From our ionization chamber measurements with the target replaced by a clean aluminum plate we conclude that the only source of neutrons in the accelerating tube was the target itself. It is conceivable, however, that with some experimental arrangements the scattering from the walls of the room might play an important role. It is also well known that in certain accelerating tubes an appreciable number of neutrons may be created at points other than the target. We feel that a low energy group of neutrons whose intensity amounts to as much as 10 percent of the main group should be evident in Fig. 2 if it exists.

We do not believe that the data presented here are sufficient to warrant a precise determination of the energy evolved in the $d-d$ reaction. We have, however, plotted an integral number-energy curve in Fig. 3. The extrapolation of this curve gives an energy of about 2.8 Mev for the neutron energy, corresponding to an extrapolated range of about 10.9 cm for the recoil protons. This agreement with Bonner's extrapolated range of 10.6 cm is satisfactory.

We wish to express our appreciation for the generous cooperation of various members of the staff in this work.

JULY 1, 1940 PHYSICAL REVIEW VOLUME 58

Note on a Field Theory of Nuclear Forces*

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(Received May 3, 1940)

An interaction between nuclear particles and a field of light particles which obey Fermi-Dirac statistics is considered. It is shown that the energy of interaction between heavy and light particles can be calculated without resorting to perturbation theory in an approximation which neglects electric forces and the kinetic reaction of the heavy particles. Results which contain the effect of the rest mass of the field particles are presented for one heavy particle and for two heavy particles.

It is the purpose of this note to extend the method which has been applied to the electron-positron-field theory of nuclear forces[1] to take account of effects dependent on the rest mass of the field particles. In the previous theory, the rest mass of the light particles was negligible. If the light particle field be that of mesons which obey Fermi-Dirac statistics, however, it is necessary to take the rest mass (~ 180 electron masses) into account, even in the approximation which neglects electric forces and the recoil of the heavy particles. The method which we present may then be used, for example, to derive the nuclear forces due to the meson-field introduced by Marshak,[2] without the use of perturbation theory.

The fundamental assumptions of the present method are (I) a heavy particle (neutron or proton) interacts strongly with a light particle (meson) if the light particle occupies a state of one particular space-dependence, $u(\mathbf{x})$, but does not influence the energy of mesons in any orthogonal state; (II) the state of the heavy particle is not changed by the interaction. Both assumptions require a nonrelativistic treatment of the heavy particle and permit the simplified device of fixing a heavy particle at the origin of the coordinate system and investigating its effect on the light particle states. Condition (II) must be changed to exclude spin direction if

* Publication assisted by the Ernest Kempton Adams Fund for Physical Research of Columbia University.

[1] Wigner, Critchfield and Teller, Phys. Rev. **56**, 530 (1939).

[2] R. E. Marshak, Phys. Rev. **57**, 1101 (1940).

spin-dependence of nuclear forces is to be explained but the modification is easily made[3] and will not be considered here. It is furthermore assumed for simplicity that the spatial dependence of $u(\mathbf{x})$ is spherically symmetrical about the position of the heavy particle.

Let $\psi(\mathbf{x})$ be the operator of the meson state as customarily defined in the formulism of quantized wave functions.[4] $\psi(\mathbf{x})$ has four spin-components which will be transformed by $\boldsymbol{\alpha}$ and β in the way required by their use in the relativistic wave equation of Dirac. With a heavy particle remaining stationary at the origin explicit reference to its state need not be made and the operator of the proposed addition to the Hamiltonian of the mesons is

$$J=-\eta\int d\mathbf{x}\int d\mathbf{x}'\psi(\mathbf{x})^{\dagger}\beta\psi(\mathbf{x}')u(\mathbf{x})^{*}u(\mathbf{x}'), \quad (1)$$

where η is the constant giving the strength of the interaction. We shall take μc^2 as the unit of energy; μ is the mass of the meson. Making a Fourier transformation $u(\mathbf{x})$ may be replaced by $v(\mathbf{p})$ and $\psi(\mathbf{x})$ by $\phi(\mathbf{p})$

$$J=-\eta\int d\mathbf{p}\int d\mathbf{p}'\phi(\mathbf{p})^{\dagger}\beta\phi(\mathbf{p}')v(\mathbf{p})^{*}v(\mathbf{p}') \quad (2)$$

and $v(p)$ is normalized so that

$$\int_0^{\infty} v(\mathbf{p})^2 4\pi p^2 dp = 1. \quad (3)$$

If $H^0\phi^{\dagger}\phi$ be the usual relativistic Hamiltonian (in quantized form) the Hamiltonian $H\phi^{\dagger}\phi=H^0\phi^{\dagger}\phi+J$ in which Coulomb interactions have been disregarded leads to a wave equation for the single light particles:

$$[E-(\boldsymbol{\alpha},\mathbf{p})-\beta]\phi(\mathbf{p})+\eta v(\mathbf{p})\beta\int v(\mathbf{p}')^{*}\phi(\mathbf{p}')d\mathbf{p}'=0. \quad (4)$$

$\mathbf{p}$ is also measured in units of μc^2. Although we follow the main lines of the method used for electron-positron pairs we shall, however, omit the transformation to eigenstates of kinetic energy. This step was useful in the former paper[1] to get a description of the interaction (1) in the language of pair-emission but it is not essential to the solution of (4).

$$\phi(\mathbf{p})=-\frac{E+(\boldsymbol{\alpha},\mathbf{p})+\beta}{E^2-1-p^2}\eta v(\mathbf{p})\beta\int v(\mathbf{p}')^{*}\phi(\mathbf{p}')d\mathbf{p}'. \quad (5)$$

Multiply (5) by $v(\mathbf{p})^{*}d\mathbf{p}$ and integrate over all angles and over all values of $|p|$.

$$\int v(p)^{*}\phi(\mathbf{p})d\mathbf{p}=-\int\frac{E+\beta}{E^2-1-p^2}\eta v(p)^2 4\pi p^2 dp\beta\int v(p')^{*}\phi(\mathbf{p}')dp'. \quad (6)$$

$v(p)$ is equivalent to $v(\mathbf{p})$ because of the assumed spherical symmetry. The integral equation (6) can be satisfied only if

$$4\pi\beta\eta\int\frac{E+\beta}{E^2-1-p^2}v(p)^2p^2dp=-1. \quad (7)$$

Let $q\equiv|(E^2-1)^{\frac{1}{2}}|$; the integral in (7) is then of the form which was evaluated in the former paper.[1] The method of evaluation is one by which the integral is replaced by a sum over momentum states quantized in a sphere of radius, L, so that $p_n=\pi\mu c^2\hbar n/L$. As a result of the interaction with the heavy particle the $(E^2-1)^{\frac{1}{2}}$ of an eigenstate with energy E, determined by Eq. (7), is changed by an amount $x\pi\mu c^2\hbar/L$, $|x|<1$. The x for a given E is determined by

$$(\beta E+1)\eta\{2\pi^2 qv(q)^2\cot\pi x+f(q)\}=-1,\quad f(q)\equiv 4\pi\int_0^{\infty}[p^2v(p)^2-q^2v(q)^2](q^2-p^2)^{-1}dp. \quad (8)$$

[3] C. L. Critchfield, Phys. Rev. **56**, 540 (1939).
[4] P. Jordan and E. Wigner, Zeits. f. Physik **47**, 631 (1928).

The factor β can take either of two values, $+1$ or -1. Let x be the fractional shift of $(E^2-1)^{\frac{1}{2}}$ when $\beta=1$ and let x' be that for $\beta=-1$

$$\begin{aligned}\tan \pi x &= 2\pi^2 q(E+1)v(q)^2/[-\eta^{-1}-(E+1)f(q)],\\ \tan \pi x' &= 2\pi^2 q(E-1)v(q)^2/[\eta^{-1}-(E-1)f(q)].\end{aligned} \tag{9}$$

Let $dq=\pi\mu c^2\hbar/L$; then the combined displacement of the value of q for levels of essentially the same E is $(x+x')dq$ and the corresponding net change in energy for these levels is $(x+x')d|E| = (x+x')qdq/|E|$. The total change in the energy of all states of negative energy, ΔE, is then

$$\Delta E = -\frac{2}{\pi}\int_0^\infty \frac{qdq}{(q^2+1)^{\frac{1}{2}}}\,\mathrm{arctg}\,\frac{4\pi^2 q^2 v(q)^2[q\dot{f}(q)-(\eta q)^{-1}]}{\eta^{-2}-q^2f(q)^2+4\pi^4q^4v(q)^4+2f(q)/\eta}. \tag{10}$$

The factor 2 represents the sum over spin directions of the mesons and ΔE is understood to be in units of μc^2.

Equation (10) gives the total interaction energy between one heavy particle and the field of all mesons in negative energy states. When two heavy particles are considered ΔE will depend upon their separation and forces will arise which might be identified with the forces in nuclei. The dependence of ΔE on the distance between heavy particles may be obtained by the same method which led to Eq. (10). The outlines of this application will now be sketched.

We modify the assumptions made above when considering one heavy particle only by considering two heavy particles of equal mass with their center of gravity at the origin of coordinates. Let the position vector of one be $c\hbar\mathbf{X}$ and of the other $-c\hbar\mathbf{X}$. An addition to the Hamiltonian of the type (2) must now be made for each heavy particle. This may be done by translating $v(p)$ from the origin to $c\hbar\mathbf{X}$ so that it becomes $e^{i(p,X)}v(p)$ in one case and from the origin to $-c\hbar\mathbf{X}$ in the other. The wave equation for mesons then becomes:

$$[E-(\boldsymbol{\alpha},\mathbf{p})-\beta]\phi(\mathbf{p})+\eta e^{i(p,X)}v(p)\beta\int v(p')^*e^{-i(p,X)}\phi(\mathbf{p}')d\mathbf{p}' + \eta e^{-i(p,X)}v(p)\beta\int v(p')^*e^{i(p,X)}\phi(\mathbf{p}')d\mathbf{p}'=0. \tag{11}$$

We define the spinors:

$$\xi_1\equiv\int e^{-i(p,X)}v(p)^*\phi(p)dp,\quad \xi_2\equiv\int e^{i(p,X)}v(p)^*\phi(p)dp \tag{12}$$

and solve (11) for $\phi(\mathbf{p})$

$$\phi(\mathbf{p}) = -\frac{E+(\boldsymbol{\alpha},\mathbf{p})+\beta}{E^2-p^2-1}\eta\beta\{e^{i(p,X)}v(p)\xi_1+e^{-i(p,X)}v(p)\xi_2\}. \tag{13}$$

$\mathbf{X}$ is simply a set of three numbers so that the exponentials and other functions of p commute. By multiplying (13) by $e^{-i(p,X)}v(p)^*d\mathbf{p}$ and integrating we get one equation in ξ_1 and ξ_2 and by multiplying (13) with $e^{i(p,X)}v(p)^*d\mathbf{p}$ and integrating we get another. These equations may be written formally:

$$\xi_1=-Q\xi_1-R\xi_2,\quad \xi_2=-R^*\xi_1-Q\xi_2. \tag{14}$$

The energy levels of (11) are then determined by the secular equation:

$$(1+Q)^2-R^*R=0 \tag{15}$$

and with the same notation as in Eq. (8)

$$Q=(\beta E+1)\eta\{2\pi^2 qv(q)^2\cot\pi x+f(q)\}, \tag{16}$$

$$R=-\int\frac{E+\beta}{q^2-p^2}\eta\beta 4\pi p^2 v(p)^2\frac{\sin pr}{pr}dp+i(\boldsymbol{\alpha},\mathbf{X})\int\frac{2\eta\beta}{r^2(q^2-p^2)}4\pi p^2 v(p)^2\left[\frac{\sin pr}{pr}-\cos pr\right]dp, \tag{17}$$

where $r\equiv 2|X|$, is $1/\hbar c$ times the separation of the heavy particles. It is evident that the integrals in (17) may be calculated by the same method by which Eqs. (8) and (16) were obtained if $v(p)^2$ be replaced by $v(p)^2\sin pr/pr$ for the first integral and by $v(p)^2[(\sin pr/pr)-\cos pr]$ for the second. In this way R may be represented by

$$R=-\eta(\beta E+1)\{2\pi^2 qv_1(q)^2\cot\pi x+f_1(q)\}+i(\boldsymbol{\alpha},\mathbf{X})\frac{2\beta}{r^2}\eta\{2\pi^2 qv_2(q)^2\cot\pi x+f_2(q)\} \tag{18}$$

with

$$v_1(q)^2=v(q)^2\sin qr/qr,\qquad f_1(q)=4\pi\int[p^2v_1(p)^2-q^2v_1(q)^2]/(q^2-p^2)dp,$$

$$v_2(q)^2=v(q)^2[\sin qr/qr-\cos qr],\quad f_2(q)=4\pi\int[p^2v_2(p)^2-q^2v_2(q)^2]/(q^2-p^2)dq.$$

The secular equation then becomes:

$$\eta^2[(\beta E+1)^2(v^4-v_1^4)-v_2^4/r^2](2\pi^2 q\cot\pi x)^2+2[\eta(\beta E+1)v^2+\eta^2(\beta E+1)^2(v^2f-v_1^2f_1)-\eta^2v_2^2f_2/r^2]$$
$$\times(2\pi^2q\cot\pi x)+1+2\eta(\beta E+1)f+\eta^2(\beta E+1)^2(f^2-f_1^2)-\eta^2f_2^2/r^2=0. \tag{19}$$

This quadratic equation has the roots $\cot\pi x_1$ and $\cot\pi x_2$ corresponding to changes in q for two orthogonal eigenstates of energy E. Since we are interested only in the sum of all changes in energy we add x_1 and x_2

$$\tan\pi(x_1+x_2)=4\pi^2q[\eta(\beta E+1)v^2+\eta^2(\beta E+1)^2(v^2f-v_1^2f_1)-\eta^2v_2^2f_2/r^2]$$
$$\times\{4\pi^4q^2\eta^2[(\beta E+1)^2(v^4-v_1^4)-v_2^4/r^2]-1+\eta^2f_2^2/r^2-2\eta(\beta E+1)f-\eta^2(\beta E+1)^2(f^2-f_1^2)\}^{-1}. \tag{20}$$

Let x_1 and x_2 represent the changes in q when $\beta=1$ and x_1', x_2' represent those when $\beta=-1$. Furthermore, we define K, L, M, and N in such a way that the tangent in Eq. (20) becomes $(K+\beta L)/(M+\beta N)$; then

$$\tan\pi(x_1+x_2)=(K+L)/(M+N),\quad \tan\pi(x_1'+x_2')=(K-L)/(M-N), \tag{21}$$

$$x_1+x_2+x_1'+x_2'=(1/\pi)\operatorname{arctg}2(KM-LN)/(M^2+L^2-N^2-K^2),$$
$$\Delta E=-\int_0^\infty 2(x_1+x_2+x_1'+x_2')qdq/(q^2+1)^{\frac{1}{2}}. \tag{22}$$

For a particular function $v(p)$, therefore, it is possible to express ΔE as an integral analogous to expression (10) but the explicit form of the one for two particles is necessarily complicated and is not presented.

The method presented above is an alternative to the usual perturbation method of calculating forces between heavy particles. In practice the integral (22) will be discouraging except perhaps for certain particular assumptions for $v(p)$.

The authors are indebted to Professor Wigner of Princeton University for many helpful discussions on this approach to a theory of nuclear forces. It is a pleasure to acknowledge Professor Wigner's interest and help.