K are not the same, so that the two calculations will give the same result only if f is independent of $\mathbf{k}_e$. This condition is satisfied for allowed transitions with the three couplings commonly employed, although not for forbidden transitions. A detailed study of the radiative beta-activity and radiative capture might thus give further information on the actual form of the beta-coupling.

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The Scattering of Thermal Neutrons by Deuterons*

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The cross section for the scattering of very slow neutrons by deuterons is calculated by numerical methods. Polarization is completely neglected and the wave equation for the process is set up in such a form as to take correctly into account exchange effects between the incident neutron and the neutron initially in the deuteron. This wave equation is then replaced by an integral equation the solution of which is correctly symmetrized and has the right asymptotic value to describe the scattering process. The numerical integration is performed by replacing the integral equation by a finite set of simultaneous linear algebraic equations. The work is greatly simplified by the use of a sum of two Gauss functions to approximate the ground state deuteron wave function. It is assumed throughout this paper that the interactions between like and unlike particles are equal and are of the general form

$$V_{ij}=-[(1-g-g_1-g_2)P_{ij}+gP_{ij}Q_{ij}+g_1+g_2Q_{ij}]J(r_{ij}),$$

where the symbols have their usual meanings and where $J(r_{ij})$ is a Gauss function. The calculation is carried out for two sets of g's. For the first set, $g_1=g_2=0$, $g=0.2$, the cross section is found to be equal to 4.57×10^{-24} cm², and for the second set of g's, $g_2=2$, $g=0.22-g_2$, $g_1=0.25-0.8g_2$, the value of the cross section is found to be equal to 6.91×10^{-24} cm². The experimental value is at least 20 percent smaller than the first of these values.

INTRODUCTION

SINCE the scattering of neutrons by deuterons involves a fundamental process in which only three particles take part, the solution of this problem can be expected to throw a great deal more light on the nature of the forces between elementary particles than is now available. This, coupled with the fact that an exact theoretical treatment is impossible because of the complexity of the equations, must justify a step-by-step attack on the problem in which various simplified models are considered in some detail. A complete treatment must take into account polarization as well as exchange effects. If polarization is entirely neglected, as has been done in the present calculation, it is possible to set the problem up in such a form that one can obtain a numerical solution without an undue amount of work. Calculations in which polarization has been neglected have already been carried out by Schiff[1] and by Yukawa and Sakata[2] who proceeded somewhat indirectly by introducing auxiliary potentials which enabled them to simplify their equations considerably. The present paper differs from the aforementioned ones in two respects: the calculations are carried through using the most general type of interaction between the particles (we assume only that the forces between like and unlike particles are equal); exchange effects are accurately taken account of to the order of approximation employed.

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† National Research Fellow.

[1] L. I. Schiff, Phys. Rev. **52**, 149 (1937).

[2] H. Yukawa and S. Sakata, Proc. Phys.-Math. Soc. Japan **19**, 542 (1937).

The Integral Equation of the Collision Process

The general form of the wave equation for a system consisting of three particles of equal mass and having a total spin S is

$$[-(\hbar^2/2M)(\Delta_1+\Delta_2+\Delta_3)+V_{12}+V_{13}+V_{23}-E] \times\Psi_S(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)=0, \quad (1)$$

where $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are the vector coordinates of the three particles, M is the common mass and V_{ij} are the interaction energies. In the present calculation we have taken the most general form for the potential[3]

$$V_{ij}=-[(1-g-g_1-g_2)P_{ij}+gP_{ij}Q_{ij} +g_1+g_2Q_{ij}]J(r_{ij}). \quad (2)$$

P_{ij} is the operator which exchanges the positions of particles i and j, and Q_{ij} is the spin exchange operator, which we write in the form $\frac{1}{2}(1+\sigma_i\cdot\sigma_j)$, where the σ_i, σ_j are Pauli spin matrices of unit amplitude.

We now transform the wave equation to the center of mass coordinate system, which is defined by the equations (see Fig. 1)

$$\mathbf{r}=\mathbf{r}_1-\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_3), \quad \boldsymbol{\rho}=\mathbf{r}_2-\mathbf{r}_3, \quad \mathbf{R}=\tfrac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3). \quad (3)$$

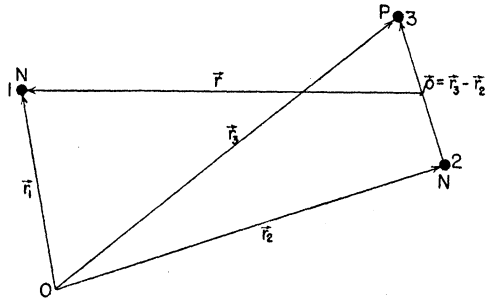

FIG. 1.

The subscript 1 refers to the incoming neutron and 2 to the neutron initially in the deuteron, so that $\boldsymbol{\rho}$ is the coordinate of the deuteron system. On factoring out the motion of the center of mass, we obtain the wave equation in the coordinates $\boldsymbol{\rho}$ and $\mathbf{r}$

$$[-(\hbar^2/2M)(2\Delta_\rho+\tfrac{3}{2}\Delta_r)+V_{12}+V_{13}+V_{23}-E] \times\Psi_S(\boldsymbol{\rho}, \mathbf{r})=0, \quad (4)$$

where the V_{ij}'s are now functions of ρ and r only.

If $\psi_0(\boldsymbol{\rho})$ and $\psi(\boldsymbol{\rho}, \epsilon)$ are the deuteron ground state and continuum wave functions, respectively, then $\Psi_S(\boldsymbol{\rho}, \mathbf{r})$ can be written as the following expansion:

$$\Psi_S(\boldsymbol{\rho}, \mathbf{r})=F_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+\chi_S(123)\int_0^\infty F(\mathbf{r}, \epsilon)\psi(\boldsymbol{\rho}, \epsilon)d\epsilon, \quad (5)$$

where $\chi_S(123)$ is the spin wave function for a definite value S of the total spin of the system. In order to take exchange effects into account, we set down a similar expansion for $P_{12}Q_{12}\Psi_S(\boldsymbol{\rho}, \mathbf{r})$

$$P_{12}Q_{12}\Psi_S(\boldsymbol{\rho}, \mathbf{r})=G_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+\chi_S(123)\int_0^\infty G(\mathbf{r}, \epsilon)\psi(\boldsymbol{\rho}, \epsilon)d\epsilon. \quad (6)$$

Substituting (5) and (6), both of which are solutions, into the wave equation, we obtain the two equations

$$\begin{aligned}
&-(\hbar^2/2M)(2\Delta_\rho+\tfrac{3}{2}\Delta_r)[F_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+F_S(\mathbf{r}\ \boldsymbol{\rho})]+[\sum_{i=2,3}V_{1i}+V_{23}][F_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123) \\
&\qquad +F_S(\mathbf{r}, \boldsymbol{\rho})]=E[F_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+F_S(\mathbf{r}, \boldsymbol{\rho})] \\
&-(\hbar^2/2M)(2\Delta_\rho+\tfrac{3}{2}\Delta_r)[G_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+G_S(\mathbf{r}, \boldsymbol{\rho})]+[\sum_{i=2,3}V_{1i}+V_{23}] \\
&\qquad \times[G_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+G_S(\mathbf{r}, \boldsymbol{\rho})]=E[G_0(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)+G_S(\mathbf{r}, \boldsymbol{\rho})],
\end{aligned} \quad (7)$$

where

$$F_S(\mathbf{r}, \boldsymbol{\rho})\equiv\chi_S(123)\int_0^\infty F(\mathbf{r}, \epsilon)\psi(\boldsymbol{\rho}, \epsilon)d\epsilon, \quad G_S(\mathbf{r}, \boldsymbol{\rho})\equiv\chi_S(123)\int_0^\infty G(\mathbf{r}, \epsilon)\psi(\boldsymbol{\rho}, \epsilon)d\epsilon.$$

[3] G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936).

These equations can now be reduced to simpler form by making use of the deuteron wave equation. If the binding energy of the deuteron is E_0, then $\psi_0(\boldsymbol{\rho})$ satisfies the following equation:

$$-(\hbar^2/M)\Delta_\rho\psi_0(\boldsymbol{\rho})+V_{23}\psi_0(\boldsymbol{\rho})=E_0\psi_0(\boldsymbol{\rho}).$$

Multiplying (7) by $\psi_0^*(\boldsymbol{\rho})\chi_S^*(123)$, integrating over the configuration space of $\boldsymbol{\rho}$, and summing over spins, we obtain with the aid of the deuteron equation

$$\begin{aligned}&\frac{3}{4}\frac{\hbar^2}{M}\Delta_r F_0(\mathbf{r})+\sum_{\text{spins}}\sum_{i=2,3}\chi_S^*(123)V_{1i}F_0(\mathbf{r})\chi_S(123)\\&\qquad+\sum_{\text{spins}}\sum_{i=2,3}\int d\tau_\rho\chi_S^*(123)\psi_0^*(\boldsymbol{\rho})V_{1i}F_S(\mathbf{r},\boldsymbol{\rho})=(E-E_0)F_0(\mathbf{r}),\\&\frac{3}{4}\frac{\hbar^2}{M}\Delta_r G_0(\mathbf{r})+\sum_{\text{spins}}\sum_{i=2,3}\chi_S^*(123)V_{1i}G_0(\mathbf{r})\chi_S(123)\\&\qquad+\sum_{\text{spins}}\sum_{i=2,3}\int d\tau_\rho\chi_S^*(123)\psi_0^*(\boldsymbol{\rho})V_{1i}G_S(\mathbf{r},\boldsymbol{\rho})=(E-E_0)G_0(\mathbf{r}).\end{aligned}\tag{8}$$

If in these equations we replace $F_S(\mathbf{r},\boldsymbol{\rho})$ and $G_S(\mathbf{r},\boldsymbol{\rho})$ by the equivalent expressions obtained from (5) and (6), and then take the difference of the resulting equations, we have left the equation

$$\begin{aligned}&\frac{3}{4}\frac{\hbar^2}{M}\Delta_r[F_0(\mathbf{r})-G_0(\mathbf{r})]+\sum_{\text{spins}}\sum_{i=2,3}\int d\tau_\rho\chi_S^*\psi_0^*(\boldsymbol{\rho})V_{1i}[\Psi_S(\mathbf{r},\boldsymbol{\rho})-P_{12}Q_{12}\Psi_S(\mathbf{r},\boldsymbol{\rho})]\\&\qquad=(E-E_0)[F_0(\mathbf{r})-G_0(\mathbf{r})].\end{aligned}\tag{9}$$

This equation is exact, but before we can proceed further, we must introduce an approximation which is equivalent to neglecting polarization effects. We place

$$\Psi_S(\mathbf{r},\boldsymbol{\rho})-P_{12}Q_{12}\Psi_S(\mathbf{r},\boldsymbol{\rho})=F_0(\mathbf{r})-G_0(\mathbf{r})-P_{12}Q_{12}[F_0(\mathbf{r})-G_0(\mathbf{r})]\equiv\phi(\mathbf{r})-P_{12}Q_{12}\phi(\mathbf{r}).\tag{10}$$

With the aid of (10) we can now obtain an equation for $\phi(\mathbf{r})$,

$$\frac{3}{4}\frac{\hbar^2}{M}\Delta\phi(\mathbf{r})+\sum_{\text{spins}}\sum_{i=2,3}\int d\tau_\rho\chi_S^*\psi_0^*(\boldsymbol{\rho})V_{1i}[\phi(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)-P_{12}Q_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S(123)]=E'\phi(\mathbf{r}),\tag{11}$$

where we have placed $E-E_0=E'$. If polarization is neglected, then $\phi(\mathbf{r})$ is the correctly symmetrized wave function of the system.

We can simplify (11) by carrying out the summation over the spins. To do this we first introduce the expression (2) for the potential functions, and thus obtain the equation

$$\begin{aligned}&\frac{3}{4}\frac{\hbar^2}{M}\Delta\phi(\mathbf{r})-\sum_{\text{spins}}\int d\tau_\rho\chi_S^*\psi_0^*(\boldsymbol{\rho})\{(g_1+\mathfrak{M}P_{12}+g_2Q_{12}+gP_{12}Q_{12})J(r_{12})[\phi(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S\\&\qquad-P_{12}Q_{12}(\phi(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S)]+(g_1+\mathfrak{M}P_{13}+g_2Q_{13}+gP_{13}Q_{13})\\&\qquad\times J(r_{13})[\phi(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S-P_{12}Q_{12}(\phi(\mathbf{r})\psi_0(\boldsymbol{\rho})\chi_S)]\}=E'\phi(\mathbf{r}),\end{aligned}\tag{12}$$

where we have placed $\mathfrak{M}=1-g-g_1-g_2$.

On performing the multiplications indicated in the integrand, we see that the following terms are present.

(a) Three terms involving no exchange operators

$$g_1\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{12})\psi_0(\boldsymbol{\rho}),$$

$$-g\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{12})\psi_0(\boldsymbol{\rho}), \tag{13}$$

$$g_1\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})\psi_0(\boldsymbol{\rho}).$$

(b) Three terms involving only spin exchange operators

$$g_2\phi(\mathbf{r})\sum_{\text{spins}}\chi_S^*Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{12})\psi_0(\boldsymbol{\rho}),$$

$$-\mathfrak{M}\phi(\mathbf{r})\sum_{\text{spins}}\chi_S^*Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{12})\psi_0(\boldsymbol{\rho}), \tag{14}$$

$$g_2\phi(\mathbf{r})\sum_{\text{spins}}\chi_S^*Q_{13}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})\psi_0(\boldsymbol{\rho}).$$

(c) Three terms involving only space exchange operators

$$\mathfrak{M}\int d\tau_\rho J(r_{12})\psi_0^*(\boldsymbol{\rho})P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}),$$

$$-g_2\int d\tau_\rho J(r_{12})\psi_0^*(\boldsymbol{\rho})P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}), \tag{15}$$

$$\mathfrak{M}\int d\tau_\rho J(r_{13})\psi_0^*(\boldsymbol{\rho})P_{13}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}).$$

(d) Seven terms involving both spin and space exchange operators

$$-g_1\sum_{\text{spins}}\chi_S^*Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{12})P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}),$$

$$g\sum_{\text{spins}}\chi_S^*Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{12})P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}),$$

$$-g_1\sum_{\text{spins}}\chi_S^*Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}),$$

$$\mathfrak{M}\sum_{\text{spins}}\chi_S^*Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})P_{13}P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}), \tag{16}$$

$$-g_2\sum_{\text{spins}}\chi_S^*Q_{13}Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}),$$

$$g\sum_{\text{spins}}\chi_S^*Q_{13}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})P_{13}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}),$$

$$-g\sum_{\text{spins}}\chi_S^*Q_{13}Q_{12}\chi_S\int d\tau_\rho\psi_0^*(\boldsymbol{\rho})J(r_{13})P_{13}P_{12}\phi(\mathbf{r})\psi_0(\boldsymbol{\rho}).$$

To evaluate the sums over the spins, we need only consider the term $\sum\chi_S^*Q_{12}\chi_S$, since $Q_{13}=Q_{13}Q_{12}=Q_{12}$. This is immediately evident from the fact that the spins of the neutron and proton in the deuteron are parallel. We have in fact

$$\begin{aligned} Q_{12}&=\tfrac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\\ &=\tfrac{1}{2}[1+\boldsymbol{\sigma}_1\cdot(\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)/2]\\ &=\tfrac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_D)=Q_{13}, \end{aligned} \tag{17}$$

where $\boldsymbol{\sigma}_D$ is the spin matrix for the deuteron. Also

$$\begin{aligned} Q_{13}Q_{12}\chi_S(\overline{123})=Q_{13}\chi_S(\overline{213})&=\chi_S(\overline{231})\\ &=\chi_S(\overline{213})\\ &=Q_{12}\chi_S(\overline{123}), \end{aligned} \tag{18}$$

where the bar has been used to indicate the deuteron unit. We can now calculate the matrix element $\sum\chi_S^*Q_{12}\chi_S$ by making use of (17). We have

$$\begin{aligned} \sum_{\text{spins}}\chi_S^*Q_{12}\chi_S&=\tfrac{1}{2}(\sum\chi_S^*\chi_S+\tfrac{1}{2}\sum\chi_S^*\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_D\chi_S)\\ &=\tfrac{1}{2}(1+\tfrac{1}{2}\sum\chi_S^*\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_D\chi_S). \end{aligned} \tag{19}$$

We can evaluate the sum in (19) by noting that

$$\begin{aligned} \tfrac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_D&=\tfrac{1}{4}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_D)^2-\tfrac{1}{4}\boldsymbol{\sigma}_1^2-\tfrac{1}{4}\boldsymbol{\sigma}_D^2\\ &=(\tfrac{1}{2}\boldsymbol{\sigma}_1+\tfrac{1}{2}\boldsymbol{\sigma}_D)^2-(\tfrac{1}{2}\boldsymbol{\sigma}_1)^2-(\tfrac{1}{2}\boldsymbol{\sigma}_D)^2. \end{aligned} \tag{20}$$

If we remember that the spin of the deuteron is one and denote by S the total spin of our system, then we obtain from (20)

$$\begin{aligned} \tfrac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_D&=S(S+1)-(3/4)-2\\ &=S(S+1)-(11/4). \end{aligned}$$

Hence

$$\begin{aligned} \sum\chi_S^*Q_{12}\chi_S=\sum\chi_S^*Q_{13}\chi_S&=\sum\chi_S^*Q_{12}Q_{13}\chi_S\\ &=\tfrac{1}{2}[S(S+1)-(7/4)]. \end{aligned} \tag{21}$$

We must now consider the space exchange operators. From Fig. 1 and Eqs. (3) it can be seen that

$$\begin{aligned} P_{12}\mathbf{r}=P_{12}[\mathbf{r}_1-\tfrac{1}{2}(\mathbf{r}_3+\mathbf{r}_2)]&=\mathbf{r}_2-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_3)\\ &=\mathbf{r}-(3/2)[\mathbf{r}+\tfrac{1}{2}(\mathbf{r}_3-\mathbf{r}_2)]\\ &=-\tfrac{1}{2}(\mathbf{r}+(3/2)\boldsymbol{\varrho}),\\ P_{12}\boldsymbol{\varrho}=P_{12}(\mathbf{r}_3-\mathbf{r}_2)=\mathbf{r}_3-\mathbf{r}_1&=\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}\equiv-\mathbf{r}_{13},\\ P_{13}\mathbf{r}=\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)&=\tfrac{1}{2}(3\boldsymbol{\varrho}/2-\mathbf{r}),\\ P_{13}\boldsymbol{\varrho}=\mathbf{r}_1-\mathbf{r}_2&=-(\mathbf{r}+\tfrac{1}{2}\boldsymbol{\varrho})\equiv\mathbf{r}_{12},\\ P_{13}P_{12}\mathbf{r}=P_{12}\mathbf{r}&=-\tfrac{1}{2}(\mathbf{r}+3\boldsymbol{\varrho}/2),\\ P_{13}P_{12}\boldsymbol{\varrho}=P_{13}(\mathbf{r}_3-\mathbf{r}_1)&=-(\mathbf{r}_1-\mathbf{r}_3)=-P_{12}\boldsymbol{\varrho}=\mathbf{r}-\tfrac{1}{2}\boldsymbol{\varrho}. \end{aligned} \tag{22}$$

If we now introduce (21) and (22) into (14), (15), and (16) and collect terms, we obtain the follow-

ing integrals:

$$[g_1-g+\tfrac{1}{2}(g_2-\mathfrak{M})\{S(S+1)-(7/4)\}]\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\varrho}|)\psi_0(\boldsymbol{\varrho}),$$

$$[g_1+\tfrac{1}{2}g_2\{S(S+1)-(7/4)\}]\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}|)\psi_0(\boldsymbol{\varrho}),$$

$$[\mathfrak{M}-g_2+\tfrac{1}{2}(g-g_1)\{S(S+1)-(7/4)\}]\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\mathbf{r}+\tfrac{1}{2}\boldsymbol{\varrho}|)\phi(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\varrho})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}), \tag{23}$$

$$[\mathfrak{M}+\tfrac{1}{2}g\{S(S+1)-(7/4)\}]\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}|)\phi(\tfrac{3}{4}\boldsymbol{\varrho}-\tfrac{1}{2}\mathbf{r})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}+\mathbf{r}),$$

$$-[\mathfrak{M}+g_1+g_2+g]\tfrac{1}{2}[S(S+1)-(7/4)]\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}|)\phi(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\varrho})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}).$$

Since all the functions appearing under the integral signs depend only on the magnitudes of their vector arguments, we can combine the first and second and the third and fourth of the expressions (23). We are then left with three terms

$$[2g_1-g+(2g_2-\mathfrak{M})\kappa]\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}+\mathbf{r}|)\psi_0(\boldsymbol{\varrho}),$$

$$[2\mathfrak{M}-g_2+(2g-g_1)\kappa]\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}+\mathbf{r}|)\phi(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\varrho})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}), \tag{24}$$

$$-\kappa\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}|)\phi(\tfrac{3}{4}\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{r})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}).$$

In these expressions we have introduced the abbreviation $\kappa=[\frac{1}{2}S(S+1)-(7/4)]$, and we have also made use of the definition of $\mathfrak{M}$.

On substituting (24) into the wave equation, we obtain

$$\Delta\phi(\mathbf{r})+k^2\phi(\mathbf{r})=(2\mu/\hbar^2)\Big\{\kappa\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r}|)\phi(\tfrac{3}{4}\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{r})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r})$$
$$-[2\mathfrak{M}-g_2+(2g-g_1)\kappa]\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}+\mathbf{r}|)\phi(\tfrac{1}{2}\mathbf{r}+\tfrac{3}{4}\boldsymbol{\varrho})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\mathbf{r})$$
$$-[2g_1-g+(2g_2-\mathfrak{M})\kappa]\phi(\mathbf{r})\int d\tau_\rho\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}+\mathbf{r}|)\psi_0(\boldsymbol{\varrho})\Big\} \tag{25}$$

where we have placed $\mu=\frac{2}{3}M$ and $k^2=(2\mu/\hbar^2)E'$. Since we are seeking the solution of this equation which has the asymptotic form

$$\phi(\mathbf{r})\rightarrow\exp[i\mathbf{k}\cdot\mathbf{r}]+(1/r)e^{ikr}f(\theta),$$

we may replace (25) by the integral equation[4]

$$\phi(\mathbf{r})=\exp[i\mathbf{k}\cdot\mathbf{r}]+\frac{\mu}{2\pi\hbar^2}\Big\{[2\mathfrak{M}-g_2+(2g-g_1)\kappa]\int\int\frac{d\tau_\xi d\tau_\rho}{|\mathbf{r}-\xi|}\exp[ik|\mathbf{r}-\xi|]\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}+\xi|)\psi(\tfrac{3}{4}\boldsymbol{\varrho}+\tfrac{1}{2}\boldsymbol{\xi})$$
$$\times\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\boldsymbol{\xi})+[2g_1-g+(2g_2-\mathfrak{M})\kappa]\int\int d\tau_\xi d\tau_\rho\frac{\exp[ik|\mathbf{r}-\xi|]}{|\mathbf{r}-\xi|}\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}+\boldsymbol{\xi}|)\psi_0(\boldsymbol{\varrho})\phi(\boldsymbol{\xi})$$
$$-\kappa\int\int d\tau_\xi d\tau_\rho\frac{\exp[ik|\mathbf{r}-\xi|]}{|\mathbf{r}-\xi|}\psi_0^*(\boldsymbol{\varrho})J(|\tfrac{1}{2}\boldsymbol{\varrho}-\boldsymbol{\xi}|)\phi(\tfrac{3}{4}\boldsymbol{\varrho}+\tfrac{1}{2}\boldsymbol{\xi})\psi_0(\tfrac{1}{2}\boldsymbol{\varrho}-\boldsymbol{\xi})\Big\}. \tag{26}$$

[4] N. E. Mott and H. S. Massey, *The Theory of Atomic Collisions* (Oxford Univ. Press, New York, 1933), Chapter IV.

We can simplify this equation slightly by introducing new variables. We first replace ϱ by $-\varrho$, which results in the following changes:

$$J(|\tfrac{1}{2}\varrho+\xi|)\rightarrow J(|\xi-\tfrac{1}{2}\varrho|),$$
$$\phi(\tfrac{3}{4}\varrho+\tfrac{1}{2}\xi)\rightarrow\phi(\tfrac{1}{2}\xi-\tfrac{3}{4}\varrho),$$
$$\psi_0(\tfrac{1}{2}\varrho-\xi)\rightarrow\psi_0(\tfrac{1}{2}\varrho+\xi).$$
$$J(|\tfrac{1}{2}\varrho-\xi|)\rightarrow J(|\tfrac{1}{2}\varrho+\xi|),$$

all the other quantities in the integrands remaining unchanged. We next introduce the new variables r' and r'' defined by

$$\mathbf{r}'=\tfrac{3}{4}\varrho-\tfrac{1}{2}\xi;\quad \mathbf{r}''=\xi$$

in the first and third integrals and by

$$\mathbf{r}'=\xi,\quad \mathbf{r}''=\varrho$$

in the second integral. (26) now becomes

$$\phi(\mathbf{r})=\exp[i\mathbf{k}\cdot\mathbf{r}]+\frac{32\mu}{27\pi\hbar^2}[2\mathfrak{M}-g_2+(2g-g_1)\varkappa]\int\int\frac{d\tau' d\tau''}{|\mathbf{r}-\mathbf{r}''|}\exp[ik|\mathbf{r}-\mathbf{r}''|]\psi_0^*((4/3)\mathbf{r}'+\tfrac{2}{3}\mathbf{r}'')$$
$$\times J(|\tfrac{2}{3}(\mathbf{r}''-\mathbf{r}')|)\psi_0(\tfrac{2}{3}\mathbf{r}'+(4/3)\mathbf{r}'')\phi(\mathbf{r}')+\frac{\mu}{2\pi\hbar^2}[2g_1-g+(2g_2-\mathfrak{M})\varkappa]\int\int\frac{d\tau' d\tau''}{|\mathbf{r}-\mathbf{r}'|}$$
$$\times\exp[ik|\mathbf{r}-\mathbf{r}'|]\psi_0^*(\mathbf{r}'')J(|\tfrac{1}{2}\mathbf{r}''+\mathbf{r}'|)\psi_0(\mathbf{r}'')\phi(\mathbf{r}')-\frac{32\mu}{27\pi\hbar^2}\varkappa\int\int\frac{d\tau' d\tau''}{|\mathbf{r}-\mathbf{r}''|}$$
$$\times\psi_0^*((4/3)\mathbf{r}'+\tfrac{2}{3}\mathbf{r}'')J(|\tfrac{2}{3}\mathbf{r}'+(4/3)\mathbf{r}''|)\psi_0(\tfrac{2}{3}\mathbf{r}'+(4/3)\mathbf{r}'')\phi(\mathbf{r}'). \quad (27)$$

In this paper we shall consider only the scattering of very slow neutrons, so that we may place $k=0$ in (27). On doing this we obtain the integral equation in its final form

$$\phi(\mathbf{r})=1+\frac{32\mu}{27\pi\hbar^2}[2\mathfrak{M}-g_2+(2g-g_1)\varkappa]\int\int\frac{d\tau' d\tau''}{|\mathbf{r}-\mathbf{r}''|}\psi_0^*((4/3)\mathbf{r}'+\tfrac{2}{3}\mathbf{r}'')J(|\tfrac{2}{3}\mathbf{r}''-\tfrac{2}{3}\mathbf{r}'|)\psi_0(\tfrac{2}{3}\mathbf{r}'+(4/3)\mathbf{r}'')$$
$$\times\phi(\mathbf{r}')+\frac{\mu}{2\pi\hbar^2}[2g_1-g+(2g_2-\mathfrak{M})\varkappa]\int\int\frac{d\tau' d\tau''}{|\mathbf{r}-\mathbf{r}'|}\psi_0^*(\mathbf{r}'')J(|\tfrac{1}{2}\mathbf{r}''+\mathbf{r}'|)\psi_0(\mathbf{r}'')\phi(\mathbf{r}')$$
$$-\frac{32\mu}{27\pi\hbar^2}\varkappa\int\int\frac{d\tau' d\tau''}{|\mathbf{r}-\mathbf{r}''|}\psi_0^*((4/3)\mathbf{r}'+\tfrac{2}{3}\mathbf{r}'')J(|\tfrac{2}{3}\mathbf{r}'+(4/3)\mathbf{r}''|)\psi_0(\tfrac{2}{3}\mathbf{r}'+(4/3)\mathbf{r}'')\phi(\mathbf{r}'). \quad (28)$$

Transforming the Integral Equation to a Form Suitable for Numerical Integration

Before we can proceed to the numerical integration of (28), we must specify the functional forms of J and $\psi_0(r)$. To facilitate the numerical work, we must choose these so as to make the integrands fall off very rapidly. For this reason we have chosen a Gauss function for J

$$J(|\mathbf{r}_{ij}|)=V_0\exp[-r_{ij}^2/a^2]$$

with the depth V_0 and the range a the same for the interactions between like and unlike particles.

For a Gauss potential the ground state wave function of the deuteron can be obtained only by quadratures, so that we have a graph but not an analytical expression for this function. To simplify the calculations we have replaced the numerical solution for ψ_0 by a sum of two Gauss functions.

For $V_0=72mc^2$ and for $a=0.25(\hbar/(Mm)^{\frac{1}{2}}c)$ cm [5]

[5] In a recent paper, Phys. Rev. **55**, 1018 (1939) Breit, Thaxton and Eisenbud have suggested a somewhat shorter range and deeper well than those here used. Since our calculations had already proceeded considerably at the time of the publication of this paper, we did not use these values. An examination of the integral equation (31) would seem to indicate that the use of the smaller range would reduce and thus somewhat improve the value of the cross section obtained.

the function

$$\psi_0(r)=(1.23\pi^{\frac{3}{2}}a^3)^{-\frac{1}{2}} \times(0.80e^{-0.60(r^2/a^2)}+0.12e^{-0.056(r^2/a^2)}) \quad (29)$$

approximates the exact wave function sufficiently well for our purpose. The graphical fit is extremely good for small values of r/a, and only for $r/a>4$ does the deviation from the exact curve begin to become important. As a check on the wave function (29), we calculated the amount of binding energy given by it and found that it gives all but about 6 percent.

With these expressions for J and ψ_0 we can now simplify (28) by integrating over the configuration space of r''. All the integrations which have to be performed are of the type

$$\int \frac{d\tau_{\mathbf{X}}}{X}\exp\left\{-\frac{1}{a^2}\{(m^2\beta+n^2\alpha+p^2)r'^2+(n^2\beta+m^2\alpha+q^2)r^2+(n^2\beta+m^2\alpha+q^2)X^2 +2mn(\beta+\alpha+pq/mn)\mathbf{r}\cdot\mathbf{r}'-2[mn(\alpha+\beta+pq/mn)\mathbf{r}'+(m^2\alpha+\beta^2+q^2)\mathbf{r}]\cdot\mathbf{X}\},\right.$$

where we have placed $\mathbf{X}=\mathbf{r}-\mathbf{r}''$ and where the constants m, n, p, q are the coefficients of $\mathbf{r}'$ and $\mathbf{r}''$ in the arguments of ψ_0 and J. α and β are the constants which appear in the exponents of (29) which we rewrite in the form

$$\psi_0(r)=A\ \exp\left[-\alpha(r/a)^2\right]+B\ \exp\left[-\beta(r/a)^2\right].$$

The integration over $\mathbf{X}$ can be carried out by transforming to spherical coordinates and introducing

$$2\{mn[\alpha+\beta+(pq/mn)]\mathbf{r}'+(m^2\alpha+\beta^2+q^2)\mathbf{r}\}$$

as the polar axis. After the integration over $\mathbf{X}$ is performed, there remains an integral over the $\mathbf{r}'$ space.

Since $\phi(\mathbf{r}')$ is independent of angle, we can integrate over the angles and thus obtain the integral equation in a form which is suitable for numerical calculation. Since the details of the analytical integration are straightforward and uninteresting, we shall merely give the equation in its final form

$$\begin{aligned}\phi(r)=1+\frac{24\pi}{\hbar^2}\frac{\mu V_0a^4}{r}\Bigg\{&[2\mathfrak{M}-g_2+(2g-g_1)\kappa]\int_0^\infty dr'r'\phi(r')\sum_{i,j=1,2}A_{ij}\frac{\exp\left[-\frac{4(\alpha^{ij}+\beta^{ij}+\alpha^{ij}\beta^{ij})}{a^2(4\alpha^{ij}+\beta^{ij}+1)}r'^2\right]}{(2\alpha^{ij}+2\beta^{ij}-1)(4\alpha^{ij}+\beta^{ij}+1)}\\ &\times\Omega\left[\frac{2(2\alpha^{ij}+2\beta^{ij}-1)r'+2(4\alpha^{ij}+\beta^{ij}+1)r}{3a(4\alpha^{ij}+\beta^{ij}+1)^{\frac{1}{2}}}\right]+\frac{2\pi^{\frac{1}{2}}}{3}[2g_1-g+(2g_2-\mathfrak{M})\kappa]\int_0^\infty dr'r'^2\phi(r')\\ &\times\sum_{i,j=1,2}A_{ij}\frac{\exp\left[-\frac{4}{a^2}\frac{(\alpha^{ij}+\beta^{ij})}{(4\alpha^{ij}+4\beta^{ij}+1)}r'^2\right]}{(1+4\alpha^{ij}+4\beta^{ij})^{\frac{3}{2}}}-\kappa\int_0^\infty dr'r'\phi(r')\\ &\sum_{i,j=1,2}A_{ij}\frac{\exp\left[-\frac{4}{a^2}\frac{\beta^{ij}(\alpha^{ij}+1)}{(4\alpha^{ij}+\beta^{ij}+4)}r'^2\right]}{(2\alpha^{ij}+2\beta^{ij}+2)(\beta^{ij}+4\alpha^{ij}+4)}\Omega\left[\frac{2(2\alpha^{ij}+2\beta^{ij}+2)r'+2(\beta^{ij}+4\alpha^{ij}+4)r}{3a(\beta^{ij}+4\alpha^{ij}+4)^{\frac{1}{2}}}\right]\Bigg\},\end{aligned} \quad (30)$$

where we have used the following notation:

$$A_{11}=A^2;\quad A_{12}=A_{21}=AB;\quad A_{22}=B^2;\quad \alpha^{11}=\beta^{11}=\alpha;\quad \alpha^{12}=\beta^{12}=\alpha;\quad \beta^{21}=\alpha^{21}=\beta;\quad \alpha^{22}=\beta^{22}=\beta.$$

The function $\Omega(br'+cr)$ which appears in the integrands of (30) is an abbreviation for the difference of the integrals of error functions

$$\Omega(br'+cr)=\Lambda(br'+cr)-\Lambda(br'-cr),$$

where

$$\Lambda(x)=\int_0^x E(y)dy,\quad E(y)=\int_0^y e^{-z^2}dz.$$

If we place $r/a=x$ and $r'/a=\xi$, and introduce the numerical values for A, B, α, β into (30), we obtain

$$\phi(ax)=1+\frac{24}{\hbar^2}\frac{\mu}{\pi^{\frac{1}{2}}}\frac{V_0a^2}{(1.23)x}\Big\{[2\mathfrak{M}-g_2+(2g-g_1)\varkappa]\int_0^\infty d\xi\xi\phi(a\xi)[0.115e^{-1.56\xi^2}\Omega(1.33x+0.467\xi)$$
$$+0.0890e^{-0.798\xi^2}\Omega(1.24x+0.112\xi)+0.169e^{-1.51\xi^2}\Omega(0.900x+0.154\xi)$$
$$+0.0145e^{-0.360}\Omega(0.900x+0.458\xi)]+\frac{2\pi^{\frac{1}{2}}}{3}[2g_1-g+(2g_2-\mathfrak{M})\varkappa]\int_0^\infty d\xi\xi^2\phi(a\xi)$$
$$\times[0.0113e^{-0.828\xi^2}+0.0294e^{-0.724\xi^2}+0.00826e^{-0.309\xi^2}]$$
$$-\varkappa\int_0^\infty d\xi\xi\phi(a\xi)[0.0208e^{-0.549\xi^2}\Omega(1.76x+1.11\xi)+0.00449e^{-0.0555\xi^2}\Omega(1.70x+0.869\xi)$$
$$+0.00601e^{-0.525\xi^2}\Omega(1.46x+1.01\xi)+0.00144e^{-0.0553\xi^2}\Omega(2.03x+0.487\xi)]\Big\}. \quad (31)$$

The Numerical Integration

We integrated (31) numerically for two different sets of constants $\mathfrak{M}$, g, g_1, g_2. This involved solving four sets of equations because both the doublet scattering ($S=\frac{1}{2}$) and the quartet scattering ($S=\frac{3}{2}$) had to be calculated separately for each set of constants. The method adopted for solving the equations was that of replacing the integrals in (31) by sums extended over a finite number of rapidly converging terms. A brief examination of the exponentials and the functions $\Omega(bx+c\xi)$ which appear in the integrands (tables of these functions were prepared with little labor) shows that for $\xi \gtrless 13$ the contributions to the integrals are zero to our order of accuracy.

By means of Simpson's one-third rule we replaced each integral by a sum of twenty terms extended over the interval from $\xi=0$ to $\xi=13$. In this way we obtained a set of twenty simultaneous equations for each of the four groups of constants. Because of the presence of the Ω functions and the rapid convergence of the other factors in the integrands, the coefficients of the $\phi(ax_i)$ $(i=0, 1, \cdots)$ in the later equations approach constant values quite rapidly. This enabled us to reduce our equations to the following form with very little labor:

$$\begin{array}{lllll}
a_{00}\phi(0)+\sum_{i=1}^{19} a_{0i}\phi(x_ia) & & & & =C_0,\\
0 \quad + & a_{11}\phi(x_1a) & +\sum_{i=2}^{19} a_{1i}(x_ia) & & =C_1,\\
0 \quad + & 0 & +\ a_{22}\phi(x_2a) & +\sum_{i=3}^{19} a_{2i}\phi(x_ia) & =C_2,\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
0 \quad + & 0 & +\ 0 & +\cdots+a_{19,19}\phi(13a) & =C_{19}.
\end{array}$$

From the last of these equations the value of ϕ(13a) can be read off directly, and since this is the asymptotic value of the wave function, the total cross section can be expressed in terms of it as follows:

$$\sigma_t=\int\sigma(\theta)d\omega=2\pi\int|f(\theta)|^2\sin\theta d\theta$$
$$=2\pi\int(13a)^2|\phi(13a)-1|^2\sin\theta d\theta. \quad (32)$$

We shall now give the results of our calculations for two special choices of the potential.

Case I. $g_1=g_2=0$; $g=0.2$; $\mathfrak{M}=0.8$.

It has been shown by Breit and Feenberg[3] that the values of the g's which may be selected in an interaction of type (2) are restricted by certain inequalities which are obtained from the consideration of the stability of nuclei. The simplest set of values satisfying all the requirements is obtained by placing $g_1=g_2=0$. Combining this with the empirical value $g+g_2=0.2$, we obtain $g=0.2$.

If we substitute this set of values in (31), the numerical integration can be performed for each value of $\varkappa$ separately. Since the doublet and quartet states are orthogonal,[1] no doublet→quartet or quartet→doublet transitions take place during the process. For this reason we may calculate the doublet and quartet cross sections separately and then combine the two in the ratio 2 : 1 to give the total cross section.

For the doublet (d) scattering we have $S=\frac{1}{2}$, $\varkappa=-\frac{1}{2}$, and we find from our equations $\phi_d(13a)=1.055$. Hence we have

$$|f_d(\theta)|=|\phi_d(13a)-1|(13a)$$
$$=1.59\times10^{-13}\text{ cm}. \quad (33)$$

For the quartet (q) scattering we have $S=\frac{3}{2}$, $\varkappa=1$, and we obtain $\phi_q(13a)=0.749$. This gives us

$$|f_q(\theta)|=|\phi_q(13a)-1|(13a)$$
$$=7.30\times10^{-13}\text{ cm}. \quad (34)$$

From (33) and (34) we can now obtain the total cross section

$$\sigma_t=(4\pi/3)(2|f_q(\theta)|^2+|f_d(\theta)|^2)$$
$$=4.57\times10^{-24}\text{ cm}^2. \quad (35)$$

Case II. $g=0.22-g_2$; $g_1=0.25-0.8g_2$; $g_2=2$.

Another possible choice of g values has been given by Inglis.[6] The restrictions which are to be placed on the g's can be summarized by means of the inequality

$$1.25\geq 1+5g_1+4g_2.$$

If a low enough binding energy is to be obtained for Li^6, the choice of g's must be limited to a one-parameter family defined by the relations

$$g=0.22-g_2; \quad g_1=0.25-0.8g_2.$$

With these relations the demands of the experimental data are best met by choosing $g_2=2$. It should be noted that for this case the potential function is a mixture of all four types of interactions.

Proceeding just as we did for case I, we obtain the following results:

$$\phi_d(13a)=0.820; \quad f_d(\theta)=5.24\times10^{-13}\text{ cm}$$
$$\phi_q(13a)=0.715; \quad f_q(\theta)=8.29\times10^{-13}\text{ cm}. \quad (36)$$

From (36) we obtain the total cross section for this case

$$\sigma_t=(4\pi/3)(2|f_q(\theta)|^2+|f_d(\theta)|^2)$$
$$=6.91\times10^{-24}\text{ cm}^2. \quad (37)$$

Comparison with Experiment

The best experimental data on the scattering of slow neutrons by deuterons are to be found in the recent work of Carroll and Dunning (unpublished),[7] who obtained a value of 5.7×10^{-24} cm^2 for the cross section. If one takes molecular binding into account, this value has to be reduced by a factor that lies between 1.5 and 2[1,2] before it can be compared with the theoretical values. The actual value of the reduction factor is determined by the strength of the molecular binding; it is equal to 2 for infinite binding but may be as small as 1.5 for weak binding.

We see that the experimental value agrees best with the theoretical value obtained for case I. This would seem to argue in favor of the first set of g's (Heisenberg and Majorana forces only) as against that given by Inglis (Wigner and Bartlett forces in addition to the other two).

Even if we take 1.5 as the reduction factor, we see that the experimental value is about 20 percent lower than the theoretical value found for case I. This discrepancy can certainly not be accounted for by the approximations introduced

[6] D. R. Inglis, Phys. Rev. **51**, 531 (1937).

[7] We should like to express our thanks to Mr. Carroll and Professor Dunning for informing us of their results before publication.

to enable us to carry out the numerical integration of Eq. (28). These were twofold: The ground state wave function of the deuteron was approximated by a sum of two Gauss functions, and the integrals were replaced by sums over finite intervals. The first of these approximations can at most account for a few percent of the discrepancy because the assumed wave function for the deuteron deviates measurably from the true wave function only for large values of r/a, and it is just for these values of r/a that the contributions to the integrals are negligible. As we have already seen, the second approximation is not serious because of the rapid convergence of the integrals.

Since we have neglected polarization in this paper, it may well be that taking it into account will get rid of most of the discrepancy, provided an interaction energy of type (2) is adequate for the process we are considering. Although a calculation taking polarization into account would be exceedingly difficult, its undertaking at the present time seems warranted.

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Multiple Scattering of Electrons. II

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The series developed in a previous paper representing the distribution for the multiple scattering of electrons has been evaluated numerically for a large number of cases; the results are given in Table I. An approximate expression is found for the value of sin θ averaged over the distribution per unit solid angle, $f(\theta)$. This expression, which agrees within a few percent with the exact computation, is

$$w\langle\sin\theta\rangle_{Av}\sim 1.76A^{\frac{1}{2}}(5.60-\tfrac{1}{3}\log Z+\tfrac{1}{2}\log A)^{\frac{1}{2}}, \quad (18)$$

in which w is the energy in units mc^2 and $A=24.8\times10^{-26}Z^2Nt$. For the scattering intensity per unit solid angle at 0°, that is $f(0)$, an approximate relation is

$$4\pi f(0)/w^2\sim 0.43/A(5.60-\tfrac{1}{3}\log Z+\tfrac{1}{2}\log A). \quad (19)$$

The accurate calculations show also that $f(\theta)/w^2$ is almost independent of the energy. A series formula is derived for the projected scattering distribution as observed in a cloud chamber. The averages of $w\sin\alpha$, α being the projected angle, are given in Table VI. These averages are smaller than the values computed by Williams and show a variation with energy. It is believed that the largest inaccuracy remaining in the results given is due to uncertainties in the single scattering law.

1. INTRODUCTION

IN a previous paper[1] we have treated the statistical problem of multiple electron scattering by thin foils. The principal purpose of the present article is to bring the results of that paper into a form which can be more easily compared with experimental data.

We consider an electron of total energy w (in units mc^2) which has traveled a path length t through scattering material of atomic number Z containing N atoms per cc. The normalized probability that the electron will be deflected into the angle between θ and $\theta+d\theta$ is given by the following series in Legendre polynomials

$$2\pi f(\theta)\sin\theta d\theta = \tfrac{1}{2}\sum(2l+1)G_lP_l(\cos\theta)\sin\theta d\theta. \quad (1)$$

The coefficients G_l depend only upon two param-

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[1] S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940).