the brackets can be performed with the aid of the invariance properties of the diamagnetic susceptibility shown by Van Vleck.[11] (It should be noted that the symmetry of the molecule is such that $|(0|M_{\xi 0}|n)|^2=\frac{1}{2}|P(0,n)|^2$ where $P(0,n)$ are the total angular momentum matrix elements used by Van Vleck.) The result is

$$\mu=\frac{(M_2^2+M_1^2)M}{M_1M_2(M_1+M_2)}J - \frac{2}{m\rho^2}\frac{(M_1+M_2)M}{M_1M_2}\left[\sum{}'\frac{|(0|M_{\xi c}|n)|^2}{E_n-E_0}\right]J - \frac{1}{\rho^2}\frac{(M_1+M_2)M}{M_1M_2}[\sum(\langle r_0^2\rangle_{Av}-\langle r_c^2\rangle_{Av})]J, \quad (5)$$

where $(0|M_{\xi c}|n)$ is the matrix element of electron angular momentum about an axis which is the perpendicular bisector of the line joining the two nuclei and is consequently the same for H_2, D_2, and HD except for second-order effects and where $\langle r_0^2\rangle_{Av}$ and $\langle r_c^2\rangle_{Av}$ are the mean square distances of the electron distribution from the center of mass and the midpoint of the molecule, respectively. However, if d is the separation of the midpoint from the center of mass $\langle r_0^2\rangle_{Av}-\langle r_c^2\rangle_{Av}=d^2$. From this and the value of $d=\frac{1}{2}(M_2-M_1)/(M_1+M_2)$ the above equation reduces to

$$\mu=\frac{(M_1+M_2)M}{2M_1M_2}\times\left\{1-\frac{4}{m\rho^2}\left[\sum{}'\frac{|(0|M_{\xi c}|n)|^2}{E_n-E_0}\right]\right\}J, \quad (6)$$

whence μ_R should be inversely proportional to the reduced mass as found experimentally.

AUGUST 1, 1940 PHYSICAL REVIEW VOLUME 58

On the Spinor Equations for Particles with Arbitrary Spin and Rest Mass Zero

J. S. DE WET*
Institute for Advanced Study, Princeton, New Jersey
(Received April 20, 1940)

The spinor equations for arbitrary spin and rest mass zero are examined in some detail. Fierz has shown that for all values of the spin $f(>\frac{1}{2})$ there exists only two "really" independent plane wave solutions instead of $(2f+1)$ when the rest mass is not zero. Fierz later showed, in rather a complicated way, that these two plane waves correspond to components of spin $\pm f$ along the momentum vector of the wave. We will arrive here at the same result but in a much simpler and more direct way.

INTRODUCTION

M. FIERZ[1] has given a detailed discussion of the spinor wave equations for particles with arbitrary spin proposed by Dirac.[2] Fierz showed that when the rest mass is zero there is a certain degeneracy of the equations, and that for a given spin f there are only two really independent plane wave solutions instead of $(2f+1)$. The interpretation of this was not given in his first paper. In a subsequent paper,[3] however, he discusses the rest mass zero case in more detail, and arrives at the result that the two plane waves correspond to axial spins $\pm f$ about the momentum vector of the plane waves. His work is rather complicated. Here we will arrive at the same result in a much simpler way. For coherence it is necessary to give here some of the work appearing in Fierz's paper. The first section will be largely a restatement of Fierz's results, although the notation I will use will be slightly different. The second section will contain a detailed study of the rest mass zero case. In the present discussion no distinction will be made between the cases of integral and half-odd spin.

* Now in the Applied Mathematics Department, University of Cape Town, South Africa.
[1] M. Fierz, Helv. Phys. Acta **12**, 3 (1939).
[2] P. A. M. Dirac, Proc. Roy. Soc. **A155**, 47 (1936).
[3] M. Fierz, Helv. Phys. Acta **13**, 45 (1940).

Such a distinction is quite unnecessary, since the work is exactly the same for both. The only essential difference between the two cases is the form of the energy momentum tensor and charge current vector, but these do not enter in the present work.

The spinor notation which I shall use is that of Veblen. A very good concise account of this formalism has been given by A. H. Taub,[4] to which the reader is referred.

The present work was developed during the course of a number of reports on Fierz's paper that I gave to a seminar on spinor theory, conducted by Professors O. Veblen and J. von Neumann. I am much indebted to Professors Veblen and von Neumann for helpful discussion.

1. The Spinor Equations for Particles with Arbitrary Spin and Rest Mass Nonzero

The equations are stated most simply for our purposes in the following form:

$$g^{\sigma\tau}\psi(0)^{A_1A_2\cdots A_r\dot{B}_1\dot{B}_2\cdots\dot{B}_s}_{\qquad;\sigma\tau} = \kappa^2\psi(0)^{A_1A_2\cdots A_r\dot{B}_1\dot{B}_2\cdots\dot{B}_s} \quad (\sigma, \tau=1, 2, 3, 4), \quad (A, B=1, 2), \tag{1}$$

where $\psi(0)^{A_1A_2\cdots A_r\dot{B}_1\dot{B}_2\cdots\dot{B}_s}$ is a spinor field of the type indicated by its indices and is symmetrical in all its dotted and undotted indices, $g^{\sigma\tau}$ is the metric tensor which has the components

$$\|g^{\sigma\tau}\| = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}$$

in all Galilean frames, and $\kappa=mc/\hbar$ where m is the rest mass of the particle. Furthermore

$$\psi(0)^{A_1A_2\cdots A_r\dot{B}_1\dot{B}_2\cdots\dot{B}_s}_{\qquad;\sigma\tau} \text{ means } \frac{\partial^2\psi(0)^{A_1A_2\cdots A_r\dot{B}_1\dot{B}_2\cdots\dot{B}_s}}{\partial x^\sigma\partial x^\tau}.$$

In addition to (1) the spinor $\psi(0)$ (spinor indices are omitted when not necessary) is to satisfy the following condition:

$$\psi(1)^{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}}$$

defined by

$$\psi(1)^{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}} = g^{\sigma\dot{B}_{s+1}}{}_{A_1}\psi(0)^{A_1A_2\cdots A_r\dot{B}_1\cdots\dot{B}_s}{}_{;\sigma} \tag{2}$$

is symmetric in all its dotted and undotted indices, where the $g^{\sigma\dot{A}}{}_B$ are the components of the four basic spinor matrices. (See Taub, reference 4.)

For the case of *integral* spin $f=r$, we must have $s=r$ in the above equations.

For the case of *half-odd spin* $f=r+\frac{1}{2}$, we must have $s=r+1$ in the above equations.

The above two statements will become clearer in the later work.

The most important property of the matrices $\|g^{\sigma\dot{A}}{}_B\|$ is

$$\tfrac{1}{2}(g^{\sigma\dot{A}}{}_B\bar{g}^{\tau\dot{B}}{}_C + g^{\tau\dot{A}}{}_B\bar{g}^{\sigma\dot{B}}{}_C) = g^{\sigma\tau}\delta^{\dot{A}}{}_{\dot{C}} \tag{3}$$

and the complex conjugate of (3). From this it follows that

$$\bar{g}^{\rho\dot{A}_1}{}_{B_{s+1}}\psi(1)^{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}}{}_{;\rho} = \kappa^2\psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}. \tag{4}$$

The second-order equation (1) may thus be replaced by the first-order equations (2) and (4) where the $\psi(1)$ and $\psi(0)$ are spinors symmetric in all their dotted and undotted indices. This is the usual form in which the equations are given.

We can now define in general a $\psi(m)$ $(m=1, 2, \cdots r)$ by

$$\psi(m)^{A_{m+1}\cdots A_r\dot{B}_1\cdots\dot{B}_{s+m}} = g^{\sigma_m\dot{B}_{s+m}}{}_{A_m}\psi(m-1)^{A_m\cdots A_r\dot{B}_1\cdots\dot{B}_{s+m-1}}{}_{;\sigma_m}$$
$$= g^{\sigma_1\dot{B}_{s+1}}{}_{A_1}\cdots g^{\sigma_m\dot{B}_{s+m}}{}_{A_m}\psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}{}_{;\sigma_1\sigma_2\cdots\sigma_m}. \tag{5}$$

It is easily seen that $\psi(m)$ is symmetric in all its dotted and undotted indices in virtue of the like symmetry of $\psi(1)$ and $\psi(0)$. Furthermore the $\psi(m)$ all satisfy equations like (4), *viz.*,

$$\bar{g}^{\rho\dot{A}_{m+1}}{}_{B_{s+m+1}}\psi(m+1)^{A_{m+2}\cdots A_r\dot{B}_1\cdots\dot{B}_{s+m+1}}{}_{;\rho} = \kappa^2\psi(m)^{A_{m+1}\cdots A_r\dot{B}_1\cdots\dot{B}_{s+m}}. \tag{6}$$

We note that in particular

$$\psi(r)^{\dot{B}_1\cdots\dot{B}_{s+r}}$$

[4] A. H. Taub, Ann. of Math. **40**, 937 (1939). For typographical reasons the usual dotted indices will here be replaced by indices with dots put in front of the indices.

has only dotted indices and is symmetric in all these indices.

In a similar way we can define spinors $\psi(-m)$ $(m=1, 2, \cdots s)$ by

$$\psi(-m)^{A_1\cdots A_{r+m}\dot{B}_{m+1}\cdots\dot{B}_s} = \bar{g}^{\sigma_m\dot{A}_{r+m}} B_m\psi(-[m-1])^{A_1\cdots A_{r+m-1}\dot{B}_m\cdots\dot{B}_s}{}_{;\sigma_m}$$

$$= \bar{g}^{\sigma_1\dot{A}_{r+1}} B_1\cdots\bar{g}^{\sigma_m\dot{A}_{r+m}} B_m\psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}{}_{;\sigma_1\cdots\sigma_m}$$

$$= \frac{1}{\kappa^2}\bar{g}^{\sigma_1\dot{A}_{r+1}} B_1\cdots\bar{g}^{\sigma_m\dot{A}_{r+m}} B_m\bar{g}^{\sigma\dot{A}_1} B_{s+1} \times\psi(1)^{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}}{}_{;\sigma_1\sigma_2\cdots\sigma_m\sigma}, \quad (7)$$

using (4) to get the last part of the equation. Again from the symmetry of $\psi(1)$ and $\psi(0)$ in all their dotted and undotted indices, it follows from (7) that $\psi(-m)$ has like symmetry.

In particular

$$\psi(-s)^{A_1A_2\cdots A_{r+s}}$$

is a spinor with undotted indices only, and is symmetric in all of them. The $\psi(-m)$ satisfy the following equations (analogous to (6)) in virtue of (1):

$$g^{\rho\dot{B}_m} A_{r+m}\psi(-m)^{A_1\cdots A_{r+m}\dot{B}_{m+1}\cdots\dot{B}_s}{}_{;\rho} = \kappa^2\psi(-[m-1])^{A_1\cdots A_{r+m-1}\dot{B}_m\cdots\dot{B}_s} \quad (8)$$

We can write the above equations (dropping the spinor indices) as follows:

$$g^\sigma\psi(0)_{;\sigma}=\psi(1), \quad \bar{g}^\sigma\psi(1)_{;\sigma}=\kappa^2\psi(0),$$
$$g^\sigma\psi(1)_{;\sigma}=\psi(2), \quad \bar{g}^\sigma\psi(2)_{;\sigma}=\kappa^2\psi(1),$$

and so on up to

$$g^\sigma\psi(r-1)_{;\sigma}=\psi(r), \quad \bar{g}^\sigma\psi(r)_{;\sigma}=\kappa^2\psi(r-1),$$

and (9)

$$\bar{g}^\sigma\psi(0)_{;\sigma}=\psi(-1), \quad g^\sigma\psi(-1)_{;\sigma}=\kappa^2\psi(0),$$
$$\bar{g}^\sigma\psi(-1)_{;\sigma}=\psi(-2), \quad g^\sigma\psi(-2)_{;\sigma}=\kappa^2\psi(-1),$$

up to

$$\bar{g}^\sigma\psi(-[s-1])_{;\sigma}=\psi(-s),$$
$$g^\sigma\psi(-s)_{;\sigma}=\kappa^2\psi(-[s-1]).$$

The notation will be clear if we note that the first two equations of (9) are Eqs. (2) and (4).

From any of the above pairs of equations, together with the symmetry in dotted and undotted indices of the two ψ's involved, can be deduced all the other equations as well as the symmetry of the rest of the ψ's in dotted and undotted indices. From this it appears that we could have started with Eq. (1) involving a ψ with any number of dotted and undotted indices, rather than the $\psi(0)$ we started with, having either $r=s$ or $r=s+1$. One reason why the $\psi(0)$ has a preferred position is that the physical quantities like the energy momentum tensor and charge current vector are most simply expressed in terms of $\psi(0)$. As we will see later, when the rest mass is zero the ψ that we start with definitely has a preferred position. If we start with any ψ other than $\psi(0)$, the energy momentum tensor and charge current vector formed from it is identically zero. This follows from the fact that these tensors defined in terms of ψ's other than $\psi(0)$ involve second and higher derivatives, i.e., terms like

$$g^{\sigma\tau}\psi_{;\sigma\tau\rho}$$

say, and these are zero in virtue of (1) with $\kappa=0$.

Let us now consider plane wave solutions of the equations.

$$\psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s} = \Psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s} e^{i\kappa_\sigma x^\sigma} \quad (10)$$

will be a solution if

$$-g^{\sigma\tau}\kappa_\sigma\kappa_\tau=\kappa^2 \quad (11)$$

and the components of $\Psi(0)$ are constants.

In addition $\Psi(1)$ given by

$$\Psi(1)^{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}} = i\kappa_\sigma g^{\sigma\dot{B}_{s+1}} A_1\Psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s} \quad (12)$$

must be symmetric in all its dotted and undotted indices. $\Psi(0)$ has $(r+1)\cdot(s+1)$ independent components, considering only its symmetry in its spinor indices. The symmetry of the right-hand side of (12) imposes $r\cdot s$ conditions on the components of $\Psi(0)$, however, so that it has $(r+1)(s+1)-rs=r+s+1$ independent components.

The same reasoning applies to all the $\Psi(m)$ $(m=r, r-1, \cdots 1, 0, -1, -2, \cdots -s)$. Each has $r+s+1$ independent components, and any

$\Psi(m)$ is expressible in terms of

$$\Psi(r)^{\dot{B}_1\cdots\dot{B}_{s+r}}$$

by the following relation:

$$\Psi(r-m)^{\dot{A}_1\cdots\dot{A}_m\dot{B}_1\cdots\dot{B}_{s+r-m}} = \frac{1}{\kappa^{2m}}\bar{g}^{A_1}{}_{\dot{B}_{s+r}}\cdots \times\bar{g}^{\dot{A}_m}{}_{B_{s+r-m+1}}\Psi(r)^{\dot{B}_1\cdots\dot{B}_{s+r}} \quad (m=0, 1, 2, \cdots, r+s), \quad (13)$$

where

$$g^{\dot{A}}{}_B = i\kappa_\sigma g^{\sigma\dot{A}}{}_B. \quad (14)$$

By (13) the components of any $\Psi(m)$ are expressed in terms of the $(r+s+1)$ independent components of $\Psi(r)$.

Let us now choose a special coordinate system in such a way that

$$\kappa_1=\kappa_2=\kappa_3=0, \quad \kappa_4=\pm\kappa.$$

Then

$$g^{\dot{A}}{}_B = i\kappa_4 g^{4\dot{A}}{}_B.$$

If we now apply a space rotation to the system (rotation of the x^1, x^2, x^3, leaving x^4 invariant) the corresponding spinor transformation is a unimodular linear transformation leaving $g^{4\dot{A}}{}_B$ numerically invariant, i.e., a unitary transformation; since in a special spin coordinate system the matrix $g^{4\dot{A}}{}_B$ (obtained from $g^{4\dot{A}}{}_B$ by lowering the dotted index) has the components

$$\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}.$$

Thus under a space rotation for a system with zero momentum ($\kappa_1=\kappa_2=\kappa_3=0$) the connection between the $(r+s+1)$ independent components of $\Psi(r)$ and the components of $\Psi(r-m)$ is numerically invariant. Now the components of $\Psi(r)$ transform according to the irreducible representation $D_{\frac{1}{2}(r+s)}$ of the rotation group under the unitary spin transformation associated with a space rotation. Further since the $g^{4\dot{A}}{}_B$ are numerically invariant under the unitary spin transformation, the $(r+s+1)$ independent components of $\Psi(r-m)$—and in particular $\Psi(0)$—also transform according to $D_{\frac{1}{2}(r+s)}$.[5]

[5] See B. L. van der Waerden, *Gruppentheoretische Methode in der Quantenmechanik* (Springer, Berlin, 1932), Chap. III.

Hence the equations correspond to a spin $\frac{1}{2}(r+s)$. From this follows the earlier statement that when $s=r$, the corresponding spin is $f=r$, and when $r=s+1$, the corresponding spin is $r+\frac{1}{2}$.

2. Rest Mass Zero Case

The work of the preceding section has been largely a restatement of Fierz's results. We now consider the rest mass zero case. Fierz showed that when the rest mass is zero, the $\Psi(0)$ had only two "really independent" components, since it was possible to add to $\psi(0)$ a set of plane wave solutions of the equations depending on $(r+s-1)$ independent constants, without changing the energy momentum tensor or charge current vector for the system. The transformation of the ψ's by the addition of these terms he calls "gauge transformations." The supposition was that by a suitable choice of the gauge transformation one could reduce all but two of the components of $\Psi(0)$ to zero. Here we will investigate the nature of the two components of $\Psi(0)$ which are not zero. Working in a special coordinate system for which $\kappa_2=\kappa_3=0$ we will show that there are two components of $\Psi(0)$ which are "gauge invariant" and nonzero, i.e., by no choice of the gauge transformation can these two components be reduced to zero. These two components then are the two "really" independent components of $\Psi(0)$. We will then show that under axial rotations about the x^1 axis through an angle θ, these two components are multiplied by

$$e^{\frac{1}{2}i(r+s)\theta} \quad \text{and} \quad e^{-\frac{1}{2}i(r+s)\theta},$$

so that they correspond to components of spin $\pm f=\pm\frac{1}{2}(r+s)$ along the x^1 axis. But the momentum vector lies along the x^1 axis (since we have chosen coordinates in such a way that $\kappa_2=\kappa_3=0$). Thus the two gauge invariant states correspond to components of spin $\pm f$ in the direction of the momentum vector. Thus we here arrive at the result of Fierz's second paper in a simple and direct manner.

The equation for the rest mass zero case is (1) with $\kappa=0$, together with the condition that $\psi(1)$ defined by (2) is symmetric in all its dotted and undotted indices. Then $\psi(1)$ satisfies (4) with

$\kappa=0$. The $\psi(m)$ $(m=1, 2, \cdots, r)$ are defined as before by (5) and satisfy (6) with $\kappa=0$.

The $\psi(-m)$ $(m=1, 2, \cdots, s)$ are defined as before in terms of $\psi(0)$ by the first part of Eqs. (7), but the symmetry of $\psi(-m)$ in dotted and undotted indices no longer follows by our previous argument, since there the last part of (7) (which no longer applies since $\kappa=0$) was used. It is still true, however, as we will see later, that $\Psi(-m)$ is symmetric in dotted and undotted indices in virtue of the like symmetry of $\Psi(0)$ and $\Psi(1)$. So that for the rest mass zero case the equations (9) are replaced by similar equations with $\kappa=0$.

Let us now consider plane wave solutions of the equations as before

$$\overset{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}{\psi(0)} = \overset{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}{\Psi(0)}\, e^{i\kappa_\sigma x^\sigma} \tag{15}$$

where now

$$g^{\sigma\tau}\kappa_\sigma\kappa_\tau=0. \tag{16}$$

We then have the following state of affairs: $\Psi(0)$ still has $(r+s+1)$ independent components —as before a result of the symmetry of $\Psi(1)$ and $\Psi(0)$ in their dotted and undotted indices. We will show that for a special coordinate system all the remaining components of $\Psi(0)$ are zero. Further, in this coordinate system $\Psi(m)$ has $(r+1-m)$ nonzero components and in particular $\Psi(r)$ has only *one* nonzero component. Similarly $\Psi(-m)$ has $(s+1-m)$ nonzero components and in particular $\Psi(-s)$ has only *one* nonzero component.

In what follows we wish to be quite explicit and give the actual matrices $\|g^{\sigma\dot{A}}{}_B\|$ in a particular spin coordinate system. Let the

$$\|g^{\sigma\dot{A}}{}_B\| = \begin{Vmatrix} g^{\dot{1}}{}_1 & g^{\dot{1}}{}_2 \\ g^{\dot{2}}{}_1 & g^{\dot{2}}{}_2 \end{Vmatrix}^{\sigma}$$

be given by

$$\begin{aligned} \|g^{1\dot{A}}{}_B\| &= \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad \|g^{2\dot{A}}{}_B\| = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}, \\ \|g^{3\dot{A}}{}_B\| &= \begin{Vmatrix} i & 0 \\ 0 & i \end{Vmatrix}, \quad \|g^{4\dot{A}}{}_B\| = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}. \end{aligned} \tag{17}$$

Then let us choose a special space time frame of reference in such a way that

$$\kappa_2=\kappa_3=0, \quad \kappa_1{}^2=\kappa_4{}^2. \tag{18}$$

Then we have

$$\begin{aligned} \|i\kappa_\sigma g^{\sigma\dot{A}}{}_B\| &= \begin{Vmatrix} 0 & 0 \\ 2i\kappa_1 & 0 \end{Vmatrix} \quad (\kappa_1=+\kappa_4) \\ &= \begin{Vmatrix} 0 & 2i\kappa_1 \\ 0 & 0 \end{Vmatrix} \quad (\kappa_1=-\kappa_4). \end{aligned} \tag{19}$$

For definiteness let us take $\kappa_1=\kappa_4$ and denote the matrix

$$\|i\kappa_\sigma g^{\sigma\dot{A}}{}_B\| = \begin{Vmatrix} 0 & 0 \\ 2i\kappa_1 & 0 \end{Vmatrix} \quad \text{by} \quad \|g^{A}{}_B\|.$$

Now $\Psi(1)$ is by definition given by

$$\overset{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}}{\Psi(1)} = i\kappa_\sigma g^{\sigma\dot{B}_{s+1}}{}_{A_1}\overset{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}{\Psi(0)} \tag{20}$$

In the special coordinate system, we have from (19) and (20)

$$\overset{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_s\dot{1}}{\Psi(1)} = 0. \tag{21}$$

Now $\Psi(1)$ is symmetric in all its dotted indices. Hence the only nonzero components of $\Psi(1)$ are:

$$\overset{A_2\cdots A_r\dot{2}\dot{2}\cdots\dot{2}}{\Psi(1)}$$

and there are r such nonzero components.* It follows further from (20) and (19) and the dotted and undotted index symmetry of $\Psi(1)$ that the following components of $\Psi(0)$ are zero in our special coordinate system:

$$\begin{aligned} &\text{all terms} \quad \overset{1A_2\cdots A_r\dot{B}_1\cdots\dot{B}_s}{\Psi(0)} \\ &\text{excepting} \quad \overset{1A_2\cdots A_r\dot{2}\dot{2}\cdots\dot{2}}{\Psi(0)} \end{aligned} \qquad (A, B=1, 2). \tag{22}$$

The number of terms of $\Psi(0)$ that are zero is thus $r\cdot(s+1)-r=rs$. Since the total number of terms of $\Psi(0)$ is $(r+1)(s+1)$ the number of nonzero terms is $(r+s+1)$, i.e., just the number of independent components of $\Psi(0)$.

* In the following we shall frequently use *component* and *term* for *independent component* and *independent term*.

Similarly we have (in the special coordinate system)

$$\Psi(m)^{A_{m+1}\cdots A_r \dot{B}_1\cdots\dot{B}_{s+m}} = g^{\dot{B}_{s+1}}{}_{A_1}\cdots g^{\dot{B}_{s+m}}{}_{A_m}\Psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s} \quad (23)$$

and it follows from the form of $\|g^{\dot{A}}{}_B\|$ and the symmetry of $\Psi(m)$ in all dotted and undotted indices, that the only nonzero terms of $\Psi(m)$ are

$$\Psi(m)^{A_{m+1}\cdots A_r\dot{2}\dot{2}\cdots\dot{2}}, \quad (r+1-m) \text{ in number.}$$

In particular the *only* nonvanishing term of $\Psi(r)$ is

$$\Psi(r)^{\dot{2}\dot{2}\cdots\dot{2}} = (2i\kappa_1)^r \Psi(0)^{11\cdots1\dot{2}\cdots\dot{2}}. \quad (24)$$

Similar results will hold for the $\Psi(-m)$ provided $\Psi(-1)$ is symmetric in all dotted and undotted indices. That this is so is easily verified in the special coordinate system. For

$$\Psi(-1)^{A_1\cdots A_{r+1}\dot{B}_2\cdots\dot{B}_s} = \bar{g}^{\dot{A}_{r+1}}{}_{B_1}\Psi(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}. \quad (25)$$

Now from (25)

$$\Psi(-1)^{A_1\cdots A_{r-1}1\dot{2}\dot{B}_2\cdots\dot{B}_s} = -2i\kappa_1\Psi(0)^{A_1\cdots A_{r-1}1\dot{1}\dot{B}_2\cdots\dot{B}_s}$$

$$= 0 \text{ from (22)} \quad \text{and}$$

$$\Psi(-1)^{A_1\cdots A_{r-1}2\dot{1}\dot{B}_2\cdots\dot{B}_s} = 0 \quad \text{from (25) and (19).}$$

So the symmetry of $\Psi(-1)$ in dotted and undotted indices follows in a special coordinate system, and hence in any spin coordinate system.

The rest of the argument then proceeds as before. $\Psi(-m)$ has $(s+1-m)$ nonzero components and in particular $\Psi(-s)$ has only *one* nonzero component given by

$$\Psi(-s)^{22\cdots2} = (-2i\kappa_1)^s \Psi(0)^{2\cdots2\dot{1}\cdots\dot{1}} \quad (26)$$

Let us now consider the effect of the "gauge transformations" of Fierz on the $\Psi(0)$, $\Psi(m)$ and $\Psi(-m)$.

Fierz showed that the energy momentum tensor and charge current vector for the system was unchanged if one replaced the plane waves

$$\Psi(0)e^{i\kappa_\sigma x^\sigma} \quad \text{by} \quad [\Psi(0)+\Psi'(0)]e^{i\kappa_\sigma x^\sigma},$$

where the $\Psi'(0)$ is given by

$$\Psi'(0)^{A_1A_2\cdots A_r\dot{B}_1\cdots\dot{B}_s} = i\kappa_\sigma g^{\sigma\dot{B}_1A_1}\Phi(0)^{A_2\cdots A_r\dot{B}_2\cdots\dot{B}_s}$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_1A_2}\Phi(0)^{A_1A_3\cdots A_r\dot{B}_2\cdots\dot{B}_s} + \cdots$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_2A_1}\Phi(0)^{A_2\cdots A_r\dot{B}_1\dot{B}_3\cdots\dot{B}_s}$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_2A_2}\Phi(0)^{A_1A_3\cdots A_r\dot{B}_1\dot{B}_3\cdots\dot{B}_s} + \cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_sA_1}\Phi(0)^{A_2\cdots A_r\dot{B}_2\cdots\dot{B}_{s-1}\dot{B}_1}$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_sA_2}\Phi(0)^{A_1A_3\cdots A_r\dot{B}_2\cdots\dot{B}_{s-1}\dot{B}_1} + \cdots, \quad (27)$$

where

$$\Phi(0)^{A_2\cdots A_r\dot{B}_2\cdots\dot{B}_s}$$

is symmetric in all its dotted and undotted indices and in addition $\Phi(1)$ defined by

$$\Phi(1)^{A_3A_4\cdots A_r\dot{B}_2\cdots\dot{B}_{s+1}} = i\kappa_\sigma g^{\sigma\dot{B}_{s+1}}{}_{A_2}\Phi(0)^{A_2\cdots A_r\dot{B}_2\cdots\dot{B}_s} \quad (28)$$

is symmetric in all its dotted and undotted indices.

We must verify that $\Psi'(0)$ and $\Psi'(1)$ are symmetric in all their dotted and undotted indices. For $\Psi'(0)$ this is apparent from the definition (27). $\Psi'(1)$ is given by

$$\Psi'(1)^{A_2\cdots A_r\dot{B}_1\cdots\dot{B}_{s+1}} = i\kappa_\sigma g^{\sigma\dot{B}_{s+1}}{}_{A_1}\Psi'(0)^{A_1\cdots A_r\dot{B}_1\cdots\dot{B}_s}$$

$$= i\kappa_\sigma g^{\sigma\dot{B}_1A_2}\Phi(1)^{A_3\cdots A_r\dot{B}_2\cdots\dot{B}_{s+1}}$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_1A_3}\Phi(1)^{A_2A_4\cdots A_r\dot{B}_2\cdots\dot{B}_{s+1}} + \cdots$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_2A_2}\Phi(1)^{A_3\cdots A_r\dot{B}_1\dot{B}_2\cdots\dot{B}_{s+1}}$$

$$+ i\kappa_\sigma g^{\sigma\dot{B}_2A_3}\Phi(1)^{A_2A_4\cdots A_r\dot{B}_1\dot{B}_3\cdots\dot{B}_{s+1}} + \cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots. \quad (29)$$

In deriving the above we use (3) and (16). Thus $\Psi'(1)$ is given in terms of $\Phi(1)$ in the same way as $\Psi'(0)$ in terms of $\Phi(0)$, and is symmetric in all its dotted and undotted indices. The same applies for all $\Psi'(m)$ for $m \leqslant r-1$. All the $\Psi'(m)$ are given by means of a relation like (29) in terms of $\Phi(m)$, where $\Phi(m)$ is defined in terms of $\Phi(0)$ in the same way as $\Psi(m)$ in terms of $\Psi(0)$.

For $m=r$, however, $\Psi'(r)=0$. This is perhaps most easily seen from (27), considering the definition of $\Psi'(r)$ in terms of $\Psi'(0)$ and noting (3) and (16).

The position then is as follows: $\Psi(0)$ has $(r+s+1)$ independent nonzero components. To this may be added $\Psi'(0)$ depending on $\Psi(0)$ which has $(r+s-1)$ independent nonzero components, so that by a suitable choice of $\Phi(0)$ all but two of the components of $\Psi(0)$ can be made zero.

Further $\Psi(m)$ $(m=1, 2, \cdots, r)$ has $(r+1-m)$ nonzero independent components and $\Phi(m)$ has $(r-m)$ nonzero independent components, so that each $\Psi(m)$ has only *one* "really" independent component. In particular $\Psi(r)$ has only one nonzero component, and $\Psi'(r)=0$ as was shown earlier. A similar state of affairs holds for the $\Psi(-m)$.

It is now clear that the single nonzero components of $\Psi(r)$ and $\Psi(-s)$ cannot be transformed away by a gauge transformation. But these nonzero components are given in terms of the components

$$\overset{11\cdots1\cdot2\cdots\cdot2}{\Psi(0)} \quad \text{and} \quad \overset{22\cdots2\cdot1\cdots\cdot1}{\Psi(0)}$$

of $\Psi(0)$ by (24) and (26).

Hence it is clear that these two components of $\Psi(0)$ are gauge invariant and are the two really independent components of $\Psi(0)$.

All that remains is to interpret these states. With the choise of the matrices $\|g^{\sigma\cdot A}{}_B\|$ that we have made, an axial rotation about the x^1 axis through an angle θ corresponds to the spin transformation $\Gamma^A{}_B$ given by

$$\|\Gamma^A{}_B\| = \begin{Vmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{Vmatrix}.$$

Under this spin transformation

$$\overset{1\cdots1\cdot2\cdots\cdot2}{\Psi(0)}$$

is multiplied by a factor $e^{\frac{1}{2}i(r+s)\theta}$, while

$$\overset{2\cdots2\cdot1\cdots\cdot1}{\Psi(0)}$$

is multiplied by a factor $e^{-\frac{1}{2}i(r+s)\theta}$.

From this follows that the two components of $\Psi(0)$ correspond to components of spin $\pm\frac{1}{2}(r+s)$ along the x^1 axis. But we have chosen the frame of reference in such a way that the x^1 axis coincides with the momentum vector. So that the two states correspond to components of spin $\pm f$ along the momentum vector (where $f=\frac{1}{2}(r+s)$). This is the result given by Fierz. A further result of Fierz is that the (total angular momentum)^2 for the system is always $\geqslant f(f+1)$. The reason for this is easily seen. The orbital angular momentum for the system always has component zero along the momentum vector. Thus it follows that the total angular momentum always has the components $\pm f$ along the momentum vector, and thus the (magnitude)^2 of the total angular momentum must always be $\geqslant f(f+1)$.