the square of the absolute value of a sum of dispersion-like terms. Even if this were accomplished the result would not necessarily be useful for the above types of questions because it is impossible to compute all the terms in such formulas starting with expressions for nuclear

[8] P. L. Kapur and R. Peierls, Proc. Roy. Soc. **166**, 277 (1938). For the one-body problem at low energies this theory gives levels which correspond to $d\mathfrak{F}/dr=0$ at the nuclear radius. There are no perceptible resonances at such energies in the deuteron problem. This is due to the fact that the KP levels depend on the total energy and the caution, which must be used in attributing to their levels ordinary resonance properties, may be emphasized. These KP levels show pronounced resonance to change of depth of the potential well but not to change of total energy.

forces. It is practically hopeless to analyze data in terms of resonance levels having variable positions depending on the energy of incident particles, as well as on the assumed nuclear radius, as becomes necessary in the formulas of Kapur and Peierls.

The unsatisfactory state of the theory of nuclear resonance makes it desirable to consider some typical examples. Only partial answers to some of the above questions will be arrived at below.

One-Body Problem

(A) Method of complex eigenvalues

For the one-body problem a single and reasonably sharp level in the continuum has simple properties. The absolute value of the wave function, normalized to unit amplitude outside the nucleus, becomes a maximum inside the nucleus for the resonance energy. An incident plane wave of particles asymptotic to $\exp(i\rho\cos\theta)$ and scattered by a central field is represented by

$$\psi=\sum i^L(2L+1)P_L(\cos\theta)\mathfrak{F}_L(\rho)/\rho. \quad (1'')$$

The functions $\mathfrak{F}_L(\rho)$ are normalized to unit amplitude at $\rho=\infty$ and, therefore, a sharp maximum of $\mathfrak{F}_L$ inside the nucleus corresponds also to a maximum of ψ. The asymptotic form of $\mathfrak{F}_L$ is

$$\mathfrak{F}_L\sim e^{iK_L}\sin(\rho-L\pi/2+K_L).$$

For a given L a resonance can be expected in the neighborhood of a complex eigenvalue of the energy. The possibility of using complex eigenvalues of the energy has been first introduced into quantum theory by Gamow.[9] A connection with resonances has been described by Breit and Yost.[10] The latter connection has recently been pointed out again by Siegert.[11] A discussion of the one-body resonances making no use of complex eigenvalues has been given by Wigner and Breit.[2] Kapur and Peierls[8] have given a discussion of one-body resonances also using complex eigenvalues but along different lines.

By the method of complex eigenvalues one considers the asymptotic form of the solution of

$$d^2\Psi_E/dr^2+(2\mu/\hbar^2)(E-V)\Psi_E=0, \quad (2)$$

which is

$$\Psi_E\sim Ae^{ikr}+Be^{ikr}, \quad k=(2\mu E)^{\frac{1}{2}}/\hbar. \quad (2.1)$$

The normalization of Ψ_E is such that

$$(d\Psi_E/dr)_{r=0}=C$$

is kept constant for all energies E. If

$$(d\Psi_E/dr)_{r=0}=0$$

then one uses for C the first nonvanishing derivative of Ψ_E. The function $A(E)$ is supposed to have a root $E_0-i\Gamma_0$ so that

$$A(E_0-i\Gamma_0)=0. \quad (2.2)$$

For small values of $E-E_0+i\Gamma_0$ one has

$$\begin{aligned} A(E)&=(E-E_0+i\Gamma_0)(dA/dE)_{A=0},\\ B(E)&=(E-E_0-i\Gamma_0)(dA/dE)^*_{A=0}. \end{aligned} \quad (2.3)$$

It is assumed that for real E, $A^*=B$. The connection with Gamow's picture of an exponentially decaying state is obtained by forming a wave packet out of the solutions of (2) with real values of the energy. One has then as in Eqs. (20) of BY the following wave packet

$$\frac{\Gamma_0}{\pi}\int_{-\infty}^{+\infty}\Psi_E\frac{\exp(-iEt/\hbar)}{(E-E_0)^2+\Gamma_0^2}dE=\Psi. \quad (2.4)$$

This wave packet outside the nucleus i.e., in the region where (2.1) applies is given by

$$(t>0);\ \Psi=\begin{cases}=2A^*(E_0)\exp\left\{\dfrac{i}{\hbar}(-E_0t+\mu v_0r)-\dfrac{\Gamma_0}{\hbar}(t-r/v_0)\right\} & (t>r/v_0)\\ =0 \quad (t<r/v_0),\end{cases} \quad (2.5)$$

$$(t<0);\ \Psi=\begin{cases}=2A(E_0)\exp\left\{-\dfrac{i}{\hbar}(E_0t+\mu v_0r)+\dfrac{\Gamma_0}{\hbar}\left(t+\dfrac{r}{v_0}\right)\right\} & (t+r/v_0<0)\\ =0 \quad (t+r/v_0>0).\end{cases} \quad (2.6)$$

[9] G. Gamow, Zeits. f. Physik **51**, 204 (1928); H. Casimir, Physica **1**, 193 (1934).
[10] G. Breit and F. L. Yost, Phys. Rev. **48**, 203 (1935). Referred to as BY; G. Breit, Phys. Rev. **40**, 127 (1932).
[11] A. J. F. Siegert, Phys. Rev. **56**, 750 (1939).

Here v_0 is the classical value of the velocity of the particle outside the nucleus and is obtained from

$$E_0=\tfrac{1}{2}\mu v_0^2.$$

According to Eq. (2.5) the wave packet flows out of the nucleus for $t>0$ as a diverging wave. The place at which Ψ becomes zero moves with the constant velocity v_0. For a fixed r the amplitude of Ψ decreases exponentially at the rate $\exp(-\Gamma_0 t/\hbar)$. The sharp break in Ψ at $r=v_0 t$ is not strictly correct. It is brought in by the approximation in the calculation of BY in the neglect of terms in $(E-E_0)^2$ in the expansion of k in powers of $E-E_0$. For $t<0$ one obtains only incoming waves as indicated by $-i\mu v_0 r/\hbar$ in Eq. (2.6) and for a fixed r the function increases exponentially with t up to $t=rv_0$. After this time $\Psi=0$ at this r. The wave packet (2.4) represents particles flowing out of the nucleus for $t>0$ and flowing into it for $t<0$.

The mean life of the nucleus is τ given by

$$1/\tau=2\Gamma_0/\hbar=4\pi\Gamma_0/h, \quad (\Gamma_0 \text{ in energy units}) \tag{2.7}$$

$$1/\tau=4\pi\Gamma_0. \quad (\Gamma_0 \text{ in frequency units}) \tag{2.8}$$

In energy units Γ_0 is the coefficient of $-i$ in the complex eigenvalue. This is convenient and this quantity will not be denoted here by $\gamma/2$ as is often done. The value of Γ_0 can be computed in some cases by finding the root of Eq. (2.2). This is not necessary, however, because one has [Eq. (21″) of BY]

$$1/\tau=v_0\Big/\int_0^r|\mathfrak{F}_{E_0}(\rho)|^2dr, \tag{3}$$

where $\mathfrak{F}_{E_0}$ is the real solution of Eq. (2) for $E=E_0$ normalized to have unit amplitude at ∞. The integral in Eq. (3) is taken through the nucleus and through a part of the barrier outside the nucleus along the rapidly decreasing portion of $\mathfrak{F}$. The qualitative meaning of Eq. (3) is clear when it is noted that the numerator is the velocity of the escaping particle while the denominator is the probability of the escaping particle being inside the nucleus. The correctness of the numerical factor 1 in the formula is not immediately obvious since the real function

$$\mathfrak{F}_{E_0}(\rho)$$

represents a standing wave and corresponds to a density (per unit thickness of a spherical shell) of $\frac{1}{2}$ rather than 1. The correctness of the factor 1 is shown in BY. It may also be seen from the fact that inside the nucleus there is little difference between $\mathfrak{F}_{E_0}(\rho)$ and the solution of Eq. (2) for $E=E_0-i\Gamma_0$ normalized so as to make

$$2|A(E_0)|=1 \tag{3.1}$$

in Eq. (2.5). If we assume for the moment that this is the case, the coefficient $2A^*(E_0)$ which multiplies the exp { } in Eq. (2.5) has absolute value one. The rate at which particles flow out for this wave packet is v_0 and Eq. (3) follows. The fact that $\int_0^r|\mathfrak{F}|^2dr$ is practically the same for the normalization (3.1) of the complex eigenvalue solution as for $E=E_0$ and $(|\mathfrak{F}|^2)_{Av}=\frac{1}{2}$ follows from the assumed sharpness of resonance. This in fact makes $\int_0^r|\mathfrak{F}|^2dr$ dependent practically only on C and not on E as long as E is not changed by amounts greater in absolute value than Γ_0; and the real function $\mathfrak{F}_{E_0}(\rho)$ normalized so as to be a sine function of unit amplitude at $\rho=\infty$ satisfies Eq. (3.1) so that $\Psi_{E_0}=\mathfrak{F}_{E_0}$ makes the coefficient of the exp in (2.5) have absolute value 1 in agreement with Eq. (3).

The correctness of the coefficient $2A^*(E_0)$ in (2.5) can be seen without integration if one observes that this coefficient must be $B(E_0-i\Gamma_0)$ and that

$$B(E_0-i\Gamma_0)=A^*(E_0-i\Gamma_0+2i\Gamma_0)=2A^*(E_0).$$

Equation (3) is definite only if the upper limit can be defined without ambiguity. For cases where this cannot be done a more precise formulation has been given by BW. This makes no use of the complex eigenvalue point of view and allows a better definition of the upper limit. It follows from Eqs. (2.3) that for the wave (1″) the probability of finding the particle inside the nucleus is proportional to

$$1/[(E-E_0)^2+\Gamma_0^2]. \tag{3.2}$$

In fact outside the nucleus the normalization of Eq. (3.1) holds. Hence according to (2.3), $|C|^2$ is proportional to (3.2) and $|\mathfrak{F}|^2$ inside the nucleus is, therefore, also proportional to (3.2). A nuclear reaction which is proportional to the probability of finding the incident particle inside

the nucleus is also proportional to this expression and is thus represented by a resonance formula.

According to Eqs. (2.3) the phase of $\mathfrak{F}$ changes by $\sim\pi$ as E changes from $E_0-n\Gamma_0$ to $E_0+n\Gamma_0$ for n appreciably >1. One goes through a resonance for scattering in the vicinity of E_0. The explicit connection with the direct calculation of phase shifts is seen through Eqs. (9), (9′), (10′), (13), (17) of BY. The discussion of BY and of BW at this point is perhaps too brief and the discussion of Siegert brings in a distinction (mentioned in reference 13) between resonance and potential scattering which is rather artificial and does not correspond to the usual one. It may be permissible, therefore, to consider the connections again and to extend them so as to include the combined effect of several complex eigenvalues for the energy. It may be possible to represent the function $A(E)$ defined by Eq. (2.1) as

$$\frac{1}{A(E)}=\frac{1}{\alpha(E)}+\sum_{n=1}^{N}\frac{1}{A'(E_n-i\Gamma_n)}\left(\frac{1}{E-E_n+i\Gamma_n}+\frac{1}{E_n-i\Gamma_n}\right) \tag{3.3}$$

and to have $1/\alpha(E)$ a sufficiently smooth function of the energy to make its more detailed consideration unnecessary.[12] The chance of finding the particle in the nucleus, according to Eqs. (1″) and (2.1) is proportional to

$$|1/A(E)|^2$$

and shows, therefore, the interference of N terms of the resonance type. For special functions $A(E)$ it is possible to have $\alpha(E)=$ const. and $N=\infty$. The analytic character of $A(E)$ is not always easy to ascertain and it appears preferable, therefore, not to assume that $\alpha(E)$ is a constant. The representation of $1/A(E)$ by the sum on the right amounts then to splitting $1/A(E)$ into a smoothly varying part $1/\alpha(E)$ and a part represented by $\sum$ which is designed to take out the bumps of $1/A(E)$. The partial scattering cross section corresponding to angular momentum L is

$$\sigma_L=(2L+1)(\lambda^2/\pi)\sin^2 K_L. \tag{3.4}$$

The phase shift K_L is obtained from (2.1) which yields

$$\exp\{i(K_L-L\pi/2)\}=iB/|B|=iA^*/|A|$$

so that

$$\sin K_L=\frac{i^L}{2|A|}[A^*+(-)^L A] \tag{3.5}$$

and

$$\sigma_L=\frac{(2L+1)\lambda^2}{4\pi}\left[2+(-)^L\frac{A^*}{A}+(-)^L\frac{A}{A^*}\right]=\frac{(2L+1)\lambda^2}{4\pi}\frac{|1/A+(-)^L/A^*|^2}{|1/A|^2}. \tag{3.6}$$

Equation (3.6) shows that the partial scattering cross section σ_L is a fraction in which both the numerator and denominator are sums of the type of Eq. (3.3). This form of the scattering cross section is different from the formulas frequently given in which σ_L is represented as the square of the absolute value of a sum of terms of the type

$$\left|\sum\frac{a_n}{E-E_n'+i\Gamma_n'}\right|^2. \tag{3.7}$$

Even if the form (3.6) should be reducible to (3.7) there is no reason to expect $E_n-i\Gamma_n$ to be equal to $E_n'-i\Gamma_n'$. The representation of partial cross section by "dispersion theoretic formulas" is seen to be somewhat arbitrary in the case of interfering levels. The interference is seen to take place in two sums occurring in the numerator and the denominator of a fraction separately as in Eq. (3.6) rather than in a single sum such as is written in the form (3.7).

[12] Professor G. Y. Rainich has called the author's attention to the fact that for entire functions α may be set $=$ const. and the summation is then an infinite series.

In the vicinity of a single sharp resonance level one may write

$$\frac{1}{A(E)}=\frac{1}{\beta(E)}+\frac{1/A'(E_n-i\Gamma_n)}{E-E_n+i\Gamma_n} \tag{3.8}$$

and consider $1/\beta(E)$ a smooth function of E. Since the whole scattering is due to the potential V no reasonable distinction between $1/\beta(E)$ and $1/A(E)-1/\beta(E)$ as referring to "potential" and "resonance" scattering can be made in this problem.[13] Both terms are in this case due to potential scattering. For large $\beta(E)$ one has

$$\sigma_L \leqq (2L+1)\frac{\Lambda^2}{\pi}\frac{[b_n\Gamma_n-a_n(E-E_n)]^2}{(a_n^2+b_n^2)[(E-E_n)^2+\Gamma_n^2]}, \quad (L \text{ even}) \tag{3.9}$$

where

$$1/A'(E_n-i\Gamma_n)=a_n-ib_n$$

and a similar formula for odd L. Thus even neglecting $1/\beta(E)$ in (3.8) one obtains a shift of the maximum of σ_L with respect to the maximum of

$$1/[(E-E_n)^2+\Gamma_n^2].$$

If we neglect the variation of Λ with E the maximum of σ_L is at an energy

$$E=E_n-a_n\Gamma_n/b_n$$

and is displaced with respect to $E=E_n$ by an amount of the order Γ_n. There is thus no exact agreement between the maximum of the scattering cross section and the maximum of the probability of the bombarding particle being in the nucleus. The two maxima are displaced relatively to each other by an amount of the order of magnitude of the width of the level. A displacement of the same order of magnitude can be expected between the maximum of the γ-ray yield curve and the maximum of the scattering cross section. Additional shifts are produced by $1/\beta(E)$ which has been neglected in Eq. (3.9). In the case of Coulomb scattering there are additional shifts which are due to the interference of the Coulomb wave with resonance scattering. These have not been considered above.

(B) The method of regular and irregular functions

The method of complex eigenvalues has some advantages in introducing from the beginning the quantities $E_n-i\Gamma_n$ which appear also in the final formulas. It is not a convenient method, however, for the computation of special problems. The values $E_n-i\Gamma_n$ are troublesome to find and an analysis of the series in Eq. (3.3) by Cauchy's theorem is complicated. The possibility of a continuous distribution of poles in $1/A(E)$ may also enter. For physical applications one needs $1/A(E)$ only for real E. For this reason the method of complex eigenvalues has been partially related to the straightforward use of wave functions for real E by BY to the extent of Eq. (3) of the present paper and by BW in a more thorough manner. The frequency shifts, discussed at the end of the preceding section have not been covered in these papers and will now be considered.

The value of $\mathfrak{F}_L$ outside the potential well ($r>a$) is obtained from the requirement of the continuity of $\mathfrak{F}_L$ and its derivative. This requirement gives

$$\begin{aligned}\frac{1}{\mathfrak{F}_L}=\frac{1}{F_L}(1-F_LG_L\delta-iF_L^2\delta)&=\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}(G_L+iF_L)-(G_L'+iF_L')\\&=\frac{1}{G_L}\left[G_L^2\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{G_L'}{G_L}\right)+iF_LG_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{F_L'}{F_L}\right)\right]; \quad (r>a)\end{aligned} \tag{4}$$

$$\delta=F_L'/F_L-\mathfrak{F}_L'/\mathfrak{F}_L, \quad F_L'=dF_L/d\rho, \quad G_L'=dG_L/d\rho.$$

[13] Such a distinction is made in the one-body problem by Siegert, reference 11.

The second of the three forms for $1/\mathfrak{F}_L$ follows from the invariance of $\mathfrak{F}_L'(G_L+iF_L)-\mathfrak{F}_L(G_L'+iF_L')=1$ as is seen for large ρ. The functions $F_L(\rho)$, $G_L(\rho)$ are the regular and irregular solutions of the wave equation

$$\frac{d^2F_L}{d\rho^2}-\frac{L(L+1)}{r^2}F_L+\frac{2\mu}{\hbar^2}EF_L=0; \quad k^2=2\mu E/\hbar^2$$

defined by their asymptotic forms $\sin(\rho-L\pi/2)$, $\cos(\rho-L\pi/2)$. The only potential barrier considered for these functions is that due to the term $-L(L+1)/r^2$ in the wave equation. The nuclear potential is supposed to be zero for $r>a$. The maximum of $\mathfrak{F}_L$ for a high potential barrier corresponds approximately to

$$1-F_LG_L\delta=0, \quad G_L'/G_L=\mathfrak{F}_L'/\mathfrak{F}_L. \tag{4.1}$$

The first and second forms of the above equation are equivalent on account of

$$F_L'G_L-F_LG_L'=1 \tag{4.2}$$

and both give for the resonance of $\sigma_{\text{scatt}}/\Lambda^2$

$$\mathfrak{F}_L=iG_L. \tag{4.3}$$

This relation means that at resonance the wave function outside the nucleus is a multiple of the irregular function G_L. For the resonance conditions given by Eqs. (4.1), (4.3) one has

$$\tan K_L=\infty, \quad \sin^2 K_L=1. \tag{4.4}$$

These conditions correspond exactly to maxima of

$$\sigma_L/\Lambda^2 \ (=\text{maximum}) \tag{4.5}$$

and for sharp resonance give also maxima of the scattering cross section. Equations (4.1), (4.3) have been used by BY and BW both in the above sense and as an approximate way of finding the resonance energy of the maximum of $|\mathfrak{F}_L|^2$ inside the nucleus. For future reference there is given below also the formula for K_L

$$\tan K_L=\frac{\mathfrak{F}_LF_L'-\mathfrak{F}_L'F_L}{\mathfrak{F}_L'G_L-\mathfrak{F}_LG_L'}=\frac{F_L^2\delta}{1-F_LG_L\delta}. \tag{4.6}$$

The value of $\mathfrak{F}_L'/\mathfrak{F}_L$ at $r=a$ is determined by the wave equation and the boundary condition at $r=0$. Equations (4.6) and (4) may be used, therefore, for the calculation of $\tan K_L$ and $\mathfrak{F}_L$. The invariance of $\mathfrak{F}_LF_L'-\mathfrak{F}_L'F_L$, $\mathfrak{F}_L'G_L-\mathfrak{F}_LG_L'$ (Wronskians) for changes in ρ together with Eq. (4.2) shows the correctness of the second form of Eq. (4) and the first form of Eq. (4.6) practically without calculation. One has from the asymptotic form for

$$\mathfrak{F}_L=[F_L\cos K_L+G_L\sin K_L]\exp(iK_L), \quad (r>a) \tag{4.7}$$

which can also be verified by means of Eqs. (4.6) and (4).

For high barriers (i.e., high values of $L(L+1)/r^2$) one has

$$G_L(ka)\gg F_L(ka). \tag{5}$$

Equation (4.7) shows then that the maximum of $|\mathfrak{F}_L(ka)|^2$ should fall approximately at $\cos K_L=0$, $\sin K_L=1$. From Eq. (4) one obtains the condition for a maximum or minimum of $|\mathfrak{F}_L|^2$

$$\frac{\partial|\mathfrak{F}_L|^{-2}}{2\partial E}=(F_LG_L\delta-1)\left[\frac{\delta\partial G_L}{F_L\partial E}+\frac{\partial F_L}{F_L^3\partial E}+\frac{G_L\partial\delta}{F_L\partial E}\right]+\delta F_L\frac{\partial(\delta F_L)}{\partial E}. \tag{5.1}$$

Resonance to scattering maximum of σ_{sc}/Λ^2 occurs for $F_LG_L\delta=1$. At the energy for which this

happens one has

$$\frac{\partial|\mathfrak{F}_L|^{-2}}{\partial E}=\frac{\partial(F_L^2\delta^2)}{\partial E}\neq 0 \tag{5.2}$$

and the maximum of the wave function inside the nucleus is not reached at the same energy as the maximum of σ_{sc}/Λ^2. An estimate of the difference in energy for the two maxima will now be made. From the second of the three forms for $1/\mathfrak{F}_L$ given in Eq. (4) one obtains

$$\frac{1}{|\mathfrak{F}_L|^2}=\left[G_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{G_L'}{G_L}\right)\right]^2+\left[F_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{F_L'}{F_L}\right)\right]^2. \tag{5.3}$$

One has the relations

$$\frac{\partial}{\partial E}\left(\frac{\partial F_L}{F_L\partial r}\right)=-\frac{k^2}{EF_L^2}\int_0^a F_L^2dr,\quad \frac{\partial}{\partial E}\left(\frac{\partial\mathfrak{F}_L}{\mathfrak{F}_L\partial r}\right)=-\frac{k^2}{E\mathfrak{F}_L^2}\int_0^a \mathfrak{F}_L^2dr \tag{5.4}$$

and the resonance width is determined by

$$\frac{E}{\Delta E}=k\int_0^a|\mathfrak{F}_L|^2dr+\left[\frac{G_L^2E\partial}{k\partial E}\frac{\partial G_L}{G_L\partial r}\right]_{r=a} \tag{5.5}$$

with ΔE determined so that at a distance ΔE from the resonance energy $|\mathfrak{F}_L|^2$ has one-half its maximum value. In the derivation of Eq. (5.5) it has been assumed that the resonance is so sharp that the maximum of $|\mathfrak{F}_L|^2$ occurs at the same energy as the maximum of σ_{sc}/Λ^2. It was found by means of Eqs. (5.4) and

$$-\Delta E\frac{\partial}{\partial E}\left[G_L^2\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{G_L'}{G_L}\right)\right]=1, \tag{5.5'}$$

which is suggested for sharp resonance by the last of the three expressions for $\mathfrak{F}_L$ in Eq. (4). At $E=E_0\pm\Delta E$ the scattering cross section is also approximately $\frac{1}{2}$ of its maximum value as is seen from the fact that (5.5') makes $\tan K_L=1$. For large $|G_L/F_L|$ the scattering cross section and $|\mathfrak{F}_L(ka)|$ are nearly proportional to each other. This is seen from Eq. (4.7) which gives

$$|\mathfrak{F}_L|^2=[1+(F_L/G_L)\cot K_L]^2G_L^2\sin^2 K_L. \tag{5.6}$$

The condition which must be satisfied for the proportionality of the scattering cross section in units Λ^2 and the probability of finding the particle in the nucleus is

$$|\mathfrak{F}_L|^2/\sin^2 K_L=G_L^2[1+(F_L/G_L)\cot K_L]^2\doteqdot\text{const.} \tag{5.7}$$

Close to the resonance peak one may neglect $(F_L/G_L)\cot K_L$. The ratio is then G_L^2 which varies with the energy. For high barriers G_L^2 is likely to decrease with the energy. The scattering cross section is relatively higher, therefore, on the higher energy side of the peak around its main part. There is thus no exact correspondence between the maximum chance of the bombarding particle penetrating the nucleus and the scattering cross section and the correspondence becomes poor when

$$|\tan K_L|\sim|F_L/G_L|$$

on account of the first factor in Eq. (5.6). By means of Eqs. (5.4) one obtains

$$\frac{\partial}{\partial E}\left[G_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{G_L'}{G_L}\right)\right]=-\frac{kG_L}{E_0\mathfrak{F}_L^2}\int\mathfrak{F}_L^2dr-\frac{G_L\partial}{k\partial E}\frac{\partial G_L}{G_L\partial r},$$

$$\frac{\partial}{\partial E}\left[F_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{F_L'}{F_L}\right)\right]=-\frac{kF_L}{E_0\mathfrak{F}_L^2}\int\mathfrak{F}_L^2dr-\frac{F_L\partial}{k\partial E}\frac{\partial G_L}{G_L\partial r}-\frac{\partial(1/G_L)}{\partial E},$$

$$F_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{F_L'}{F_L}\right)=-\frac{1}{G_L},\quad (E=E_0).$$

In these formulas E_0 is the energy for which the scattering resonance condition (4.1) is satisfied. The derivatives $\partial/\partial E$ are supposed to be evaluated for $E=E_0$. The same convention will be followed in the following approximate calculations of the energy shift. In the vicinity of $E=E_0$ one has according to Eq. (5.3)

$$\frac{1}{|\mathfrak{F}_L|_E^2}=(E-E_0)^2\left\{\frac{\partial}{\partial E}\left[G_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{G_L'}{G_L}\right)\right]\right\}^2+\frac{1}{G_L^2}-2\frac{E-E_0}{G_L}\frac{\partial}{\partial E}\left[F_L\left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L}-\frac{F_L'}{F_L}\right)\right].$$

The second derivative of $[F_L(\mathfrak{F}_L'/\mathfrak{F}_L-F_L'/F_L)]^2$ is neglected. Minimizing the above expression, one obtains the condition

$$E-E_0=-\frac{E_0F_L}{G_L^3}\frac{\frac{1}{\mathfrak{F}_L^2}\int\mathfrak{F}_L^2d\rho+\frac{E_0\partial}{k\partial E}\frac{\partial G_L}{G_L\partial r}+\frac{E_0\partial(1/G_L)}{F_L\partial E}}{\left[\frac{1}{\mathfrak{F}_L^2}\int\mathfrak{F}_L^2d\rho+\frac{E_0\partial}{k\partial E}\frac{\partial G_L}{G_L\partial r}\right]^2}. \tag{5.8}$$

Similarly the corresponding approximate condition for the minimum of $|G_L/\mathfrak{F}_L|^2$ is

$$E'-E_0=\frac{-F_LE_0}{G_L^3\left[\frac{1}{\mathfrak{F}_L^2}\int\mathfrak{F}_L^2d\rho+\frac{E_0\partial}{k\partial E}\frac{\partial G_L}{G_L\partial r}\right]}=-\frac{F_L}{G_L}\Delta E. \tag{5.9}$$

Here ΔE is introduced by means of Eq. (5.5). For high barriers $|F_L/G_L|\ll 1$ and $|E'-E_0|\ll\Delta E$. The *shift* of the peak in such a case is *small in comparison with the resonance width* and the barrier penetration factors enter in the same power in $(E'-E_0)/\Delta E$ as in $\Delta E/E$. According to Eq. (5.8) $E-E_0$ is of the same order of magnitude as $E'-E_0$. The term $(E_0/F_L)\partial(1/G_L)/\partial E$ contains the penetration factors in the same power (zero) as the other two terms in the numerator of Eq. (5.8) and there is no special general distinction between them. It is thus seen that the energy shifts indicated by Eq. (3.9) are the less important the smaller $|F_L/G_L|$.

This conclusion can also be seen from Eq. (5.6). For $E=E_0+\Delta E$ the value of G_L^2 is

$$\sim(G_L^2)_{E_0}+2G_L(\partial G_L/\partial E)E\Big/\int G_L^2d\rho\sim G_L^2+2/\rho.$$

The latter estimate is very rough. The point is that the addition to G_L^2 is of zero degree in G_L^2. The fractional change of the factor G_L^2 at $E=E_0\pm\Delta E$ is roughly $1/G_L^2$ and becomes negligible for high barriers.

The essential point of the approximation to resonance by the method of complex eigenvalues and the method of regular and irregular functions is the same. Resonance occurs nearly at the energy for which $|1/\mathfrak{F}_L|^2$ has a minimum. At the minimum this quantity can be approximated by a parabola and in this approximation formulas of the resonance type are obtained for cross sections.

One-body problem with strong repulsion

If the nucleus is surrounded by a strong repulsive field acting in addition to the centrifugal barrier[14] then it is more natural to modify the above procedure and to introduce wave functions in the repulsive field. The repulsive field has to be continued to $r=0$ and there is some arbitrariness involved in this step. The procedure is illustrated in Fig. 1. The heavy line is the potential energy curve. The light line is the continuation of the

[14] The repulsive field considered here is not the Coulomb field. The calculations are easily modified to include this case also. The notation used here is convenient for applications to many dimensional problems.

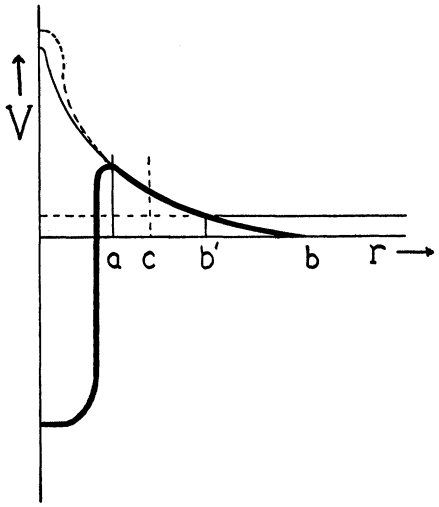

FIG. 1. Potential well and potential barrier. Heavy line represents the actual potential. The light and dotted lines represent auxiliary repulsive potentials. The height of the straight horizontal line is the energy.

repulsive potential to $r=0$. At $r=a$ the two curves part. The region from $r=a$ to $r=b$ is the potential energy barrier and the region $a<r<b'$ is the region of negative kinetic energy. The auxiliary repulsive potential could be drawn in also as the dotted curve and in an infinite number of other ways.

In the auxiliary repulsive potential one obtains a regular solution f_L for r times the radial function. For $r>b$ this function is a linear combination of corresponding functions without any potential. One has

$$f_L=F_L\cos\delta_L+G_L\sin\delta_L,$$
$$g_L=-F_L\sin\delta_L+G_L\cos\delta_L. \tag{6}$$

Here δ_L is the phase shift of f_L with respect to F_L and the function g_L is defined so as to be asymptotically the cos of the same argument of which f_L is the sine. For $r<b$ the function g_L is defined by the radial equation and continuity of the function and its derivative at $r=b$.

From Eq. (6) one obtains

$$f_L'g_L-f_Lg_L'=F_L'G_L-F_LG_L'=1. \tag{6.1}$$

The phase shift of $\mathfrak{F}_L$ with respect to f_L will be denoted by κ_L and one has

$$K_L=\kappa_L+\delta_L. \tag{6.2}$$

For $r>b$ one has according to Eq. (4.7) and Eq. (6)

$$\mathfrak{F}_L=(f_L\cos\kappa_L+g_L\sin\kappa_L)\exp\{i(\kappa_L+\delta_L)\}. \tag{6.3}$$

But from $r=a$ to $r=b$ the actual potential and the auxiliary repulsive potential are the same. Therefore Eq. (6.3) holds also for all $r>a$ and can be applied at $r=a$ where the attractive region begins.

The usefulness of this model is seen in cases in which the region $r<a$ can be pictured as an approximation to the interior of the nucleus, somewhat as in the case of the Coulomb barrier. In such a case one is interested in $|\mathfrak{F}_L|^2$ in this region and one can obtain an idea of these by studying $|\mathfrak{F}_L|^2$ at $r=a$ because in $0<r<a$ the shape of $\mathfrak{F}_L$ does not depend critically on the energy. At $r=a$ one has, on the other hand, $|g_L/f_L|\gg1$ for high barriers and one obtains, therefore, higher reaction yields for approximately those energies for which $\sin\kappa_L=\pm1$. The scattering cross section depends, on the other hand, on $\sin^2K_L$ and one has thus an energy shift between the peaks of the scattering cross section curve and the curve for $|\mathfrak{F}_L|^2$ which is due to δ_L. This shift is of the order of magnitude of the width of the resonance curve multiplied by δ_L/π because the half-value width of the resonance curve corresponds to a change in K_L by $\pi/2$.

It is natural to expect a shift between the energy of maximum penetration and the energy of maximum scattering. Suppose, for instance, that the maximum of $|\mathfrak{F}_L(kc)|^2$ occurs at the same energy (Fig. 1) as the maximum of $\sin^2K_L$. The maximum of $|\mathfrak{F}_L(ka)|^2$ will not occur at the same energy because the barrier between c and a affects the wave function. The ratio $|\mathfrak{F}_L(ka)/\mathfrak{F}_L(kc)|^2$ varies by large amounts in the vicinity of resonance, provided a and c are sufficiently far apart and an energy shift results.

Since the repulsive potential can be continued to the left of a in an infinite number of ways the method of the auxiliary repulsive potential is not unique. Even though the functions f_L, g_L are natural ones to use they are only auxiliary quantities, while $\mathfrak{F}_L$ is the actual wave function.

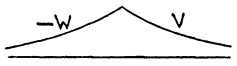

FIG. 2. Illustrating the cusp at the intersection of $-w$ and v.

One has

$$\tan \kappa_L = \frac{\mathfrak{F}_L f_L' - \mathfrak{F}_L' f_L}{\mathfrak{F}_L' g_L - \mathfrak{F}_L g_L'} = \frac{f_L^2 \delta'}{1 - f_L g_L \delta'} \left(\delta' = \frac{f_L'}{f_L} - \frac{\mathfrak{F}_L'}{\mathfrak{F}_L} \right), \tag{6.41}$$

$$\frac{1}{|\mathfrak{F}_L|^2} = \left[g_L \left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L} - \frac{g_L'}{g_L} \right) \right]^2 + \left[f_L \left(\frac{\mathfrak{F}_L'}{\mathfrak{F}_L} - \frac{f_L'}{f_L} \right) \right]^2 \tag{6.42}$$

analogously to Eqs. (4.6), (5.3). For the width of resonance $(\Delta E)'$ as measured by $|\mathfrak{F}_L(ka)|^2$ one has

$$\frac{E}{(\Delta E)'} = k \int_0^a |\mathfrak{F}_L|^2 dr + \left[\frac{g_L^2 E \partial}{k \partial E} \frac{\partial g_L}{g_L \partial r} \right]_{r=a} \approx \int_0^b |\mathfrak{F}_L|^2 d\rho; \quad [\text{Maximum of } |\mathfrak{F}_L(ka)|^2]. \tag{6.43}$$

This formula is analogous to Eq. (5.5). The function $\mathfrak{F}_L$ and the other functions occurring in it are evaluated for the energy which gives maximum $|\mathfrak{F}_L|^2$ inside the repulsive barrier, however, and this energy is not the same as the energy for which $\sin^2 K_L$ is a maximum. Equation (6.43) should not be applied, therefore, to an estimate of the width of the scattering resonance in the general case.

A superficial comparison of Eq. (6.43) with Eq. (5.5) appears to indicate that $(\Delta E)' = \Delta E$ because the right-hand sides do not depend on the choice of the upper limit and Eq. (6.43) could equally well be written

$$\frac{E}{(\Delta E)'} = k \int_0^b |\mathfrak{F}_L|^2 dr + \left[\frac{g_L^2 E \partial}{k \partial E} \frac{\partial g_L}{g_L \partial r} \right]_{r=b}. \tag{6.43'}$$

The integral in this formula has the same form as in the corresponding modification of (5.5) *viz.*

$$\frac{E}{\Delta E} = k \int_0^b |\mathfrak{F}_L|^2 dr + \left[\frac{G_L^2 E \partial}{k \partial E} \frac{\partial G_L}{G_L \partial r} \right]_{r=b}, \tag{5.5'}$$

suggesting that $\Delta E = (\Delta E)'$. This is not the case because the energy which must be used in Eq. (6.43') is for resonance at a while in Eq. (5.5') the energy is for resonance at b.

Two-Dimensional Problems

Green's function for separable two-dimensional differential equations

Consider the differential equation

$$\mathfrak{L}\psi(x, y) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa^2 - U(x) - V(y) \right] \psi(x, y) = 0 \tag{7}$$

in the domain

$$\begin{aligned} -a < x < a, \\ -\infty < y < +\infty \end{aligned} \tag{7.1}$$

with boundary conditions

$$\psi(a, y) = \psi(-a, y) = 0. \tag{7.2}$$

It will be required also that for $y = \pm\infty$ and fixed x the wave function should either vanish exponentially or else be asymptotic to a sum of a finite number of *sine* functions of multiples of y with a number of fixed but arbitrary phase constants. In order that the latter boundary condition be a possible one, it will be supposed that $V(y)$ approaches a constant value faster than $1/y$. This constant value can be made $= 0$ without loss of generality by changing κ^2 so as to absorb the difference and this change will be supposed to be made.

The differential equation

$$[d^2/dx^2 + \lambda_n - U(x)] u_n(x) = 0 \tag{7.3}$$

with the boundary conditions (7.2) defines in $-a<x<a$ an orthonormal set with eigenvalues λ_n. The function $U(x)$ will be supposed to have either no singularities or if it has singularities these will be supposed to be sufficiently innocent so as to allow the existence of the set $\lambda_1<\lambda_2<\lambda_3\cdots$ with a minimum $\lambda_1\neq-\infty$. A particular solution of Eq. (7) is obtained by letting

$$\psi(x, y)=u_n(x)\,Y_k(y) \tag{7.4}$$

with

$$\kappa^2=k^2+\lambda_n \tag{7.5}$$

and with

$$[d^2/dy^2+k^2-V(y)]Y_k(y)=0. \tag{7.6}$$

The dependence of k on n is not explicitly indicated here so as not to complicate the notation. There are two linearly independent solutions of Eq. (7.6). These will be

$$v_k(y),\quad w_k(y),$$

respectively. The solution $v_k(y)$ will be chosen so as to be acceptable at $y=+\infty$ and $w_k(y)$ so as to be acceptable at $y=-\infty$. Thus if at $y=\pm\infty$ the solutions of Eq. (7.6) are of the exponential type, $v_k(y)$ will be taken as the exponentially decreasing branch and similarly $w_k(y)$ will be arranged to be exponentially increasing with increasing y. This condition is schematically indicated in Fig. 2. If, on the other hand, $k^2>0$ then one may use any pair of linearly independent solutions of (7.6) for v and w. These solutions will be restricted later so as to give either diverging or standing waves. In all cases the functions $v_k(y)$, $w_k(y)$ will be normalized so as to have

$$\begin{aligned} &v_k'(y)w_k(y)-v_k(y)w_k'(y)=1,\\ &v_k'(y)=dv_k(y)/dy. \end{aligned} \tag{7.7}$$

If Eq. (7.7) holds for one y then it holds also for all y as a consequence of Eq. (7.6) according to the theorem regarding the invariance of the Wronskian. A kernel will now be defined as follows

$$\begin{aligned} K(x, y; \xi, \eta)&=\sum v_k(y)w_k(\eta)u_n(x)u_n(\xi)\quad (y>\eta),\\ K(x, y; \xi, \eta)&=\sum v_k(\eta)w_k(y)u_n(x)u_n(\xi)\quad (y<\eta), \end{aligned} \tag{8}$$

where it is understood that k varies with n in accordance with Eq. (7.6). It will now be shown that

$$\mathfrak{L}\int\int K(x, y; \xi, \eta)\rho(\xi, \eta)d\xi d\eta=\rho(x, y). \tag{8.1}$$

This equation will be first verified for the special form

$$\rho(\xi, \eta)=u_n(\xi)f(\eta). \tag{8.2}$$

One has

$$\mathfrak{L}\int\int K(x, y; \xi, \eta)u_n(\xi)f(\eta)d\xi d\eta=\mathfrak{L}u_n(x)\int_{-\infty}^{+\infty}Y_k(y, \eta)f(\eta)d\eta=u_n(x)M\int_{-\infty}^{+\infty}Y_k(y, \eta)f(\eta)d\eta,$$

where

$$M=d^2/dy^2+\kappa^2-\lambda_n-V(y) \tag{8.3}$$

and

$$\begin{aligned} Y_k(y, \eta)&=v_k(y)w_k(\eta)\quad (y>\eta),\\ Y_k(y, \eta)&=v_k(\eta)w_k(y)\quad (y<\eta). \end{aligned} \tag{8.4}$$

It remains to show that

$$M\int_{-\infty}^{+\infty}Y_k(y, \eta)f(\eta)d\eta=f(y). \tag{8.5}$$

This is the case according to the construction of Green's function for one-dimensional problems as explained in Courant-Hilbert.[15] The special case in which $v_k(y), w_k(y)$ are the same is excluded for the present. The verification of Eq. (8.5) is indicated next for the sake of completeness.

$$\frac{d^2}{dy^2}\int_{-\infty}^{+\infty} Y_k(y,\eta)f(\eta)d\eta=\frac{d^2}{dy^2}\int_{-\infty}^{y} v_k(y)w_k(\eta)f(\eta)d\eta+\frac{d^2}{dy^2}\int_{y}^{\infty} v_k(\eta)w_k(y)f(\eta)d\eta$$

$$=\frac{d}{dy}\int_{-\infty}^{y} v_k'(y)w_k(\eta)f(\eta)d\eta+\frac{d}{dy}\int_{y}^{\infty} v_k(\eta)w_k'(y)f(\eta)d\eta=[v_k'(y)w_k(y)-v_k(y)w_k'(y)]f(y)$$

$$+\int_{-\infty}^{y} v_k''(y)w_k(\eta)f(\eta)d\eta+\int_{y}^{\infty} v_k(\eta)w_k''(y)f(\eta)d\eta.$$

Substituting this into Eq. (8.5), taking into account (7.7) and $Mv_k(y)=Mw_k(y)=0$ one verifies Eq. (8.5). In order that the calculation in Eq. (8.6) be valid it is necessary that the integrals occurring in it converge. In applications $\rho(x, y)$ will vanish except in a finite range of values of y so that no difficulty will be met in this connection. Having verified Eq. (8.1) for the special form (8.2) one sees also that Eq. (8.1) holds for

$$\rho(x, y)=\sum u_n(x)f_n(y). \tag{8.7}$$

If the sum is finite the above verification applies directly. If the sum in Eq. (8.7) is an infinite series an investigation of convergence is required. This has not been carried out in the general case. No difficulty is expected in this connection in view of the fact that in special cases Eq. (8.1) can be verified by other methods.

According to Eq. (8) two different expressions for K are used depending on whether $y>\eta$ or $y<\eta$. These approach the same value $\sum v_k(y)w_k(y)u_n(x)u_n(\xi)$ as the line $y=\eta$ is approached in the two regions and $K(x, y; \xi, \eta)$ is continuous on that line. The derivative normal to the line appears to be discontinuous being given by

$$[\partial K(x, y; \xi, \eta)/\partial y]_{y=\eta+0}-[\partial K(x, y; \xi, \eta)/\partial y]_{y=\eta-0}=\sum [v_k'(\eta)w_k(\eta)-v_k(\eta)w_k'(\eta)]u_n(x)u_n(\xi). \tag{8.8}$$

On account of Eq. (7.7) the right side of the above formula becomes

$$\sum u_n(x)u_n(\xi)=\delta(x-\xi).$$

The discontinuity in the derivative occurs, therefore, only at the point

$$(x, y)=(\xi, \eta).$$

If one wishes to work with complex eigenfunctions the product $u_n(x)u_n(\xi)$ in Eq. (8) can be replaced by

$$u_n(x)u_n^*(\xi)$$

and the above argument remains essentially unchanged. It should be noted that Eq. (8.1) has been verified only for such forms of ρ which can be expanded as in Eq. (8.7). This is a restriction which must be watched especially at the boundaries of x since the boundary conditions for ψ have no direct significance for ρ.

General plan for applications of Green's function

In proper units one can apply the wave equation (7) to the discussion of a quantum-mechanical system consisting of two particles with the following interpretation of quantities: κ^2 = total energy, λ_n = energy of x particle in state n, k^2 = energy of y particle, and $u_n(x)$ = wave function of x particle in state n.

[15] R. Courant and D. Hilbert, *Methoden der mathematischen Physik* (Springer, Berlin, 2nd edition, 1931), Vol. I, p. 302.

In Fig. 3 a schematic picture of the two-dimensional condition is sketched. The wave function is confined to the strip of width $2a$ on the boundaries of which it is supposed to vanish. The wave equation (7) holds outside the shaded region A. Inside this region another and in general a nonseparable wave equation is supposed to hold. The potential energy function $V(y)$ is schematically represented as a barrier sticking out of the plane of the paper. In order not to complicate the figure $U(x)$ was taken as zero and $V(y)$ was made to vanish in the shaded region. Neither of these simplifications matters in the applicability of the method.

Outside the shaded region the wave function will be represented as

$$\psi(x, y) = \int\int K(x, y; \xi, \eta) \times \rho(\xi, \eta) d\xi d\eta + \psi_0(x, y) \quad (9)$$

and it will be supposed that

$$\rho(x, y) = 0 \text{ (outside } A); \quad \mathfrak{L}\psi_0 = 0. \quad (9.1)$$

In consequence of Eq. (8.1) one has

$$\mathfrak{L}\psi = 0, \quad \text{(outside } A). \quad (9.2)$$

The function ψ_0 will be used to represent the incident wave and $\psi - \psi_0$ will give its modification due to the interaction potential in the region A. The reflection of the waves is thus pictured as due to a source density ρ placed within the interaction region. In the analogous electrostatic problem the source density is replaced by a charge density. Since for $k^2 > 0$ the functions v_k, w_k are not uniquely determined by the boundary conditions and by Eq. (7.7) one can arrange for $\psi - \psi_0$ to have the desired asymptotic forms (outgoing, incoming, standing waves) at $y = \pm\infty$.

Some qualitative information can be obtained from the form of Eq. (9) without special assumptions concerning the location of $\rho(x, y)$. It should be pointed out, however, that in a number of cases it is possible to condense $\rho(x, y)$ to a very narrow strip on the boundary of A. This is a well-known procedure in electrostatics and it corresponds to representing the field due to a collection of charges by an equivalent surface charge density over a surface enclosing the charges. The possibility of doing so depends on the existence of solutions of elliptic differential equations in closed regions with prescribed boundary values and may be seen as follows. A will be bordered by a thin strip of uniform thickness just outside of it and shown shaded in Fig. 4. The original problem will be modified by requiring that the wave equation (7) holds everywhere except in the shaded area of the strip. The values of ψ on the boundary of A (inner boundary of strip) will be taken to be those given by Eq. (9). These values determine a new ψ inside A which will be called φ. φ is a solution of Eq. (7) and is continuous with ψ just outside the shaded border when the border is thin. The normal derivatives $\partial\psi/\partial n$, $\partial\varphi/\partial n$ are not continuous in going across the border as in electrostatics. Leaving the thickness of the strip finite but small one can join φ to ψ by smooth curves across the strip and define a function

$$\begin{aligned} &\Psi = \psi \text{ outside } A, \\ &\Psi = \varphi \text{ inside } A, \\ &\Psi = \text{joining function in strip.} \end{aligned} \quad (9.3)$$

Now form

$$\sigma(x, y) = \mathfrak{L}\Psi. \quad (9.4)$$

From the definition of Ψ it follows that $\sigma(x, y)$ has a value different from zero only in the shaded

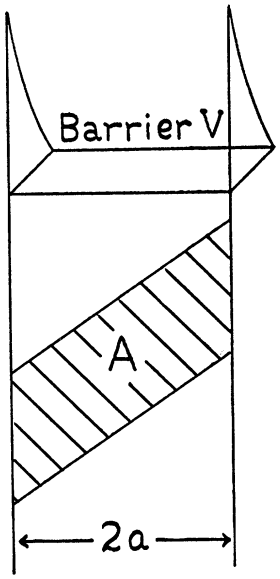

FIG. 3. Illustrating the two-dimensional strip model. The wave function vanishes within the vertical strip of width $2a$. The interaction between the particles takes place within area A.

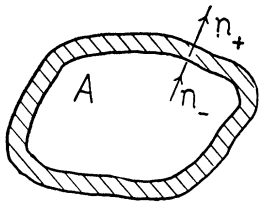

FIG. 4. Illustrating the concentration of source density to the periphery of region in which the interaction takes place.

strip. Consider

$$\chi(x, y)=\psi_0+\int\int K(x, y; \xi, \eta)\sigma(\xi, \eta)d\xi d\eta. \quad (9.5)$$

This function satisfies

$$\begin{aligned} &\mathfrak{L}\chi=0 \text{ outside strip,} \\ &\mathfrak{L}\chi=\sigma \text{ inside strip.} \end{aligned} \quad (9.6)$$

Comparing (9.6) with (9.4) one sees that χ and Ψ satisfy the same inhomogeneous equation. Also ψ_0 and K determine the asymptotic behavior of χ and ψ at $y=\pm\infty$ and one can see that this behavior is the same for ψ and χ. In fact the choice of $v_k(y)$ in (8.4) determines completely whether for $y=\pm\infty$ the nth state of the x particle is excited together with y particle moving towards $y=+\infty$ or $y=-\infty$. It is usually assumed that the specification of boundary conditions in the above sense determines the wave function. Adopting this view, it follows that

$$\begin{aligned} &\chi=\Psi, \\ &\chi=\psi \text{ outside strip.} \end{aligned} \quad (9.7)$$

According to the above one expects to be able to replace a surface distribution of sources by a distribution of sources on a contour in agreement with physical expectation. One has for an infinitely narrow strip

$$\psi=\psi_0+\int K(x, y; \xi, \eta)\sigma(\xi, \eta)ds, \quad ds^2=d\xi^2+d\eta^2. \quad (9.8)$$

The integration is carried out along the outer boundary of A and σ in (9.8) is the previous σ integrated through the strip thickness.[16]

The value of ψ on the boundary of A and the differential equation inside A determine ψ inside A and hence also $\partial\psi/\partial n$ at the boundary. The connection between the values of ψ and those of $\partial\psi/\partial n$ on the boundary is linear, therefore, and the problem reduces itself to the solution of linear equations with the density function σ of Eq. (9.8) on the one-dimensional boundary of the region A.

This connection can be shown more explicitly by introducing Green's function for the interior of A for the differential equation obeyed by ψ inside this region. In terms of it one can write

$$\psi(x, y)=\int \bar{K}(x, y; \xi, \eta)\bar{\sigma}(\xi, \eta)ds. \quad (10)$$

The continuity of ψ and its normal derivative on the boundary gives

$$\psi_0(x, y)+\int K(x, y; \xi, \eta)\sigma(\xi, \eta)ds=\int \bar{K}(x, y; \xi, \eta)\bar{\sigma}(\xi, \eta)ds, \quad (10.1)$$

$$\frac{\partial\psi_0(x, y)}{\partial n_+}+\int \frac{\partial K(x, y; \xi, \eta)}{\partial n_+}\sigma(\xi, \eta)ds=\int \frac{\partial\bar{K}(x, y; \xi, \eta)}{\partial n_-}\bar{\sigma}(\xi, \eta)ds, \quad (10.2)$$

the outward drawn normal being denoted by n_+ just outside of the boundary and the inward drawn normal by n_- just inside, as in Fig. 4. The above equations involve values of σ and $\bar{\sigma}$ over the contour. To emphasize this write s for (x, y), s' for (ξ, η) and

$$\begin{aligned} a(s, s')&=K(x, y; \xi, \eta), & \bar{a}(s, s')&=\bar{K}(x, y; \xi, \eta), \\ b(s, s')&=\partial K(x, y; \xi, \eta)/\partial n_+, & \bar{b}(s, s')&=\partial\bar{K}(x, y; \xi, \eta)/\partial n_-, \\ f_0(s)&=\psi_0(x, y), & g_0(s)&=\partial\psi_0(x, y)/\partial n_+. \end{aligned}$$

One has then

$$f_0(s)+\int a(s, s')\sigma(s')ds'=\int \bar{a}(s, s')\bar{\sigma}(s')ds', \quad (10.3)$$

$$g_0(s)+\int b(s, s')\sigma(s')ds'=\int \bar{b}(s, s')\bar{\sigma}(s')ds'. \quad (10.4)$$

[16] The possibility of singularities in σ is not excluded.

The construction of K and $\bar{K}$ thus reduces the problem to the solution of two simultaneous integral equations in a one-dimensional domain in two unknowns σ, $\bar{\sigma}$.

The linear connection between ψ and $\partial\psi/\partial n_-$ on the boundary is expressed in the above by

$$\psi=\int \bar{a}(s, s')\bar{\sigma}(s')ds', \quad \partial\psi/\partial n_-=\int \bar{b}(s, s')\bar{\sigma}(s')ds'.$$

It can also be expressed in other ways. Thus, for example, for the interior of a circle, inside of which the potential is axially symmetric, it is convenient to use polar coordinates and a Fourier expansion according to the azimuthal angle. The values of ψ on the boundary determine then the values of the radial functions by integrals involving $\psi(s)$ linearly and $\partial\psi/\partial n_-$ is a sum of such integrals. The formulas are

$$\psi(s)=\sum (e^{in\theta}/2\pi)\int_0^{2\pi}\psi(a\varphi)e^{-in\varphi}d\varphi; \quad ds=ad\theta$$

$$\partial\psi/\partial n_-=\sum [R_n'(a)/R_n(a)](e^{in\theta}/2\pi)\int_0^{2\pi}\psi(a\varphi)e^{-in\varphi}d\varphi.$$

Substituting into the last of these equations by means of $\psi(s)=f_0(s)+\int a(s, s')\sigma(s')ds'$ and $\partial\psi/\partial n_-=g_0(s)+\int b(s, s')\sigma(s')ds'$ one has a single integral equation in $\sigma(s)$. The radial function belonging to $e^{in\theta}$ is denoted here by R_n.

Degeneracy due to excitation

The spectrum of eigenvalues of Eq. (7.6) may be entirely a continuum. It is then always possible to find both v_k and w_k for every κ and n. If the spectrum of (7.6) has also a discrete part then it may happen that for certain values of κ and $n=\bar{n}$ it is not possible to find both v_{k^*} and w_{k^*}. This happens whenever $\bar{k}^2$ is an eigenvalue of Eq. (7.6) and in this case the energy of the two-particle system in the incident state ψ_0 is just equal to the sum of the energies of the x particle in the state $\bar{n}$ and the y particle in the state $\bar{k}$. The same function $v_{k^*}(y)$ is (to within a constant multiple) the only acceptable one both for $y=+\infty$ and $y=-\infty$, and it is impossible to construct a satisfactory $Y_{k^*}(y, \eta)$ by means of Eq. (8.4) on account of the failure of Eq. (7.7). This degeneracy of the kernel can be avoided by a slight shift of the total energy and no difficulty arises as long as the energy balance for simultaneous capture of the two particles is satisfied approximately rather than exactly. As this condition of resonance is approached the product of the normalizing factors in v_{k^*}, w_{k^*} must be increased in order to satisfy $v_{k^*}'w_{k^*}-v_{k^*}w_{k^*}'=1$ and $Y_{k^*}(y, \eta)\to\infty$. This does not mean, however, that one obtains an infinite probability of the formation of the state because $a(s, s')$ in Eq. (10.3) also becomes large.

For $k\approx\bar{k}$ both v_k and w_k are to within a constant factor $\approx v_{k^*}$ and it will be shown next that

$$Y_k(y, \eta)\sim v_{k^*}(y)v_{k^*}(\eta)/(k^2-\bar{k}^2).$$

This relation can be obtained as follows. From the differential equations satisfied by v_k, w_k, v_{k^*} one obtains

$$[v_{k^*}(y)v_k'(y)-v_k(y)v_{k^*}'(y)]_a^b+(k^2-\bar{k}^2)\int_a^b v_k(y)v_{k^*}(y)dy=0. \tag{11}$$

The asymptotic forms of the logarithmic derivatives of the functions are

$$v_k'(+\infty)/v_k(+\infty)=-(-k^2)^{\frac{1}{2}}; \quad w_k'(-\infty)/w_k(-\infty)=(-k^2)^{\frac{1}{2}},$$

$$v_{k^*}'(+\infty)/v_{k^*}(+\infty)=-(-\bar{k}^2)^{\frac{1}{2}}; \quad v_{k^*}'(-\infty)/v_{k^*}(-\infty)=(-\bar{k}^2)^{\frac{1}{2}},$$

and the functions vanish exponentially as follows

$$v_k(+\infty)=w_k(-\infty)=v_{k^*}(+\infty)=v_{k^*}(-\infty)=0.$$

From Eq. (11) one obtains on setting $a=y$, $b=+\infty$ a formula for $v_k'(y)/v_k(y)-v_{k^*}'(y)/v_{k^*}(y)$. Similarly forming an expression analogous to (11) with $w_k(y)$, $v_{k^*}(y)$ one obtains a formula for $w_k'(y)/w_k(y)-v_{k^*}'(y)/v_{k^*}(y)$. Subtracting the second formula from the first, one obtains

$$v_k'(y)w_k(y)-v_k(y)w_k'(y)=(k^2-\bar{k}^2)\left[\frac{w_k(y)}{v_{k^*}(y)}\int_y^\infty v_k(\eta)v_{k^*}(\eta)d\eta+\frac{v_k(y)}{v_{k^*}(y)}\int_{-\infty}^y w_k(\eta)v_{k^*}(\eta)d\eta\right].$$

In the limit of $k=\bar{k}$ the right side of this equation can be made equal to unity if one sets

$$v_k\approx w_k\approx v_{k^*}/(k^2-\bar{k}^2)^{\frac{1}{2}} \tag{11.1}$$

and normalizes

$$\int_{-\infty}^{+\infty} v_{k^*}^2(y)dy=1. \tag{11.2}$$

The two equations (11.1) can never be satisfied for all y. For fixed y, however, one can make $|k-\bar{k}|$ sufficiently small so that the shape of $v_k(\eta)$, $w_k(\eta)$ is practically the same as that of $v_{k^*}(\eta)$ for $|\eta|<|y|$. For fixed y and sufficiently small $|k-\bar{k}|$ one has then the limiting form

$$Y_k(y,\eta)\approx v_k(y)v_{k^*}(\eta)/(k^2-\bar{k}^2). \tag{11.3}$$

A special problem will now be considered. It will be required that the wave function be zero over the area of a small circle surrounding the point (ξ,η). The wave equation (7) will be supposed to hold outside the small circle. The interaction between the two particles is thus entirely inside the circle where it is infinitely great. This example will be considered in more detail later but at present, by means of Eq. (11.3) one can see the effect of resonance due to excitation. Close to resonance the values of $K(x,y;\xi,\eta)$ over the circle are[17]

$$K_C(x,y;\xi,\eta)\approx u_{n^*}^2(\xi)v_{k^*}^2(\eta)/(k^2-\bar{k}^2)+\sum c'. \tag{12}$$

Here $\sum c'$ represents the effect of other states in K and suffix C means: on circle. Denoting the incident wave by $\psi_0(x,y)$ one has the approximate solution

$$\psi(x,y)=\psi_0(x,y)-\psi_0(\xi,\eta)\frac{u_{n^*}(x)u_{n^*}(\xi)v_k(y)v_k(\eta)/(k^2-\bar{k}^2)+\sum' u_n(x)u_n(\xi)\,Y_k(y,\eta)}{u_{n^*}^2(\xi)v_{k^*}^2(\eta)/(k^2-\bar{k}^2)+\sum c'\,u_n(x)u_n(\xi)\,Y_k(y,\eta)}. \tag{12.1}$$

This formula is approximate only. Its main inaccuracy lies in the assumption that $\sum c'$ in the denominator has the same value for all points on the circle. This is the case to a first approximation, the logarithmic term in $(x-\xi)^2+(y-\eta)^2$ being predominant in $\sum c'$ for small circles. The effect of a single term in the summation in Eq. (12.1) may include in it a wave extending to $y=\infty$. The above formula shows that for $k\sim\bar{k}$ these waves become small. An individual term behaves as

$$-\frac{\psi_0(\xi,\eta)}{\sum c'}\left[1-\frac{u_{n^*}^2(\xi)v_{k^*}^2(\eta)/\sum c'}{k^2-\bar{k}^2+u_{n^*}^2(\xi)v_{k^*}^2(\eta)/\sum c'}\right]Y_k(y,\eta)u_n(x)u_n(\xi) \tag{12.2}$$

and shows a maximum for $k^2=\bar{k}^2-u_{n^*}^2(\xi)v_{k^*}^2(\eta)/\sum c'$. The latter equation cannot be satisfied accurately because $\sum c'$ is only approximately real. The resonance has, therefore, a finite width. At the maximum the above calculation breaks down because then the parts of $u_{n^*}(x)u_{n^*}(\xi)\,Y_k(y,\eta)$ which have been neglected become important. For $k=\bar{k}$, however, the above scattered wave is zero. This, of course, is also only an approximation. For $k=\bar{k}$ one has from (12.1)

$$\psi(x,y)=\psi_0(x,y)-\psi_0(\xi,\eta)u_{n^*}(x)v_{k^*}(y)/[u_{n^*}(\xi)v_{k^*}(\eta)]. \tag{12.3}$$

[17] It is natural that u_{n^*} and v_{k^*} occur symmetrically in this case.

This expression is an exact solution of the wave equation and it vanishes at (ξ, η). It is, therefore, also small on the little circle. It is understandable that the scattering due to the interaction within the circle is small at this energy.[18] The incident wave is in a sense "sucked in" to produce excitation.

Extension to several dimensions

An extension of Green's function given by Eq. (8) may be made by generalizing the variable x to a set of variables: $x_1, x_2, \cdots x_s$. The functions $u_n(x_1, x_2, \cdots, x_s)$ are supposed to satisfy

$$\left[\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_s^2}+\lambda_n-U(x_1, \cdots, x_s)\right]u_n(x_1, \cdots, x_s)=0. \tag{13}$$

The λ_n and u_n are supposed to form a complete and discrete set of eigenvalues and of eigenfunctions for this equation. The wave equation outside the region of configuration space in which the particles interact is

$$\mathfrak{L}\psi(x, \cdots, x_s; y)=0 \tag{13.1}$$

and

$$\mathfrak{L}=\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_s^2}+\frac{\partial^2}{\partial y^2}+\kappa^2-U(x_1, \cdots, x_s)-V(y). \tag{13.2}$$

Defining k^2 as before and subjecting $v_k(y)$, $w_k(y)$ to Eqs. (7.6), (7.7) one defines the kernel

$$\begin{aligned} K(x_1, \cdots, x_s, y; \xi_1, \cdots, \xi_s, \eta) &= \sum v_k(y)w_k(\eta)u_n(x_1, \cdots, x_s)u_n(\xi_1, \cdots, \xi_s) \quad (y>\eta) \\ &= \sum v_k(\eta)w_k(y)u_n(x_1, \cdots, x_s)u_n(\xi_1, \cdots, \xi_s) \quad (y<\eta). \end{aligned} \tag{13.3}$$

One verifies in the same way as before that

$$\mathfrak{L}\int K(x_1, \cdots, x_s, y; \xi_1, \cdots, \xi_s, \eta)\rho(\xi_1, \cdots, \xi_s, \eta)d\xi_1 \cdots d\xi_s d\eta=\rho(x_1, \cdots, x_s, y). \tag{13.4}$$

In applying the above generalization one may think of $x_1, x_2, \cdots x_s$ as describing the internal condition of the bombarded nucleus and of the coordinate y describing the incident particle. Green's function K is designed so as to represent coherently scattered waves and scattered waves with excitation of the bombarded nucleus by means of the source density $\rho(x_1, \cdots, x_s, y)$.

The restriction of the operator $\mathfrak{L}$ to the form of Eq. (13.2) is not essential. The method still works for

$$\mathfrak{L}=N+\kappa^2-V(y),$$

where N is the wave equation operator (multiple of Hamiltonian) for the bombarded nucleus. Equation (13) should then be replaced by

$$(N-\lambda_n)u_n(x_1, \cdots, x_s)=0.$$

The operator N may contain exchange forces. The separability of N into parts corresponding to $x_1, \cdots, x_s$ is not assumed and is not necessary for the working of the method. The degeneracy due to excitation works out very similarly in the many-dimensional and the two-dimensional models.

Modification for central fields

A schematic discussion of radial functions for particles colliding with a nucleus can be carried out along the above lines. The differential equation

$$\mathfrak{L}\psi(r_1, r)=\left[\frac{\partial^2}{\partial r_1^2}+\frac{\partial^2}{\partial r^2}+\kappa^2-V_1(r_1)-V(r)\right]\psi(r_1, r)=0 \tag{14}$$

[18] Minima in scattering can be produced also in other ways as has been brought out by M. R. MacPhail, Phys. Rev. **57**, 669 (1940) from the general point of view of interference.

will be considered subject to boundary conditions

$$\psi(0, r)=\psi(r_1, 0)=0. \tag{14.1}$$

The distances of particles 1, 2 from the center of gravity of the system are here denoted by r_1, r. The potential energies V_1, V are supposed to include the centrifugal force terms $L(L+1)/r^2$ which enter if the angular momentum $L\neq 0$. Particle 1 will be supposed to be confined to the nucleus and, as in the preceding section, the scheme may be generalized to include the case of a composite system instead of the single particle 1.

Equation (14) corresponds to the two particles being in separate fields of force described by $V_1(r_1)$, $V(r)$. By an extension of the previous scheme one can discuss the interaction between the particles provided the potential energy is taken to be a function of r_1 and r only. Actually the potential energy is also a function of the polar angles of the two particles and the present discussion is incomplete in this respect. One may hope, however, that this idealization spoils only the quantitative side of the problem but does not affect radically the qualitative conclusions regarding effects of resonance.

The eigenvalues λ_n of

$$[\partial^2/\partial r_1{}^2+\lambda_n-V_1(r_1)]u_n(r_1)=0 \tag{14.2}$$

are supposed to form a discrete set and the eigenfunctions u_n are normalized by

$$\int_0^\infty |u_n(r_1)|^2 dr_1=1.$$

The number k is defined by

$$\kappa^2=\lambda_n+k^2 \tag{14.3}$$

and functions f_k, g_k are defined by

$$\begin{aligned}&[d^2/d\rho^2+1-k^{-2}V(\rho/k)]f_k(\rho)=0,\\ &[d^2/d\rho^2+1-k^{-2}V(\rho/k)]g_k(\rho)=0.\end{aligned} \tag{14.4}$$

The function f_k will be regular at $r=0$ and will be normalized so as to be asymptotic to a sine function of unit amplitude at large ρ. The function g_k will be asymptotic to a cosine function and g_k+if_k will contain ρ only as $e^{i\rho}$. One has the relation

$$f_k'g_k-f_kg_k'=1, \tag{14.5}$$

the differentiations being performed with respect to ρ. If the functions f_k, g_k are not oscillatory ($k^2<0$) then f_k will still be understood to be regular at $r=0$ and g_k will be made to vanish exponentially at $r=\infty$. In this case the normalizing constants in f_k, g_k will be adjusted so as to satisfy Eq. (14.5). The exceptional case analogous to that already discussed under "Degeneracy due to excitation" may also arise and can be treated analogously to Eqs. (11)$\cdots$(12.3).

The function $g_k(\rho)$ is analogous to $v_k(y)$ inasmuch as it is always admissible at $+\infty$. The smaller of the two arguments of the kernel cannot occur in g_k since g_k is not admissible at $r=0$. One is thus led to

$$\begin{aligned}-K_S(r_1, r; r_1', r')&=\sum u_n(r_1)u_n(r_1')f_k(kr')g_k(kr)/k \quad (r>r')\\ &=\sum u_n(r_1)u_n(r_1')f_k(kr)g_k(kr')/k \quad (r<r').\end{aligned} \tag{14.6}$$

The above kernel has the suffix S so as to indicate that its asymptotic behavior at $r=\infty$ is that of a standing wave. The $-$ sign present in Eq. (14.6) is due to the fact that substitutions $v\to g$, $w\to f$ made in (7.7) would give -1 on the right side of Eq. (14.5). Practically the same calculation as that carried out in Eqs. (8.5), (8.6) gives

$$\mathfrak{L}\int\int K_S(r_1, r; r_1', r')\rho(r_1', r')dr_1'dr'=\rho(r_1, r). \tag{14.7}$$

Another possible kernel is

$$-K_D(r_1, r; r_1', r') = \sum_{k^2>0} u_n(r_1)u_n(r_1')f_k(kr')[g_k(kr)+if_k(kr)]/k + \sum_{k^2<0} u_n(r_1)u_n(r_1')f_k(kr')g_k(kr)/k. \quad (14.8)$$

This kernel contains for large r only diverging waves and the index D refers to "diverging." By adding to the incident plane wave solutions of the type

$$\int\int K_D(r_1, r; r_1', r')\rho(r_1', r')dr_1'dr',$$

one is sure that at a large distance the asymptotic form of the wave function is the incident plane wave plus a diverging wave.

Qualitative Applications

Large repulsive interaction in small region

The interaction between the two particles will be supposed to be confined to a small circle as in Eqs. (12), $\cdots$(12.3). On the circumference of the circle the wave function is supposed to vanish so as to correspond to an infinite repulsion within the circle. The circle will be referred to as an "obstacle" because its interior is a region into which the wave cannot penetrate.[19]

In the absence of the obstacle the wave function will be taken to be of the form

$$\psi_0(r_1, r) = u_{n0}(r_1)f_{k0}(k_0r) \exp[i\delta(k_0)] \quad (15)$$

and for any real k the asymptotic form at $r=\infty$ will be

$$f_k(kr) \sim \sin[kr - L\pi/2 + \delta(k)].$$

The functions $f_k(kr)$, $g_k(kr)$ are analogous to f_L, g_L used in the discussion of the one-body problem in Eqs. (6), $\cdots$(6.43). The function $\psi_0(r_1, r)$ replaces the term

$$u_{n0}(r_1)F_L(k_0r) \rightarrow \psi_0(r_1, r) \quad (15.1)$$

in the expansion of the plane wave

$$u_{n0}(r_1) \exp(ik_0z) = u_{n0}(r_1)\sum i^L(2L+1)P_L(\cos\theta)F_L(k_0r)/(k_0r). \quad (15.2)$$

The substitution of (15) by means of (15.1) into (15.2) gives a modified plane wave consisting of the plane wave $u_{n0}(r_1) \exp(ik_0z)$ plus a scattered wave. The initial state of the particle 1 is n_0 and the particle 2 is in the state $\exp(ik_0z)$ and the substitution indicated by Eq. (15.1) gives, therefore, the solution of the scattering problem, taking into account the central field potential $V(r)$ but neglecting the interaction between particles 1, 2. To the function $\psi_0(r_1, r)$ one may add $\int\int K_D(r_1, r; r_1', r')\rho(r_1', r')dr_1'dr'$ with $\rho(r_1', r')=0$ outside the obstacle and the result may still be substituted in place of $u_{n0}(r_1)F_L(k_0r)$ in formula (15.2) without adding anything to the wave function except diverging and exponentially decaying waves. If the obstacle is sufficiently small to make $\psi_0(r_1, r)$ practically constant on the circumference, then one may use as an addition to ψ_0 a constant multiple times $K_D(r_1, r; r_{10}, r_0)$ where (r_{10}, r_0) is the center of the circle. In this approximation one has to replace

$$\psi_0(r_1, r) \rightarrow u_{n0}(r_1)f_{k0}(k_0r) \exp[i\delta(k_0)] - \frac{u_{n0}(r_{10})f_{k0}(k_0r_0) \exp[i\delta(k_0)]K_D(r_1, r; r_{10}, r_0)}{\langle K_D(r_{10}+a\cos\theta, r_0+a\sin\theta; r_{10}, r_0)\rangle_{Av\theta}}. \quad (15.3)$$

[19] In the present model the interaction is large whenever one of the particles is on the sphere $r_1=r_{10}$ and the other on the sphere $r=r_0$. It does not correspond to a simple law of force on a two-particle model. The requirement of a vanishing wave function in the triangular region $r_1<r$ would perhaps be more natural since it would correspond to the incident particle never penetrating closer to the center than particle 1. This type of interaction is of the type considered in Eqs. (16) $\cdots$ (16.4). No such model can represent the actual case perfectly and extreme cases are, therefore, worked out.

Here a is the radius of the circle. The value of K_D on the circumference is averaged over the angle θ between the radius and the r_1 axis. For a small circle the value of $\langle K_D\rangle_{\text{Av}}$ depends logarithmically on a as for the electrostatic potential and its dependence on the energy is not especially critical. The second term in Eq. (15.3) contains a part in $u_{n0}(r_1)[g_{k0}(k_0r)+if_{k0}(k_0r)]$ which combines itself with the divergent wave part of the first term to give the coherently scattered wave. The contribution due to this wave to the plane wave in Eq. (15.2) is

$$\psi_{s,\text{coh}}=\frac{2L+1}{k_0r}\left\{\sin\delta(k_0)+\frac{\exp[i\delta(k_0)]}{k_0\langle K_D\rangle_{\text{Av}}}f_{k0}^2(k_0r_0)u_{n0}^2(r_{10})\right\}P_L(\cos\theta)u_{n0}(r_1)\exp\{i[k_0r+\delta(k_0)]\}. \quad (15.4)$$

Distinction of potential scattering.—Here the specific nuclear scattering adds itself to the potential scattering represented by $\sin\delta(k_0)$. It should be noted that in this problem a definite distinction between pure potential scattering and potential scattering + nuclear scattering can be made. For pure potential scattering one has

$$\psi_{s,\text{coh, pot}}=\frac{2L+1}{k_0r}\sin\delta(k_0)P_L(\cos\theta)u_{n0}(r_1)\exp\{i[k_0r+\delta(k_0)]\}.$$

For a known $\psi_{s,\text{coh}}$ one obtains a coefficient $\sin\delta(k_0)\exp[i\delta(k_0)]$ and the point is that an arbitrary number cannot be represented as $e^{i\delta}\sin\delta$ with a real δ. This complex number has the special property of having a square of its absolute value equal to its imaginary part.

For incoherent scattering with excitation n the above model gives

$$\psi_{s,\text{incoh}}=+\frac{2L+1}{k_0r}u_n(r_1)\exp\{i[\rho+\delta(k_0)+\delta(k)]\}\frac{P_L(\cos\theta)}{k\langle K_D\rangle}u_{n0}(r_{10})u_n(r_{10})f_k(kr_0)f_{k0}(k_0r_0). \quad (15.5)$$

Equation (15.4) gives the scattered wave for which the nucleus is left in the original state n_0. Equation (15.5) gives the scattered wave for which the nucleus has been transferred into another (e.g. an excited) state.

The denominator $\langle K_D\rangle$ contains a number of the f_k and the barrier penetrabilities enter in compensating products f_kg_k since all quantities in $\langle K_D\rangle$ refer to the inside of the nucleus. The dependence of $\langle K_D\rangle$ on energy is, therefore, much less marked than that in the numerator. One sees that primarily it is

$$u_n(r_{10})f_k(kr_0)/k,$$

that matters for the amplitude of the scattered wave in the present example and in this case $|f_k|^2$ is a measure of the probability of escape to the state n.

Repulsive interaction in large region

Some of the qualitative features of the above example hold also in more general circumstances. Several small repulsive regions inside the nucleus give a dependence on energy of the scattered wave similar to that of Eqs. (15.4), (15.5). Sources of K_D put inside each obstacle have then strengths determined by $\langle K_D\rangle^{-1}$ for each obstacle and the intensity of the incident wave + wave scattered by other sources. If the obstacles are sufficiently small and there are not too many of them the wave scattered by other sources can be neglected and the energy dependence of scattering or excitation is the same as before.

If, however, the scattered wave inside the nucleus is comparable with $\psi_0(r_1, r)$ then the energy dependence of $\psi_{s,\text{coh}}$, $\psi_{s,\text{incoh}}$ outside the nucleus is also changed. Equation (15.3) may be used for a crude estimate of some of the effects. The repulsion will be supposed to take place in a circular region having dimensions comparable with the wave-length inside the nucleus and it will be supposed high in this region. As a crude approximation the wave $\psi_0(r_1, r)$ will be canceled approximately by a source

of K_D at the center so that Eq. (15.3) still will hold. Only in this case K_D does not depend primarily on the radius of the circle and the denominator may have a decided energy dependence.

If one of the terms in Eq. (14.8) for K_D is appreciably larger than the others inside the nucleus and if for it $g_{k'}(k'r_0) \gg f_{k'}(k'r_0)$ then one may set

$$-\langle K_D\rangle \sim u_{n'}(r_{10}) f_{k'}(k'r_0) g_{k'}(k'r_0)/k'. \tag{16}$$

Here $\kappa^2=\lambda_{n'}+k'^2$ so that the state n' goes with the wave number k'. If, besides, $n'=n_0$, $k'=k_0$ i.e., if the incident state is the one that is important inside the nucleus then Eq. (15.4) becomes

$$\psi_{s,\,\mathrm{coh}} \sim \frac{2L+1}{k_0 r} u_{n0}(r_1) P_L(\cos\theta)\left\{\sin\delta(k_0) - \frac{f_{k0}(k_0r_0)}{g_{k0}(k_0r_0)} \exp\left[i\delta(k_0)\right]\right\} \exp\left\{i(k_0r+\delta(k_0)\right\} \tag{16.1}$$

and from Eq. (15.5) one obtains

$$\psi_{s,\,\mathrm{incoh}} \sim -\frac{2L+1}{kr} u_n(r_1) \frac{u_n(r_{10})}{u_{n0}(r_{10})} \frac{f_k(kr_0)}{g_{k0}(k_0r_0)} P_L(\cos\theta) \exp\left\{i\left[\rho+\delta(k_0)+\delta(k)\right]\right\}. \tag{16.2}$$

Equation (16.1) corresponds to the scattering cross section

$$\sigma_{\mathrm{coh}} \sim (\Lambda_0^2/\pi)(2L+1)\left|\sin\delta(k_0) - \left[f_{k0}(k_0r_{10})/g_{k0}(k_0r_{10})\right] \exp\left[i\delta(k_0)\right]\right|^2 \tag{16.3}$$

and (16.2) to

$$\sigma_{\mathrm{incoh}} \sim (\Lambda\Lambda_0/\pi)(2L+1)\left|u_n(r_{10})/u_{n0}(r_{10})\right|^2 \left|f_k(kr_0)/g_{k0}(k_0r_0)\right|^2. \tag{16.4}$$

The last formula can be compared with the result of the considerations of Kapur and Peierls as used by Konopinski and Bethe[20] and those by Weisskopf and Ewing.[21] Konopinski and Bethe apply the theory to $Li^7+H^1 \rightarrow He^4+He^4$ and they use

$$\Gamma_l \propto E^{\frac{1}{2}}/G_l^2, \quad \sigma \sim \Lambda^2\Gamma_l \propto E^{-\frac{1}{2}} G_L^{-\frac{1}{2}}, \tag{16.5}$$

where G_l according to their paper is the irregular function of the proton for the Coulomb field. Equation (16.4) contains the factor $\Lambda\left|g_{k0}(k_0r_0)\right|^{-2}$ which shows a similar variation with energy. The following points may be noted in a comparison of the two results.

(a) Equation (16.4) does not give a solution of the $Li^7+H^1 \rightarrow He^4+He^4$ problem in a direct way. It is obtained only as an approximate solution of the excitation problem. One may nevertheless expect some similarity between the energy dependence of the excitation and disintegration processes. Since the factors $\Lambda\left|f_k(kr_0)\right|^2 \times \left|u_n(r_{10})\right|^2$ and $\Lambda_0\left|u_{n0}(r_{10})\right|^{-2}\left|g_{k0}(k_0r_0)\right|^{-2}$ are characteristic, respectively, of the final and initial states one has reasonable grounds for expecting the dependence $\Lambda_0\left|u_{n0}(r_{10})\right|^{-2}\left|g_{k0}(k_0r_0)\right|^{-2}$ to hold also for disintegrations in the approximation in which Eq. (16.4) has been obtained.

(b) Since the formalism used is essentially the same for the case of several coordinates $r_1, \cdots, r_s$ replacing the single coordinate r_1 the above can be applied to the case of a nucleus excited by bombardment of the particle whose coordinate has been denoted by r_1. If the nucleus disintegrates into two products (with a small probability) from the state n a three-body disintegration is described.

(c) The approximation used to obtain Eq. (16.4) is very crude. Essentially it neglects the effect of all states except $u_{n0}(r_1)g_{k0}(k_0r)/k_0$ which are produced by the sources of Green's function inside the nucleus. The complete expression according to Eq. (14.8) contains other terms besides and these contribute to $\langle K_D\rangle$. In addition the approximation of replacing the many sources of diverging waves inside the interaction region by one at the center neglects interference effects which contribute to an additional energy dependence.

(d) The function $g_{k0}(k_0r)$ which enters into the present discussion is not the irregular function for the Coulomb field but the irregular function for the solution of the radial wave equation with

[20] E. J. Konopinski and H. A. Bethe, Phys. Rev. **54**, 130 (1938).
[21] V. F. Weisskopf and D. H. Ewing, Phys. Rev. **57**, 472 (1940).

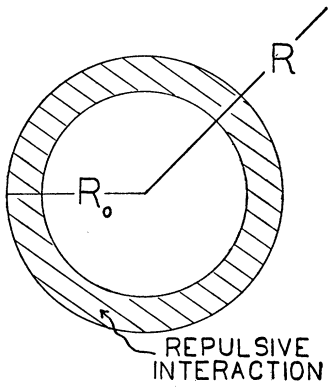

FIG. 5. The two-dimensional resonance model.

the Coulomb field modified by the average nuclear potential taken into account by the difference between $V(r)$ and the Coulomb potential. (The Coulomb field is here thought of as breaking off to zero at a reasonable distance outside the nucleus. If this is not done the above formulas must be changed to take into account the effects characteristic of the $1/r$ potential.)

Schematic representation of two-body resonance

The interaction potential will be supposed to be large and attractive in a small circular region surrounding the point (r_{10}, r_0). Outside this region there is a narrow ring [Fig. 5] within which the potential is repulsive and sufficiently high to serve as a barrier for waves emerging from (r_{10}, r_0). The two-dimensional region formed by the circle and the ring is somewhat analogous to the Gamow, Gurney-Condon model of radioactive decay. The depth of the potential well and the height of the barrier will be adjusted so that for a suitable energy the wave function is large inside the circle and small just outside the ring. For this energy the circular resonance region will be found to act as a strong source of two-dimensional waves which will give rise to strong scattering. The wave function within the circular region is in this model a schematic representation of the compound nucleus which is formed by the coupling of the two particles to each other. In this model the compound nucleus is screened against external disintegration by the repulsive region in the ring and in addition also by the barrier $V(r)$.

The resonating properties of the circle-ring system will be first defined by coupling it to an infinite plane with constant κ. In plane polar coordinates the axially symmetric solution satisfies

$$d^2\psi/dR^2+d\psi/RdR+[\kappa^2-V(R)]\psi=0, \quad (17)$$

which is the same as

$$d^2(R^{\frac{1}{2}}\psi)/dR^2+[\kappa^2-V(R)+1/4R^2] \times(R^{\frac{1}{2}}\psi)=0. \quad (17.1)$$

Here $V(R)$ is the potential within the circle-ring region and $V(R)=0$ for $R>R_0$. Outside the circle-ring system

$$\psi=J_0(\kappa R)\cos\varphi-N_0(\kappa R)\sin\varphi \quad (17.2)$$

is a solution of Eq. (17). One has

$$J_0(\kappa R)\sim(2/\pi\kappa R)^{\frac{1}{2}}\cos(\kappa R-\pi/4) \quad (\kappa R\gg 1) \quad (17.3)$$

$$N_0(\kappa R)\sim(2/\pi\kappa R)^{\frac{1}{2}}\sin(\kappa R-\pi/4),$$

so that by Eq. (17.2)

$$\psi\sim(2/\pi\kappa R)^{\frac{1}{2}}\cos(\kappa R-\pi/4+\varphi), \quad (\kappa R\gg 1). \quad (17.4)$$

Equations (17.4), (17.2) show that φ is caused by $V(R)$ in the same way as the phase shift K is produced in the single-body problem. It will be assumed, therefore, that close to the resonance

$$\tan\varphi\backsimeq\Delta E/(E-E_0), \quad (17.5)$$

where ΔE is a positive constant. This assumption is reasonable in view of Eq. (17.1) which is like the radial equation for $\mathfrak{F}$ in the central field problem. Eq. (17.2) does not apply for $R<R_0$. At $R=R_0$ it joins smoothly to the internal solution. One has

$$\left(\frac{d\psi}{\psi dR}\right)_{R_0}=\kappa\frac{J_0'(\kappa R_0)\cos\varphi-N_0'(\kappa R_0)\sin\varphi}{J_0(\kappa R_0)\cos\varphi-N_0(\kappa R_0)\sin\varphi} \quad (17.6)$$

and this formula does not depend on whether the ring is coupled to an infinite plane or to the actual potential provided one may assume ψ to depend only on R for $R<R_0$. In fact this axially symmetric part of ψ is determined by Eq. (17) and the requirement of regularity at $R=0$. The requirement of axial symmetry is good only if

$$x=\kappa R_0\ll 1.$$

For small x

$$J_0(x)=1-x^2/4+\cdots;$$
$$\pi N_0(x)/2=J_0(x)\ln(\gamma x/2)+x^2/4+\cdots.$$

The formula

$$J_0(x)N_0'(x)-N_0(x)J_0'(x)=2/\pi x$$

when used to eliminate N_0' in Eq. (17.6) gives

$$\left(\frac{d\psi}{\psi dR}\right)_{R_0}\approx-\frac{\kappa x}{2}-\frac{2/\pi R_0}{\cot\varphi-(2/\pi)\ln(\gamma x/2)}.$$

Since $x\ll 1$ this expression will be approximated by

$$\left(\frac{d\psi}{\psi dR}\right)_{R_0}\approx\frac{-2\Delta E/\pi R_0}{E-E_0-(2\Delta E/\pi)\ln(\gamma x/2)}\approx\frac{-\kappa N_0',}{\cot\varphi-N_0}, \quad (17.7)$$

where the approximation (17.5) has been used.

In the presence of $V_1(r_1)$, $V(r)$ and with the boundary conditions $\psi(0,r)=\psi(r_1,0)=0$ the approximation

$$\psi(r_1,r)=\psi_0(r_1,r)+\alpha\psi_0(r_{10},r_0)K_D(r_1,r;r_{10},r_0) \quad (18)$$

will be used. Here ψ_0 is given by Eq. (15) and it is assumed as in Eqs. (15), $\cdots$ (15.5) that the density ρ in $R<R_0$ can be replaced by a point source. This assumption is a poor one close to $R=R_0$ but the axially symmetric part of ψ can be represented by a point source. One has

$$d\psi/dR\approx\alpha\psi_0(r_{10},r_0)(dK_D/dR)_{R_0}$$

and

$$\left(\frac{d\psi}{\psi dR}\right)_{R_0}\approx\frac{(dK_D/dR)_{R_0}}{1/\alpha+(K_D)_{R_0}}. \quad (18.1)$$

The variation of ψ_0 over the circle-ring system is neglected and in (18.1) the average values of ψ, $d\psi/dR$ over $R=R_0$ are used. These values will be denoted as $\langle\psi\rangle$, $\langle d\psi/\psi dR\rangle$ and one obtains

$$\alpha=\frac{\langle d\psi/\psi dR\rangle}{-\langle K_D\rangle\langle d\psi/\psi dR\rangle+\langle dK_D/dR\rangle}\approx\left[-\langle K_D\rangle-\frac{\cot\varphi-N_0}{\kappa N_0'}\langle dK_D/dR\rangle\right]^{-1}.$$

One has approximately

$$4iK_D\sim J_0+iN_0 \quad (18.2)$$

and hence $4\langle dK_D/dR\rangle\sim\kappa N_0'$ so that

$$\alpha\approx\frac{4}{\langle N_0\rangle-\cot\varphi-4\langle K_D\rangle}\approx-\frac{4\Delta E}{E-E_0-\Delta E\langle N_0\rangle+4\Delta E\langle K_D\rangle}. \quad (18.3)$$

On account of the potential field $V(r)$ the quantity $u_{n0}(r_1)F_L(k_0r)$ in (15.2) should be replaced by $\psi_0(r_1,r)$ as in Eq. (15.1). On account of the resonance interaction in $R<R_0$ one must make the substitution

$$\psi_0\rightarrow\psi$$

with ψ given by (18), (18.3). These substitutions are made in the right side of (15.3) and $u_{n0}(r_1)\exp[ik_0z]$ is subtracted from the result. The difference is the scattered wave. Its value for large r is obtained by means of the asymptotic expressions

$$f_{k0}(k_0r)\exp[i\delta(k_0)]\sim i^{-L}\sin\delta(k_0)\exp\{i[k_0r+\delta(k_0)]\}+F_{k0}(k_0r)$$
$$g_k(kr)+if_k(kr)\sim i^{-L}\exp\{i[kr+\delta(k)]\}.$$

The scattered wave at a large distance is

$$\psi_s\sim\frac{2L+1}{k_0r}P_L\Big\{u_{n0}(r_1)\sin\delta(k_0)\exp[ik_0r+i\delta(k_0)]-\alpha u_{n0}(r_{10})f_{k0}(k_0r_0)\sum(k^2>0)u_n(r_1)u_n(r_{10})\frac{1}{k}f_k(kr_0)\exp[ikr+i\delta(k)+i\delta(k_0)]\Big\}. \quad (18.4)$$

The first term in the braces represents scattering due to $V(r)$. The remainder is due to the resonance interaction. Neglecting the former one obtains the scattering cross section due to resonance scattering

$$\sigma_{\rm res}=\sum(k^2>0)\int d\Omega\cdot r^2\int_0^\infty dr_1\cdot\frac{k}{k_0}|\psi_{sk}|^2, \quad (18.5)$$

where ψ_{sk} is the part of ψ_s due to excitation into the state n and emission of the wave g_k+if_k. The factor k/k_0 is introduced into (18.5) because the number of emitted particles is proportional to their velocity. Substituting into (18.5) one finds

$$\sigma_{\text{res}}=(2L+1)\frac{\Lambda_0^2}{\pi}|\alpha|^2u_{n0}^2(r_{10})\frac{1}{k_0}f_{k0}^2(k_0r_{10})\sum(k^2>0)u_n^2(r_{10})\frac{1}{k}f_k^2(kr_0). \tag{18.6}$$

Introducing α by means of Eq. (18.3) one has

$$\sigma_{\text{res}}=\frac{(2L+1)\Lambda_0^2\Gamma_{n0}\sum\Gamma_n}{\pi|E-E_0-\Delta E\langle N_0\rangle+4\Delta E\langle K_D\rangle|^2}, \tag{18.7}$$

where

$$\Gamma_n=4\Delta Eu_n^2(r_{10})\frac{1}{k}f_k^2(kr_0). \tag{18.8}$$

The expression (14.8) for K_D is now used to obtain $\langle K_D\rangle$. The imaginary part comes only from $\sum(k^2>0)$ and consists of a finite number of terms. The real part is an infinite series and has a logarithmic singularity like that of $N_0/4$. Denoting the real part by Re and the coefficient of the imaginary part by Im one has from Eq. (14.8)

$$4\Delta EIm\langle K_D\rangle\cong-\sum\Gamma_n, \tag{18.9}$$

which when introduced into (18.7) gives

$$\sigma_{\text{res}}=\frac{(2L+1)\Lambda_0^2\Gamma_{n0}\sum\Gamma_n}{\pi|E-E_0-\langle N_0\rangle\Delta E+4\Delta ERe\langle K_D\rangle-i\sum\Gamma_n|^2}. \tag{19}$$

This is the ordinary formula for resonance competition. It has been obtained with the following restricting assumptions.

(a) The two-dimensional region within which the interaction takes place is small. This assumption does not mean that within this region the interaction is small. The result of this assumption is to simplify the result by making Γ_n depend only on u_n^2, f_k^2 at $r_1=r_{10}, r=r_0$. (b) Resonance was assumed to be sharp.

According to Eq. (18.9) the damping constants Γ_n depend on the constant $4\Delta E$ which is characteristic of the resonance region. Besides, there is present also the dimensionless factor $u_n^2(r_{10})f_k^2(kr_0)/k$ which is characteristic of the binding of particle 1 in the residual nucleus, the energy of the escaping particle and the potential barrier through which this particle must escape. The barrier enters implicitly through $f_k^2(kr_0)$. The presence of this factor could be expected by analogy with the result of Wigner and the writer by a method analogous to the Weisskopf-Wigner theory of the absorption and emission of light. The damping constant is then proportional to the square of the matrix element of the interaction energy divided by the average energy interval, $|M_s|^2/\Delta\nu_s$ in the notation of the above reference. This quantity varies with the energy as f_k^2/k in exact agreement with the result obtained on the present model. It is claimed, on the other hand, from the considerations of Kapur and Peierls that $\Gamma\propto E^{\frac{1}{2}}/G_k^2$. The difference between the results is in a sense quantitative rather than qualitative and it is perhaps unnecessary to emphasize it. For slow neutrons both results give $\Gamma\propto E^{\frac{1}{2}}$ and for small r one has $F_kG_k\propto\rho\propto E^{\frac{1}{2}}$ so that F_k^2/k and $E^{\frac{1}{2}}/G_k^2$ vary with energy in the same way. For low energies f_k^2 and F_k^2 also vary similarly. On the other hand f_k^2/k, $E^{\frac{1}{2}}/G_k^2$ do not vary with energy in exactly the same way and it should be noted that $f_k^2(r_0)$ is here taken at $r=r_0$ i.e., at the mean r of the compound state rather than at the arbitrary nuclear radius of Kapur-Peierls.

Special Two-Dimensional Green's Functions

(A) Infinite plane

The differential equation

$$(\partial^2/\partial x^2+\partial^2/\partial y^2+1)\psi=0$$

is supposed to hold through an infinite plane. Green's function is

$$K_D=(1/4i)H_0^{(1)}([(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}), \tag{20}$$

where $H_0^{(1)}$ is the Hankel cylindrical function of the first kind. The method of Eqs. (7) will now be applied to the computation of K_D and the result will be shown to agree with Eq. (20). The point of doing so is to make sure that the method works also for a continuous spectrum of λ_n.

Quantizing in the interval $-L/2<x<L/2$ by means of the periodic boundary condition one has

$$u_n(x)=L^{-\frac{1}{2}}e^{i\tau x}$$
$$\tau=2\pi n/L;\quad n=0,\ \pm1,\ \pm2,\ \cdots$$

and

$$u_n(x)u_n^*(\xi)=L^{-1}\exp[i\tau(x-\xi)]. \tag{20.1}$$

Equation (7.6) gives

$$v_k(y)=N_ke^{iky},\quad w_k(y)=N_ke^{-iky}$$

with

$$\begin{aligned} k&=(1-\tau^2)^{\frac{1}{2}} && \text{for}\quad 1>\tau^2\\ k&=i|(1-\tau^2)^{\frac{1}{2}}| && \text{for}\quad 1<\tau^2 \end{aligned} \tag{20.2}$$

and according to the normalizing condition (7.7) one has

$$2ikN_k^2=1. \tag{20.3}$$

Substituting (20.1), (20.2), (20.3) into Eq. (8) and remembering that the interval between successive τ is $2\pi/L$ one has for large L

$$K_D(x,y;\xi,\eta)=\frac{1}{4\pi i}\int_{-\infty}^{+\infty}\frac{1}{(1-\tau^2)^{\frac{1}{2}}}\exp[i\tau(x-\xi)+i(1-\tau^2)^{\frac{1}{2}}(y-\eta)]d\tau;\quad (y>\eta). \tag{20.4}$$

According to Eq. (20) one has then

$$H_0^{(1)}((x^2+y^2)^{\frac{1}{2}})=\frac{2}{\pi}\int_0^\infty\frac{\cos\tau x}{(1-\tau^2)^{\frac{1}{2}}}\exp[i|y|(1-\tau^2)^{\frac{1}{2}}]d\tau$$
$$=\frac{2}{\pi}\int_0^1\frac{\cos(\tau x)}{(1-\tau^2)^{\frac{1}{2}}}\exp[i|y|(1-\tau^2)^{\frac{1}{2}}]d\tau+\frac{2}{\pi i}\int_1^\infty\frac{\cos\tau x}{(\tau^2-1)^{\frac{1}{2}}}\exp[-(\tau^2-1)^{\frac{1}{2}}|y|]d\tau. \tag{20.5}$$

This expression for $H_0^{(1)}$ is not obviously symmetric in x, y. It shows the importance of states with negative kinetic energy for motion in the y direction. The $\int_1^\infty$ part of the formula consists of such states. It should be noted that these states of negative kinetic energy may contribute to the asymptotic value of $H_0^{(1)}$ for $x\to\infty$. Setting $x=0$ in Eq. (20.5) one obtains

$$H_0^{(1)}(y)=\frac{2}{\pi i}\int_0^\infty e^{-y\sinh t}dt+\frac{1}{\pi}\int_0^\pi e^{iy\sin\theta}d\theta,\quad (y>0), \tag{20.6}$$

which is obtainable from Eq. (2) in Watson's *Theory of Bessel Functions*, p. 178.[22] Setting $y=0$ one has

$$H_0^{(1)}(x)=\frac{2}{\pi}\int_0^1\frac{\cos\tau x}{(1-\tau^2)^{\frac{1}{2}}}d\tau+\frac{2}{\pi i}\int_1^\infty\frac{\cos\tau x}{(\tau^2-1)^{\frac{1}{2}}}d\tau, \tag{20.7}$$

[22] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, 1922).

which is verified by means of

$$J_0(x)=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)d\theta,\quad N_0(x)=-\frac{2}{\pi}\int_0^{\infty}\cos(x\cosh t)dt,\quad H_0^{(1)}=J_0+iN_0.$$

For large x, $J_0(x)$, $N_0(x)$ have comparable values and Eq. (20.7) shows that N_0 arises in this case from waves with a negative kinetic energy of the y motion. The most important values of τ are those slightly >1 which correspond to only slightly negative kinetic energies of the y motion. In Eq. (20.6), on the other hand, the negative kinetic energies contribute the first integral and one has

$$\int_0^{\infty}e^{-|y|\sinh t}dt<\int_0^{\infty}e^{-|y|t}dt=\frac{1}{|y|}.$$

This is negligible in comparison with $H_0^{(1)}(y)$ for large $|y|$. Along the y axis, therefore, the negative kinetic energies of the y motion do not contribute to the dominating term of the outgoing wave.

(B) Quadrant $x>0$, $y>0$

The same differential equation as in case (a) is being solved subject to boundary conditions $\psi(0, y)=\psi(x, 0)=0$. Green's function can be obtained again by the general method and this time by the modification for central fields, as in Eq. (14.8). The substitution $r_1\to x$, $r\to y$ is made, the functions u_n are quantized in the finite interval $0<x<L$ and are $(2/L)^{\frac{1}{2}}\sin\tau x$ while the interval between successive τ is $\Delta\tau=\pi/L$. One obtains

$$K_D=-\frac{2}{\pi}\int_0^{\infty}\frac{\sin\tau x\sin\tau\xi}{(1-\tau^2)^{\frac{1}{2}}}\sin\left[(1-\tau^2)^{\frac{1}{2}}\eta\right]\exp\left[i(1-\tau^2)^{\frac{1}{2}}y\right]d\tau \tag{21}$$

with the understanding that for

$$\tau^2>1,\quad (1-\tau^2)^{\frac{1}{2}}=i(\tau^2-1)^{\frac{1}{2}}. \tag{21.1}$$

Rearranging expression (21) one expresses it in terms of integrals of the form (20.5) and obtains

$$K_D=\frac{1}{4i}\{H_0^{(1)}(\left[(x-\xi)^2+(y-\eta)^2\right]^{\frac{1}{2}})-H_0^{(1)}(\left[(x+\xi)^2+(y-\eta)^2\right]^{\frac{1}{2}})$$
$$-H_0^{(1)}(\left[(x-\xi)^2+(y+\eta)^2\right]^{\frac{1}{2}})+H_0^{(1)}(\left[(x+\xi)^2+(y+\xi)^2\right]^{\frac{1}{2}})\}, \tag{21.2}$$

which can be also obtained from (20) by placing negative images of the source at $(-\xi, \eta)$, $(\xi, -\eta)$ and a positive image at $(-\xi, -\eta)$.

It will be noted that in both of the above examples K_D is asymptotically an outgoing wave at a large distance even though this is not obvious in the forms (20.6), (21).

(C) Straight channel

The region considered is $-a<x<a$, $0<y<\infty$ and $\psi(x, y)$ is subject to the boundary conditions

$$\psi(a, y)=\psi(-a, y)=0$$

and for $y=\infty$ it is required that K_D be an outgoing wave. These are the boundary conditions required for Eq. (14.8) for use with central fields. The substitution

$$(r_1, r)=(x+a, y)$$

is made. The wave equation in the strip is taken to be

$$(\partial^2/\partial x^2+\partial^2/\partial y^2+\kappa^2)\psi=0.$$

One has

$$u_n(x)=a^{-\frac{1}{2}}\sin\left[\pi n(x+a)/2a\right];\quad k^2=\kappa^2-\pi^2n^2/4a^2;\quad n=1,\,2,\,3,\,\cdots \tag{22}$$

$$-K_D=\sum\,(k^2>0)\frac{1}{2ak}\left[\cos\frac{\pi n}{2a}(x-\xi)-(-)^n\cos\frac{\pi n}{2a}(x+\xi)\right]\sin\kappa\eta\cdot e^{iky}$$

$$+\sum\,(k^2<0)\frac{1}{2a|k|}\left[\cos\frac{\pi n}{2a}(x-\xi)-(-)^n\cos\frac{\pi n}{2a}(x+\xi)\right]\sinh|k|\eta e^{-|k|y}. \qquad (y>\eta)$$

For $\kappa\to0$ this expression becomes

$$-K_D=\frac{1}{2\pi}\text{R.P. ln}\left\{\frac{1+\exp\{(\pi/2a)[-(y-\eta)+i(x+\xi)]\}}{1-\exp\{(\pi/2a)[-(y-\eta)+i(x-\xi)]\}}\cdot\frac{1-\exp\{(\pi/2a)[-(y+\eta)+i(x-\xi)]\}}{1+\exp\{(\pi/2a)[-(y+\eta)+i(x+\xi)]\}}\right\}$$

$$(y>\eta). \tag{22.1}$$

The expression under the logarithm is a function of $x+iy$ and, therefore, the right side is a solution of Laplace's equation as it should be. The boundary conditions are verified directly in Eq. (22.1). Also one sees that K_D and its derivatives are continuous except at $x+iy=\xi+i\eta$. At the latter point K_D behaves as

$$K_D\approx\frac{1}{2\pi}\ln\left\{\frac{\pi}{2a}[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}\right\}. \tag{22.2}$$

Constant terms have been dropped. Except for constant terms K_D is seen to be the electrostatic potential due to an electric charge of linear density $(-1/4\pi)$ on an infinitely long filament perpendicular to the plane (x, y) and crossing it at the point (ξ, η). This is in agreement with Poisson's equation in electrostatics which is in two dimensions

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\int\int\frac{1}{2\pi}\rho(\xi,\eta)\ln\left[(x-\xi)^2+(y-\eta)^2\right]^{\frac{1}{2}}d\xi d\eta=\rho(x,y),$$

which is the same as Eq. (14.7) for K_D in the limiting case of Eq. (22.2).

If $\kappa^2>0$ but $\ll\pi^2/4a^2$ one may still apply Eq. (22.1) in the rectangle $0<y<2a$ as an approximation. Since in the form of Eq. (22) the series converges slowly one sees here the importance of taking into account many states of excitation of the residual nucleus.

For any κ the series (22) for K_D behaves for high n like the series (22.1) and as long as $|y-\eta|\neq0$ this series remains convergent when differentiated with respect to y. The convergence is due to the presence of the factor $\exp\left[-\pi n|y-\eta|/2a\right]$. It will now be verified that $\partial K_D/\partial y$ is continuous on the line $y=\eta$. For a small $\epsilon>0$ one has for $\kappa^2<\pi^2/4a^2$

$$(\partial K_D/\partial y)_{y=\eta+\epsilon}-(\partial K_D/\partial y)_{y=\eta-\epsilon}=\sum\frac{1}{2a}\left[\cos\frac{\pi n}{2a}(x-\xi)-(-)^n\cos\frac{\pi n}{2a}(x+\xi)\right]\left[e^{-|k|\epsilon}+e^{-2|k|\eta}\sinh|k|\epsilon\right].$$

The second term in square brackets gives an absolutely converging series which vanishes with ϵ and may, therefore, be disregarded. The first term also gives an absolutely converging series if $\epsilon>0$. For $\epsilon=0$ the series does not converge. As $\epsilon\to0$ the sum can be seen to $\to0$. In fact ϵ can be made so small that $\epsilon\pi N/2a\ll1$ for $N\gg2a\kappa/\pi$. The $\sum_{n=1}^{\infty}$ is split into $\sum_1^N+\sum_N^\infty$ and the series is compared with its value for the same ϵ and $\kappa=0$. The difference of the $\sum_1^N$ is negligible on account of the assumption about ϵ and the difference of $\sum_N^\infty$ is negligible on account of the assumption about N. On the other

hand for $\kappa=0$ the series can be summed and gives 0. It is thus seen that $\partial K_D/\partial y$ is continuous on the line $y=\eta$. The point $(x, y)=(\xi, \eta)$ is, of course, an exception.

The main object of the above example is to illustrate the importance of excited states in the vicinity of the region in which interaction takes place. This is seen in the limiting form of Eq. (22.1) in which the outgoing waves combine to give the electrostatic solution. If there are two obstacles, instead of one, and if their distance apart is smaller than the wave-length the wave scattered by (ξ_1, η_1) gives a contribution to ψ at (ξ_2, η_2) which contains many states of excitation.

(D) Transformation of Hankel function

The relations of Eq. (20) suggest formula (20.6) for the Hankel function $H_0^{(1)}$. This will now be verified by means of contour integration. According to Courant-Hilbert.[23]

$$H_0^{(1)}(r)=-\frac{1}{\pi}\int_{L_1} e^{-ir\sin\zeta}d\zeta, \tag{23}$$

where the path of integration is from $-i\infty$ to 0, then to $-\pi$ and finally to $-\pi+i\infty$ as indicated in Fig. 6. The differential equation for $H_0^{(1)}$ is satisfied also for other paths of integration provided

$$[\cos\zeta\, e^{-ir\sin\zeta}]_a^b=0, \tag{23.1}$$

where $\zeta=a$ and $\zeta=b$ are the limits of integration. The path will be deformed into L_1' which starts at $\xi_0-i\infty$ and goes over ξ_0, $-\pi+\xi_0$ to $-\pi+\xi_0+i\infty$ with $\pi/2>\xi_0>\infty$. [This path is not the path denoted by L_1' on p. 408 of Courant-Hilbert I.] The condition (23.1) is satisfied by L_1' because $\cos\xi_0>1$. The integrand of (23) vanishes at the ends of the path for both L_1 and L_1' and no poles of the integrand have been crossed in the deformation. One has, therefore,

$$H_0^{(1)}(r)=-\frac{1}{\pi}\int_{L_1'} e^{-ir\sin\zeta}d\zeta. \tag{23.2}$$

Setting $\zeta=\xi+i\eta$ and introducing the abbreviations

$$y=r\cos\xi_0, \quad x=r\sin\xi_0, \tag{23.3}$$

one obtains

$$H_0^{(1)}(r)=\frac{2}{\pi i}\int_0^\infty \cos(x\cosh\eta)e^{-y\sinh\eta}d\eta+\frac{1}{\pi}\int_{-\xi_0}^{\pi-\xi_0} e^{ir\sin\theta}d\theta. \tag{23.4}$$

Substitution of $\tau=\cosh\eta$ in the first integral changes it into the second integral of Eq. (20.5). The second integral in (23.4) is transformed as follows

$$\frac{1}{\pi}\int_{-\xi_0}^{\pi-\xi_0}\exp(ir\sin\theta)d\theta=\frac{1}{\pi}\int_0^\pi \exp(iy\sin\theta-ix\cos\theta)d\theta=\frac{2}{\pi}\int_0^{\pi/2} e^{iy\cos\theta}\cos(x\sin\theta)d\theta.$$

The second integral in (23.4) is thus the first integral in (20.5) so that the latter representation is equal to (23). It will be noted that the restrictions

$$0<\xi_0<\pi/2, \quad x>0, \quad y>0$$

are essential for the deformation of L_1 into L_1' and that these conditions are also essential for the applicability of the construction of $H_0^{(1)}$ as $4iK_D$.

The somewhat unexpected features of (20.5) are its dissymmetry between x and y and the fact that it is not obvious from its form that it represents outgoing waves in any direction but that of

[23] Reference 15, Vol. I, p. 407.

increasing y. The way in which (20.5) gives an outgoing wave for $y=0$ has been discussed in connection with Eq. (20.7). It may be pointed out that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \tau x}{(1-\tau^2)^{\frac{1}{2}}}d\tau=\frac{2}{\pi}\int_0^1 \frac{\sin \tau x}{(1-\tau^2)^{\frac{1}{2}}}d\tau+\frac{2}{\pi i}\int_1^\infty \frac{\sin \tau x}{(\tau^2-1)^{\frac{1}{2}}}d\tau$$

is also an outgoing wave for $x>0$. One has in fact

$$\frac{2}{\pi i}\int_1^\infty \frac{\sin \tau x}{(\tau^2-1)^{\frac{1}{2}}}d\tau=-iJ_0(x)\sim -i\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos\left(x-\frac{\pi}{4}\right)$$

and

$$\frac{2}{\pi}\int_0^1 \frac{\sin \tau x}{(1-\tau^2)^{\frac{1}{2}}}d\tau=\frac{4}{\pi}[J_1(x)+\tfrac{1}{3}J_3(x)+\tfrac{1}{5}J_5(x)+\cdots]\sim\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin\left(x-\frac{\pi}{4}\right).$$

Combining these relations one has

$$\frac{2}{\pi}\int_0^\infty \frac{\sin \tau x}{(1-\tau^2)^{\frac{1}{2}}}d\tau\sim -i\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}e^{i(x-\pi/4)}. \tag{23.5}$$

This formula is another example of an outgoing wave which is compounded of stationary waves. It and Eq. (21) show that the general method of Eq. (14.8) when applied to a continuum can give in special cases outgoing waves in all directions within the quadrant $x>0$, $y>0$. This does not mean, however, that the same is true in the general case.

Final Remarks

The calculations of the last section show that the behavior of K_D close to the singular point is in fact such as has been used in the calculation of scattering by a small obstacle in the (r_1, r) plane and the related calculations on the interaction in a large region as well as with resonance. The example of the straight channel shows how at the singularity the logarithmic dependence on $(x-\xi)^2+(y-\eta)^2$ is obtained. In fact for small values of this quantity the contributions of high terms in the series of Eq. (22) approach the limit for $\kappa=0$ and give the electrostatic term. It is clear, therefore, that close to (ξ, η) one needs a larger number of excitation states to represent the wave function.

In the two-dimensional model the singularity is logarithmic. In m dimensions the dependence of K close to the singularity is of type const. $[\sum(x_i-\xi_i)^2]^{(-m+2)/2}$. The effect of a source of a diverging wave is the more localized in the m space the higher the number of dimensions and the region in which the higher states are especially important becomes more confined as m increases. The scattered wave function outside the nucleus may be expected to be small in comparison with the difference between the actual and incident wave functions inside the nucleus in a many-dimensional diagram. The importance of the intermediate state is in this sense increased by increasing the number of interacting particles.

The two-dimensional resonance model treated above gives the expected answer. It does not

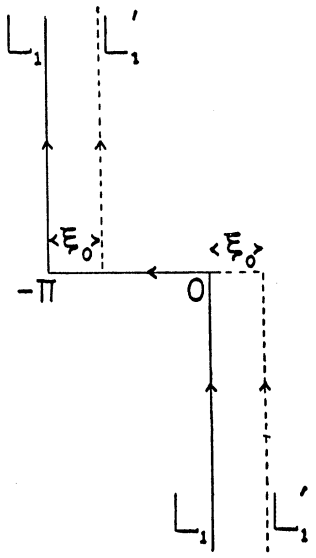

Fig. 6. Contours of integration for $H_0^{(1)}$.

include the effect of competitive emission of several particles but only the effect of competition between various types of emission of the same particle. For the discussion of more general competitive processes it is obviously more suitable to bring in the intermediate state in a more pronounced way. It is possible to do so by a generalization of the method of complex eigenvalues to many-dimensional problems. This will be treated in another publication.

The occurrence of the regular and irregular solutions of the radial equation (f_k and g_k) in the formulas for the scattered waves is not special to the two-dimensional model as is clear from the many-dimensional extension in Eqs. (13.1) $\cdots$ (13.4). The examples show that in special cases the factor f_k^2 occurs in the formulas. If the interaction is distributed within the nucleus, however, then the general form of Green's function shows that the g_k occur in the denominators. In the general case the g_k for all excited states will occur and very special assumptions about the relative importance of excited states are necessary to leave the simple factor $[g_{k0}(k_0r_0)]^{-2}$ in Eq. (16.4). Quantitative agreement with experiment obtained by application of formulas with one or another simple choice of such factors does not appear therefore to be very significant.

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The Principal M-Series Emission Lines of Tungsten and the K Absorption of Magnesium and Aluminum

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A study of the M-series lines of tungsten and of the K-absorption edges of magnesium and aluminum has been made by means of a double crystal x-ray spectrometer. The extension of the double crystal technique to these long x-ray wave-lengths has been made possible by thorough evacuation of the spectrograph and a windowless x-ray tube. The crystals were the aquamarine form of beryl in which the 1010 planes have a grating constant of 8.06A. The detector was a Geiger-Müller counter with an aluminum window of 2.5×10^{-4} cm thickness. The width and relative intensities of the prominent M-series tungsten lines were measured. Accompanying the diagram lines are some 33 satellite lines. In the study of the K-absorption edges of magnesium and aluminum, structure not heretofore resolved photographically has been recorded. A qualitative comparison of the intensity of the continuous radiation from tungsten to that from aluminum at 10A indicates that the ratio is less than one, whereas theory predicts about six.

INTRODUCTION

PRECISION measurements with the double crystal x-ray spectrometer have shown the effectiveness of the x-ray method in checking experimentally the theoretical calculations on the electronic conduction bands of solids.[1] Ruled grating methods for the longer wave-lengths have given valuable information on the conduction bands of the light elements.[2] Photographic measurements in the intermediate range have not yielded results of definitive accuracy because of low resolving power and inaccuracies of intensity measurements. This region, however, is one of considerable theoretical interest, because here the present theory of metals makes sufficiently accurate predictions that their comparison with experiment may be expected to be significant.[3]

[1] W. W. Beeman and H. Friedman, Phys. Rev. **56**, 392 (1939); H. Friedman and J. A. Bearden, Phys. Rev. **57**, 1085A (1940); T. M. Snyder and J. A. Bearden, Phys. Rev. **57**, 1085A (1940); W. W. Beeman and J. A. Bearden, Phys. Rev. **57**, 1085A (1940).

[2] See, for example, *Reports on Progress in Physics*, Vol. 5 (Physical Society, London, 1939).

[3] M. F. Manning and H. M. Krutter, Phys. Rev. **51**, 761 (1937); J. C. Slater, Phys. Rev. **45**, 794 (1934).