exist. The application of the argument to higher and higher orders shows that solutions **V**, **H**, **E** exist for which all terms are static. Thus steady currents do not encounter even the smallest resistance.

A slight modification of this argument shows that the resistance is rigorously zero for nonsteady currents as well. Equations (29–32) have simple harmonic solutions,

$$\mathbf{V}_0(x, t) = \mathbf{V}_0(x)e^{i\omega t};$$

for such solutions the inhomogeneous parts of (33–36) have factors $e^{2i\omega t}$, so that they have solutions

$$\mathbf{V}_1(x, t) = \mathbf{V}_1(x)e^{2i\omega t}$$

and in general

$$\mathbf{V}_n(x, t) = \mathbf{V}_n(x)e^{(n+1)i\omega t}.$$

Thus an alternating current or field excites only harmonics of the fundamental frequency and there are periodic solutions of the equations, which would not be the case under the working of an irreversible resistance.

The conditions upon the nth correction, $\mathbf{V}_m$, $\mathbf{H}_n$, $\mathbf{E}_n$, differ from the conditions (29–32) upon the zeroth approximation only in the inhomogeneous terms. Consequently, a uniqueness theorem, once established for the solutions of the London theory, is equally valid for the rigorous solutions of the present theory. This becomes important in view of the results of the following paper (cf. reference 8).

It is a pleasure to acknowledge the assistance of Professor Eckart, who also suggested the application of the variation principle to the problem of the superconductor.

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Complete Data and Boundary Conditions for a Superconductor

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When the applied fields and the total charge on each conductor are known, the solution of the London equations for a superconductor in the steady state is completely determined by requiring the continuity of the magnetic field and of the tangential components of the electric field. This result is incompatible with the somewhat prevalent notion that the lines of current flow rotate with the conductor. The solution for a nonsteady state is uniquely determined by the additional specification of **E**, **H**, **J** throughout the system at some particular time.

The Steady State

The purpose of the present paper is to arrive at a set of data and boundary conditions for the London theory which is at the same time mathematically complete and physically meaningful. The existence of such a set is of more than mere mathematical interest; it has the following important physical implication.

An experiment has been described[1] in which a superconducting, hollow sphere in an applied magnetic field experienced a torque when turned out of its equilibrium position. This is interpreted to mean that the currents turn with the conductor. More precisely, it means that the fields and currents in the initial and final positions are different. But if this experiment is taken at its face value, it is contrary to the present theory, for it was performed in such a manner that the fields at infinity (i.e. the applied fields) are the same in the two positions. It follows from the uniqueness theorem that the fields should therefore have been the same everywhere. The experiment thus rules out, not only this theory, but every theory which has a similar uniqueness theorem.[2] In the absence of confirmation for the experiment, it appears reasonable to assume that the sphere was not homogeneously superconducting, but had non-

[1] K. H. Onnes, Comm. Leiden Suppl., No. 50a, 8–10 (1924).

[2] In particular, it is contrary to even the rigorous form of the theory resulting from the variation principle (cf. the previous paper, Section 4).

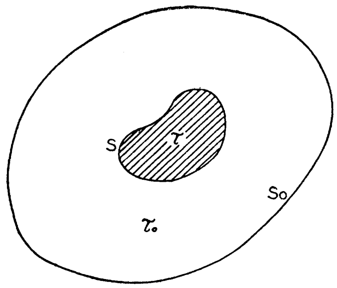

FIG. 1. A superconducting region τ, surrounded by free space τ_0, the entire system enclosed by a surface S_0.

spherical regions of ordinary conductivity in its interior.

Experience with electrodynamic theories of other types suggests that the boundary conditions across the air-metal interface are the continuity of **H** and $\mathbf{v}\times\mathbf{E}$ ($\mathbf{v}$ is the unit normal). The continuity of **H** is the physical assumption of unit permeability and zero surface density both of electric current and magnetic charge. The continuity of $\mathbf{v}\times E$ assumes unit dielectric constant and zero surface magnetic current but admits a surface electrical charge.

The imposition of these boundary conditions still leaves many solutions corresponding to the diversity of the possible physical situations. The physical situation is completely determined by the applied fields and the total charge on each conductor.[3] It is very satisfactory that the above boundary conditions, together with the values of the charge parameters (the data) and the behavior of the solutions at infinity (the boundary data), determine the mathematical solution uniquely. In all probability these various physical conditions are mathematically independent; their independence has been established in detailed calculations for an infinite cylinder.[4]

The procedure is to express the volume integral of $E^2+H^2+\lambda J^2$ in terms[5] of integrals over a surface S_0 surrounding the system and over the air-metal interface S across which discontinuities may occur (Fig. 1). These surface integrals vanish for **E**, **H**, **J** which satisfy the boundary conditions and have zero data and boundary, so such **E**, **H**, **J** are everywhere zero. The uniqueness theorem is then immediate, for the linearity of the equations enables us to consider the identically zero solution as the difference of two solutions which satisfy the boundary conditions and have the same data and boundary data.

In discussions of the steady state, the electric and magnetic fields may be treated independently. Equation (11) of the previous paper requires that $\mathbf{E}=0$ in a superconductor just as in a normal conductor; in free space the field **E** satisfies the usual Maxwell equations. The analysis for the interaction of an electrostatic field and a system of conductors is well known[6] and need not be discussed here. The result is that the boundary data,

$$\mathbf{v}\cdot\mathbf{E}, \tag{1}$$

specified on S_0, and the total charge Q,

$$Q=\int_S \mathbf{v}\cdot\mathbf{E}dS \tag{2}$$

specified for each conductor, determine the field **E** uniquely.

The usual proof of this uniqueness of the electrostatic field depends upon the existence of a scalar potential; but a generalization of the method is applicable to the magnetic field which has a vector potential. The equations governing the magnetic field are:[7]

$$\nabla\times\mathbf{H}=0,\quad \nabla\cdot\mathbf{H}=0 \quad \text{in } \tau_0, \tag{3}$$

$$\nabla\times\mathbf{H}=\mathbf{J},\quad \mathbf{H}=\nabla\times\mathbf{A},\quad \mathbf{A}+\lambda\mathbf{J}=0 \quad \text{in } \tau. \tag{4}$$

Let the indices 1,2 designate different solutions of these equations, and

$$H=\mathbf{H}_1-\mathbf{H}_2,\quad J=\mathbf{J}_1-\mathbf{J}_2,\quad A=\mathbf{A}_1-\mathbf{A}_2.$$

Putting $\mathbf{P}=\mathbf{Q}=A$ in the identity[8]

$$\nabla\cdot[\mathbf{P}\times(\nabla\times\mathbf{Q})]=(\nabla\times\mathbf{P})\cdot(\nabla\times\mathbf{Q})-\mathbf{P}\cdot\nabla\times(\nabla\times\mathbf{Q})$$

gives

$$\nabla\cdot[A\times H]=H^2$$

in free space τ_0, and

$$\nabla\cdot[A\times H]=H^2+\lambda J^2$$

in the superconducting region τ. Now, since $\mathbf{J}=0$ in τ_0,

[3] Only simply connected conductors are considered here. The permanent magnetic moment of a ring-like conductor will be discussed in a future paper.

[4] Unpublished.

[5] λ is the positive constant introduced by London.

[6] Sir James Jeans, *The Mathematical Theory of Electricity and Magnetism*, fifth edition (Macmillan Company, New York), p. 163.

[7] **A** differs from the usual Maxwellian potential which London employs (cf. the previous paper, Section 3).

[8] J. A. Stratton and L. J. Chu, Phys. Rev. **56**, 163 (1939), have emphasized the importance of this identity.

$$\int_{\tau+\tau_0}[H^2+\lambda J^2]d\tau=\int_{S_0}\nu\cdot A\times H dS + \int_S \nu\cdot A\times H\Big]_+^- dS, \quad (5)$$

where $]_+^-$ is used to designate the difference of the values of the preceding function on the two sides of the surface. If the surface integrals in Eq. (5) vanish, then $(\mathbf{E}_1, \mathbf{H}_1)=(\mathbf{E}_2, \mathbf{H}_2)$ throughout $\tau+\tau_0$.

The integral over S_0 vanishes if either $\nu\times A=0$ or $\nu\times H=0$. The latter is the more important physical case since the absolute value of $\mathbf{A}$ has no significance in free space. Thus the specification of the boundary data, $\nu\times\mathbf{H}$, corresponding to the applied field, i.e.,

$$\nu\times\mathbf{H}_1=\nu\times\mathbf{H}_2, \quad (6)$$

causes the first of the surface integrals to vanish.

The vanishing of the integral over S is not obtained quite so simply. The following identity, in which subscripts are used to denote the evaluation of the function on the positive or negative side of S,

$$\nu\cdot A\times H\Big]_+^- = H_-\cdot\nu\times(A_- - A_+) + A_+\cdot\nu\times(H_- - H_+), \quad (7)$$

insures the vanishing of this integral, provided

$$\nu\times\mathbf{A}, \text{ and } \nu\times\mathbf{H} \text{ are continuous across } S. \quad (8)$$

But the continuity of $\nu\times\mathbf{A}$ has no immediate physical interpretation; it is necessary to investigate the extent to which requirement (8) is equivalent to the expected boundary condition:

$$\nu\cdot\mathbf{H} \text{ and } \nu\times\mathbf{H} \text{ are continuous across } S. \quad (9)$$

The continuity of $\nu\times\mathbf{A}$ and $\nu\times\mathbf{H}$ insures the result

$$\int_S \nu\cdot A\times H\Big]_+^- dS=0 \quad (10)$$

through the vanishing of the integrand. The question now is whether or not the continuity of $\nu\cdot\mathbf{H}$ and $\nu\times\mathbf{H}$ also suffices. The answer is in the affirmative provided there exists in τ_0 a single valued and sufficiently continuous function χ such that

$$\nu\times\nabla\chi=\nu\times A\Big]_+^-. \quad (11)$$

For the two relations, $\nabla\times\mathbf{H}=0$ in τ_0, and $\mathbf{H}_+=\mathbf{H}_-$ on S, together with the previously assumed boundary data, $\nu\times H=0$ on S_0, then permit the evaluation of the integral:

$$\int_S \nu\cdot A\times H\Big]_+^- dS=\int_S \nu\cdot[(\nabla\chi)\times H]dS = -\int_{\tau_0}\nabla\cdot[(\nabla\chi)\times H]d\tau + \int_{S_0}\nu\cdot[(\nabla\chi)\times H]dS=0. \quad (12)$$

The question thus reduces to that of the existence of the function χ.

The function χ will now be constructed for a hollow sphere. The topological character of this construction is such that it is possible for any superconductor which can be obtained from a hollow sphere by continuous deformation, i.e. for one which is simply connected.

The notation is defined in Fig. 2. S_0 is the surface of the bounding sphere of radius r_0. S is the complete surface, inner and outer, of the hollow sphere; r_1, r_2 are its inner and outer radii. τ is the superconducting region, τ_0 is free space.

Now if S' is any part of S, and C is its contour, then the continuity of H gives

$$0=\int_{S'}\nu\cdot H\Big]_+^- dS=\int_{S'}\nu\cdot\nabla\times A\Big]_+^- = \int_C A\Big]_+^-\cdot d\mathbf{R}.$$

This enables us to define functions of the spherical coordinates (r, ϑ, φ):

$$\psi_1(\vartheta, \varphi)=\chi(r_1, \vartheta, \varphi)=\int_{0,0}^{\vartheta,\varphi}(A_- - A_+)\cdot d\mathbf{R}\Big]_{r=r_1},$$

$$\psi_2(\vartheta, \varphi)=\chi(r_2, \vartheta, \varphi)=\int_{0,0}^{\vartheta,\varphi}(A_- - A_+)\cdot d\mathbf{R}\Big]_{r=r_2},$$

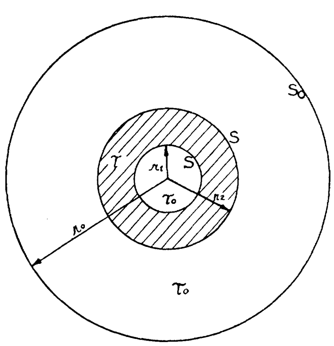

FIG. 2. A superconducting, hollow sphere τ, in free space τ_0, the entire system within a sphere of surface S_0.

which are single valued, continuous, and have the property (11).[9] There remains but to continue χ throughout the intervals $(0, r_1)$ and (r_2, r_0) so that it is defined at each point of τ_0. To this end a function $f(r)$ is introduced which is required merely to have continuous first and second derivatives and to take on the prescribed values

$$f(0)=0, \quad f(r_1)=1, \quad f(r_2)=1, \quad f(r_0) \text{ finite.}$$

The continuation

$$\chi(r, \vartheta, \varphi)=f(r)\vartheta_1(\vartheta, \varphi), \quad r\leqslant r_1,$$
$$\chi(r, \vartheta, \varphi)=f(r)\vartheta_2(\vartheta, \varphi), \quad r\geqslant r_2,$$

gives χ all the desired properties in τ_0 and on S and thus completes the proof that the conditions (6) and (9) determine the field uniquely, at least for a hollow sphere. But the spherical form of the superconductor entered only in providing a simple coordinate system for the definition of the function $\chi(r, \vartheta, \varphi)$. If the sphere is continuously deformed into an arbitrary shape, χ retains its continuity properties as well as the property (11), the only difference being that the coordinates (r, ϑ, φ) are no longer the simple spherical coordinates but are the parameters along the curves into which the original coordinate curves have been deformed. Thus conditions (6), (9) form a complete set for all simply connected superconductors.

The Nonsteady State

The general equations are

$$\left.\begin{aligned} &\nabla\times\mathbf{H}-\mathbf{E}'=0, \quad \nabla\cdot\mathbf{E}=0, \\ &\nabla\times\mathbf{E}+\mathbf{H}'=0, \quad \nabla\cdot\mathbf{H}=0, \end{aligned}\right\} \text{ in } \tau_0$$

$$\left.\begin{aligned} \nabla\times(\nabla\times\mathbf{J})+\beta^2\mathbf{J}+\mathbf{J}''=0,& \\ \mathbf{H}+\lambda\nabla\times\mathbf{J}=0,& \\ \mathbf{E}-\lambda\mathbf{J}'=0,& \end{aligned}\right\} \text{ in } \tau. \qquad (13)$$

[9] Up to this point no use had been made of the simple connectivity of the conductor. But here it is needed to obtain the single valuedness of the ψ's. In the case of a ring-like conductor, a cut must be made to render its surface simply connected. In the subsequent integral transformations, this cut contributes an additional surface integral which vanishes only when the two solutions have the same total current around the ring. The mathematics for this case has been worked out in detail and will be presented in a future paper.

The proof of the uniqueness theorem will be based upon the energy Eq. (12) of the previous paper. Terms of order higher than the second must be discarded in the linear approximation represented by Eqs. (13). There results

$$(\partial/\partial t)[\tfrac{1}{2}(\mathbf{H}^2+\mathbf{E}^2)+\tfrac{1}{2}\lambda\mathbf{J}^2]+\nabla\cdot[\mathbf{E}\times\mathbf{H}]=0,$$

which is valid in the composite region $\tau+\tau_0$ with the understanding that $\mathbf{J}=0$ in τ_0. If script letters are again used to denote the difference of two solutions, then

$$\int_{\tau+\tau_0}\int_0^t[\tfrac{1}{2}(H^2+\mathcal{E}^2)+\tfrac{1}{2}\lambda J^2]dtd\tau$$
$$=-\int_{S_0}\mathbf{v}\cdot\mathcal{E}\times HdS+\int_S\mathbf{v}\cdot\mathcal{E}\times H\Big]_-^+dS,$$

which gives immediately the result:

If two solutions, together with their first derivatives, are continuous except possibly across the air-metal interface S, satisfy the boundary conditions:

$$\mathbf{v}\times\mathbf{E} \text{ and } \mathbf{v}\times\mathbf{H} \text{ continuous for } t\geqslant 0, \qquad (14)$$
$$\mathbf{v}\cdot\mathbf{H} \text{ continuous at } t=0,$$

and have the same values for the boundary data,

$$\mathbf{v}\times\mathbf{E} \text{ or } \mathbf{v}\times\mathbf{H} \text{ for } t\geqslant 0, \qquad (15)$$

on the bounding surface S_0, as well as for the initial data

$$\mathbf{E}, \mathbf{H}, \mathbf{J}, \text{ in } \tau+\tau_0 \text{ at } t=0, \qquad (16)$$

then the solutions are identical in $\tau+\tau_0$ for $t\geqslant 0$ and $\mathbf{v}\cdot\mathbf{H}$ is continuous for each.

The continuity of $\mathbf{v}\cdot\mathbf{H}$ is not read out of the above integral transformation as are the other conclusions of the theorem; but is a consequence of the continuity of $\mathbf{v}\times\mathbf{E}$ and the initial continuity of $\mathbf{v}\cdot\mathbf{H}$. For any part S' of S,

$$\frac{\partial}{\partial t}\int_{S'}\mathbf{v}\cdot\mathbf{H}\Big]_-^+dS=-\int_{S'}\mathbf{v}\cdot\nabla\times\mathbf{E}\Big]_-^+dS$$
$$=-\int_C\mathbf{E}\Big]_-^+\cdot dR=0,$$

and so $\mathbf{v}\cdot\mathbf{H}]_-^+=0$ on S for $t\geqslant 0$.