Scattering Matrix of Radioactive States[1]

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The method of complex eigenvalues is generalized to many dimensional problems by means of the scattering matrix. It is essential to allow for the existence of a background matrix on which the resonance is superposed. The "radioactive state" (i.e., the state with complex energy for which waves in all disintegration channels are outgoing) determines the damping constants of the resonance formulas. Phase constants [Eq. (6.2)] responsible for displacements of observable resonance peaks with respect to the real part of the complex eigenvalue of the energy are also determined by the radioactive state. The background matrix is restricted by the above-mentioned damping constants and phase constants to a considerable extent, leaving free in its specification an n-1-dimensional symmetric unitary matrix for the case of n channels. The equations presented do not include the case of continuously variable energy distribution among disintegration products.

THE notion of the scattering matrix has been introduced by Wheeler[2] for the discussion of scattering in the approximation of the resonating group structure. Radioactive quantum-mechanical states with complex energy have been introduced by Gamow[3] and the resonance scattering of one body by a central field can be described conveniently by means of the radioactive state. Resonance scattering for a system having several modes of decay can also be understood by means of the radioactive state by an extension of the argument used for the one-body problem.[4] This extension is being carried out below and makes use of a general scattering matrix. It is found that the energy dependence of the scattering and disintegration cross sections is given approximately by the formula for resonance competition derived by Wigner and the writer. The scattering matrix is found to consist of resonance-like terms superposed on a background. Restricting conditions on the background are found to follow as a general consequence of the wave equation. For two competitive modes of disintegration the background part of the scattering matrix is fixed to within one real parameter. For three competitive modes there enter three real parameters. For n modes the parameters involved are those contained in the matrix elements of an n-1-dimensional symmetric unitary matrix. A partial object of the present note is to eliminate, in the discussion of the resonance formula, special considerations regarding the interaction between particles.

THE SCATTERING MATRIX

The system of particles will be assumed to have no spin and the particles will be assumed to be nonidentical. The wave equation is

$$\left[-\sum_1^m \frac{\hbar^2}{2M_i}\left(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y^2{}_i}+\frac{\partial^2}{\partial z_i^2}\right)+V-E\right]\psi=0. \quad (1)$$

The potential energy depends only on the relative distance between particles so that conservation of momentum, angular momentum and parity hold. It is supposed that it is energetically possible for the system to break up into pairs of fragments in several ways. The possibility of breakup into triplets or quadruplets of fragments

[1] Professor E. P. Wigner mentioned to the writer that one can use the scattering matrix for an improved understanding of resonance phenomena more than a year ago. At the end of 1939 he had considerable success in this direction which was mentioned briefly in conversation. These considerations will be described by him jointly with L. Eisenbud in a forthcoming paper in the *Reviews of Modern Physics*. The present note is thus not an independent investigation because of its origin in the knowledge that useful relations can be obtained by means of the scattering matrix. The reason for publishing is that the scattering matrix is here related to the method of complex eigenvalues and the approach differs in this way from that of Wigner and Eisenbud who have arrived at the results before the present author.

[2] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937).

[3] G. Gamow, Zeits. f. Physik **51**, 204 (1928).

[4] G. Breit and F. L. Yost, Phys. Rev. **48**, 203 (1935); G. Breit, *ibid.* **40**, 127 (1932); A. J. F. Siegert, *ibid.* **56**, 750 (1934); G. Breit, *ibid.* **58**, 506 (1940). See pages 509–517 in latter paper.

is neglected. The discussion does not apply, therefore, to $(n, 2n)$ reactions except insofar as the emission of neutrons may be treated as a succession of two processes.

The region of configuration space in which the system has separated into two definitely formed and noninteracting fragments will be called a *channel*. Within each channel the wave function is a sum of products of functions describing the motion of the center of mass of the whole system, the internal coordinates of each fragment and finally the relative motion of the particles with respect to each other. The latter factor may in its turn be represented as a sum of products of angular functions corresponding to different angular momenta and radial factors depending only on the distance between the fragments. The product of the angular factor, the factors representing the internal motion within the fragments and the factor containing the coordinates of the center of mass of the system will be referred to as u.

The transformation

$$X=(M_1x_1+M_2x_2)/(M_1+M_2), \quad x=x_2-x_1$$

represents the introduction of the center of mass and relative coordinates for a pair of particles. Its determinant is 1. The internal coordinates for each fragment can be introduced by a succession of such transformations. The centers of mass of each fragment are then used in such a transformation to introduce the relative coordinates of the fragments with respect to each other and the center of mass of the whole system. The Jacobian of the transformation from the original coordinates is unity and the normalization integral of the whole wave function retains its form.

Within the channels the wave function can be represented as

$$\psi=\sum_l[a_l\exp(ik_lr_l)+b_l\exp(-ik_lr_l)]u_l/r_l, \qquad k_l=\mu_lv_l/\hbar. \tag{1.1}$$

The index l specifies here the channel, the state of the fragments and the relative angular momentum of the fragments; a_l, b_l are constants; μ_l, v_l, r_l are respectively the reduced mass, relative velocity and distance between the fragments. Each term in the above sum corresponds to a mode of disintegration. All modes of disintegration in the same channel correspond to the same μ_l, r_l. The function u_l is normalized so that $|u_l|^2$ gives unity when integrated over the coordinates of the center of mass of the system, the internal coordinates of each fragment and the angular coordinates of the relative position of the fragments.

Special solutions are now considered for a system having n independent modes of disintegration. Within the channels these are given by

$$\psi^i=\sum_l\varphi_l{}^iu_l; \qquad \varphi_l{}^i=[\delta_{jl}\exp(-ik_lr_l)+a_{lj}\exp(ik_lr_l)]/r_l. \tag{1.2}$$

Each ψ^i is obtained by solving (1) subject to the boundary condition of there being only one incident wave $u_j\exp(-ik_jr_j)/r_j$. The matrix (a_{lj}) is called the scattering matrix. Although a solution of Eq. (1) is practically impossible some information can be obtained by studying the a_{lj}.

A surface is passed so as to cut across the channels. Within each channel the surface satisfies the equation $r_l=\text{const}$. The radius of a sphere inscribed into the surface is made $\gg m\times$range of force. The wave function on the surface is thus negligible except at the channels. From Eq. (1) one obtains

$$\psi^i\sum\frac{1}{M_i}\Delta_i\psi^m-\psi^m\sum\frac{1}{M_i}\Delta_i\psi^i=0.$$

This formula is integrated through the $3m$-dimensional volume contained within the surface and the integral is transformed into a surface integral by Green's theorem. The transformation from the original coordinates to the channel variables takes the form of orthogonal transformations if one uses $(M_i)^{\frac{1}{2}}x_i$ instead of x_i and $\mu^{\frac{1}{2}}$ times relative coordinate at each stage. To each r_l there correspond the coordinates X_l, Y_l, Z_l which correspond to $\mu_l{}^{\frac{1}{2}}X_l$, $\mu_l{}^{\frac{1}{2}}Y_l$, $\mu_l{}^{\frac{1}{2}}Z_l$ in the Euclidean space. In the integrand there corresponds to them

$$\sum_{x,y,z}\frac{1}{\mu_l}\frac{\partial}{\partial X_l}\left(\psi^i\frac{\partial\psi^m}{\partial X_l}-\psi^m\frac{\partial\psi^i}{\partial X_l}\right).$$

The factor μ_lv_l brought down by differentiation on account of Eq. (1.1) combines with $1/\mu_l$ leaving v_l. Only terms with the same l contribute

to the result because the u_l are orthogonal to each other and because the channels do not overlap on the surface. One obtains on substituting Eq. (1.2)

$$v_l a_{lj} = v_j a_{jl}. \tag{2}$$

This relation holds for real or complex E. For real E one obtains additional simple relations by considering ψ^{i*} and ψ^m in Green's formula *viz.*

$$\sum_k v_k a^*_{kj} a_{kl} = \delta_{jl} v_l. \tag{2.1}$$

It is convenient to introduce

$$b_{kl} = (v_k/v_l)^{\frac{1}{2}} a_{kl} \tag{2.2}$$

for which

$$b_{kl} = b_{lk}, \quad \sum b^*_{kl} b_{kj} = \delta_{lj}. \tag{2.3}$$

The latter of these equations shows that $\|b_{kl}\|$ is a unitary matrix. According to Eq. (2.2) there are $|b_{kl}|^2$ systems leaving by way of the disintegration mode k per unit incident system in state l. Relations (2.3) have as a consequence the conservation of the number of systems as has been pointed out by Wheeler. The symmetry of $\|b_{lk}\|$means detailed balance at high temperatures.

Complex Energies

For complex energies it is also possible to use a scattering matrix. Equation (2.1) does not hold in this case and the matrix $\|b_{lk}\|$ is not unitary. It is symmetric, however. The values of E will have in most of the calculation negative imaginary parts. In such cases v will be taken to have positive real and negative imaginary parts. The factors $\exp[i\mu vr/\hbar]$, $\exp[-i\mu vr/\hbar]$ will have exponentially increasing and decreasing amplitudes for large r. The energy states of the fragments still correspond to real internal energies. For values of E with a small imaginary part one has solutions of the wave equation which are similar to solutions for real E as long as one does not go too far out in the channels. If the imaginary part of E is not sufficiently small, however, there will be no predominance of the rapidity of decrease of states with negative kinetic energy in the channels over the decrease of $\exp[-i\mu vr/\hbar]$ and the distinction between the interior and exterior of the channel disappears. The imaginary part of E will be supposed to be sufficiently small so that this does not happen.

For a suitably chosen complex $E=E_c$ it may be possible to have

$$\varphi_l = \alpha_l \exp(ik_l r_l)/r_l, \tag{3}$$

so that only outgoing waves are present in the channels. Such solutions are, of course, impossible for real energies, since their existence would contradict the conservation of the number of systems. The state (3) will be referred to as the radioactive state. Its normalization is immaterial at this stage. By superposing solutions of the form (1.2) it should be possible to obtain (3). This can only happen if some of the a_{lj} become infinite at $E=E_c$. The simplest possibility for this will be considered

$$a_{lj} \cong \frac{c_{lj}}{E-E_c} + d_{lj}. \tag{3.1}$$

Here the c_{lj}, d_{lj} are constants and the equation is meant to be only an approximation in the neighborhood of the complex eigenvalue. On the other hand the number of independent solutions of the wave equation must be still n. There must exist, therefore, $n-1$ linearly independent combinations of the ψ^j for which the solution is not (3). The equation

$$\sum_{j=1}^{n} c_{lj}\lambda_j = 0 \tag{3.2}$$

must have, therefore, $n-1$ linearly independent solutions and all the minors of det $\|c_{lj}\|$ are zero. According to Eq. (2) one has

$$v_l c_{lj} = v_j c_{jl} \tag{3.3}$$

and one obtains, therefore, as the general form of c_{lj}

$$c_{lj} = (v_j/v_l)^{\frac{1}{2}} C_l C_j. \tag{3.4}$$

The wave equation (1) is real. If $E=E_c^*$ there must exist, therefore, an *antiradioactive state* consisting entirely of incoming waves and it should be possible to superpose the solutions ψ^j linearly so as to obtain this state. The required linear combination is $\sum a_j^* \psi^j(E_c^*)$ because only with this combination are the ratios of the coefficients of $\exp(-ik_l r_l)$ correct. From Eq. (3.4) it follows that one may take

$$\hbar^{\frac{1}{2}} \alpha_j = v_j^{-\frac{1}{2}} C_j; \quad c_{lj} = v_j \alpha_l \alpha_j \hbar \tag{3.5}$$

and hence

$$\sum_j v_j^{-\frac{1}{2}} C_j^* \left[\frac{c_{lj}}{E_c^* - E_c} + d_{lj} \right] = 0, \qquad (3.6)$$

because the terms $\exp(ik_l r_l)$ must disappear in the antiradioactive state. One has in Eq. (3.6) n conditions on the d_{lj}. Formally, according to this presentation, one must use complex v_j, v_l in Eqs. (3.4) and (3.6). The approximation (3.1), however, is not exactly consistent with Eq. (2) for constant c_{lj} as E is varied. Since it is important to represent scattering for real E, the real values of v_j, v_l corresponding to $E=(E_c+E_c^*)/2$ will be used.

It will be shown next that the restrictions on $\|d_{lj}\|$ imposed by Eq. (3.6) are such that $\|b_{lj}\|$ is unitary for real E. On account of Eq. (2) it is convenient to introduce

$$q_{lj} = (v_l/v_j)^{\frac{1}{2}} d_{lj} = q_{jl}, \qquad (4)$$

so that

$$b_{lj} = C_l C_j/(E-E_c) + q_{lj} \qquad (4.1)$$

and

$$\sum_i b_{ij} b_{il}^* = \frac{C_j C_l^*}{|E-E_c|^2} \sum_i |C_i|^2 + \frac{C_j}{E-E_c} \sum_i C_i q_{il}^* + \frac{C_l^*}{E-E_c^*} \sum_i C_i^* q_{ij} + \sum_i q_{ij} q_{il}^*. \qquad (4.2)$$

The first three terms in this formula depend on E and it appears strange that this could be reconciled with the unitary nature of $\|b_{lj}\|$ expressed by Eq. (2.3). The relations (3.6), however, connect the q_{lj} with the C_l in such a way that only the last term in Eq. (4.2) survives. Substituting (3.4) into (3.6) one has

$$\sum_i q_{ij} C_i^* = \frac{C_j}{E_c - E_c^*} \sum_i |C_i|^2. \qquad (4.3)$$

This formula and its complex conjugate are used in (4.2) and the first three terms are then seen to cancel. One has, therefore,

$$\sum_i q_{ij} q_{il}^* = \delta_{lj}. \qquad (4.4)$$

These, however, are the necessary and sufficient conditions for $\|q_{ij}\|$ to be a unitary matrix. It is necessary for $\|q_{ij}\|$ to be unitary in order that $\|b_{ij}\|$ be unitary. The unitary symmetric nature of $\|q_{ij}\|$ together with (4.3) is also sufficient for the unitary, symmetric nature of $\|b_{ij}\|$.

In the above presentation Eq. (4.3) has been obtained from a consideration of the antiradioactive state. It may possibly be preferred to use the notion of the radioactive state only to the extent of suggesting the form (4.1) as an approximation for the scattering matrix close to resonance. This may be done either through Eq. (3.2) as has been explained or else this form can be made reasonable by observing that for $E \cong E_c$ the ratios of the b_{lj} for fixed j are equal to the ratios of the corresponding C_l. This means that any single incident wave of type $(u_l/r_l)\exp[-ik_l r_l]$ gives rise for $E \cong E_c$ to a multiple of the radioactive state. Having decided on (4.1) as the desired approximation one can derive the relations (4.3) as well as the unitary symmetric nature of $\|q_{ij}\|$ from the consideration of the scattering matrix for real E alone as follows. Since $b_{lj}=b_{jl}$ it follows from (4.1) that $q_{lj}=q_{jl}$. Substituting (4.1) into the left side of (4.2), one obtains (4.2). On the other hand the left side of (4.2) $=\delta_{jl}$ which is independent of E. Rearranging the right side of (4.2) one has

$$\frac{C_j}{E-E_c}\left[\frac{C_l^*}{E_c-E_c^*}\sum_i |C_i|^2 + \sum_i C_i q_{il}^*\right] + \frac{C_l^*}{E-E_c^*}\left[\frac{C_j}{E_c^*-E_c}\sum_i |C_i|^2 + \sum_i C_i^* q_{ij}\right] + \sum_i q_{ij} q_{il}^* - \delta_{jl} = 0. \qquad (4.5)$$

This relation is of the form $A_1/(E-E_c)+A_2/(E-E_c^*)+B=0$ where A_1, A_2, B are independent of E and it must be satisfied for all real E. It follows that $B=0$ and that either $E_c=E_c^*$ (which is not the case) or else $A_1=A_2=0$. From $B=0$ one obtains (4.4) so that $\|q_{ij}\|$ is unitary. $A_2=0$ says that either $C_l^*=0$ or else the square bracket multiplying C_l^* in (4.5) vanishes. Varying l in (4.5) from 1 to n and keeping j fixed, one will find at least one $C_l^* \neq 0$. Otherwise there is no resonance scattering. The square bracket in (4.5) is, therefore $=0$ for any j so that Eq. (4.3) holds for all i.

The matrix $\|q_{ij}\|$ gives the *background* to the resonance scattering. It is reasonable that it turns out to be unitary and symmetric because

for large $|E-E_c|b_{ij}\gtrsim q_{ij}$. The conditions (4.3) will now be studied.

THE BACKGROUND MATRIX AND MEAN LIFE

Making use of the fact that $\|q_{ij}\|$ is unitary and symmetric one solves (4.3) for C_i^* and obtains

$$\sum q_{ij}^* C_j = \frac{E_c - E_c^*}{\sum_j |C_j|^2} C_i^* \tag{5}$$

and comparing this with the complex conjugate of (4.3) one has

$$|E_c - E_c^*| = \sum_j |C_j|^2 = \sum_j v_j |\alpha_j|^2 \hbar. \tag{5.1}$$

The quantity $\frac{1}{2}|E_c-E_c^*|$ is the absolute value of the imaginary part of E_c and $|E_c-E_c^*|$ is, therefore, the resonance half-width. According to Eq. (5.1) this half-width is equal to the sum of partial half-widths $|C_j|^2=\hbar v_j|\alpha_j|^2$ each of which is associated with one of the modes of disintegration. One has here a connection between the numerators of the dispersion terms of the scattering matrix and the resonance width. Equation (5.1) has a simple meaning in terms of mean life. For $E=E_c$ the time-dependent wave equation has a solution of the form (3) with a time factor $\exp[-iE_ct/\hbar]$. The reciprocal of the mean life of this state is $|E_c-E_c^*|/\hbar$ and according to (5.1) this transition probability can be considered as the sum of elementary transition probabilities $\hbar^{-1}|C_j|^2=v_j|\alpha_j|^2$.

On the other hand, another formula for the mean life is obtained from Eq. (1) and its complex conjugate for the radioactive state. Eliminating V by multiplication with ψ_c^* and ψ_c and subtraction and then applying Green's theorem one obtains on account of Eq. (3)

$$\frac{E_c^*-E_c}{\hbar i}\int |\psi_c|^2 d\tau \gtrsim \sum v_l |\alpha_l|^2, \tag{5.2}$$

so that comparing with (5.1)

$$\int |\psi_c|^2 d\tau \gtrsim 1. \tag{5.3}$$

This means that the normalization of the radioactive state made by setting $\alpha_j=\hbar^{-\frac{1}{2}}v_j^{-\frac{1}{2}}C_j$ is such as to give (5.3), i.e., one system in the region of configuration space out of which the wave function leaks into the channels; $|C_j|^2/\hbar$ is thus the rate of disintegration caused by the mode j.

According to (5.2) the imaginary part of E_c is negative. On account of (5.1) one has, therefore,

$$\sum_j |C_j|^2 = i(E_c - E_c^*) \tag{6}$$

and hence from Eq. (4.3) one has

$$\sum_j q_{lj} C_j^* = iC_l \tag{6.1}$$

as the conditions on $\|q_{lj}\|$ that must be required in addition to having it unitary and symmetric. These conditions can be simplified by letting

$$C_j = \rho_j \exp(-i\theta_j) \tag{6.2}$$

with ρ_j real and positive and θ_j real. Substituting this into (6.1) one has

$$\sum_j p_{lj}\rho_j = \rho_l, \tag{6.3}$$

$$p_{lj} = i^{-1} q_{lj} \exp[i(\theta_j+\theta_l)]. \tag{6.4}$$

The matrix $\|p_{lj}\|$ is symmetric and unitary as a consequence of the same properties of $\|q_{lj}\|$. Besides it is restricted by (6.3). For many modes of disintegration the restrictions (6.3) allow considerable freedom on p_{lj} because these restrictions mean that $\|p_{lj}\|$ has an eigenvalue 1 and that the vector corresponding to this eigenvalue is parallel to $(\rho_1, \rho_2, \cdots, \rho_n)$. The most general form of p_{lj} is obtained as follows.

One considers the general transformation of n variables $x_1, \cdots, x_n$, $x_i' = \sum p_{ij}x_j$ simultaneously with $x_i = \sum S_{ij}y_j$, $x_i' = \sum S_{ij}y_j'$. The matrix S_{ij} is taken to be real orthogonal. One has then

$$y_i' = \sum (S^{-1}pS)_{ij} y_j. \tag{6.5}$$

One makes besides

$$S_{1i}^{-1} = \rho_i / (\sum \rho_j^2)^{\frac{1}{2}} = S_{i1}. \tag{6.6}$$

One has

$$(S^{-1}pS)_{ij} = \sum S_{ik}^{-1} p_{kl} S_{lj} = \sum S_{ki} p_{lk} S_{jl}^{-1} = (S^{-1}pS)_{ji}.$$

Hence $S^{-1}pS$ is symmetric and unitary. Conversely if it is symmetric and unitary then p is also. On account of (6.6) one has

$$y_1' = \sum \rho_i x_i' / (\sum \rho_j^2)^{\frac{1}{2}} = \sum_{ij} \rho_i p_{ij} x_j / (\sum \rho_j^2)^{\frac{1}{2}}$$
$$= \sum_j \rho_j x_j / (\sum \rho_j^2)^{\frac{1}{2}} = y_1, \tag{6.7}$$

where $\sum\rho_i p_{ij}=\rho_j$ has been used in accordance with (6.3). Comparing (6.7) with (6.5) one sees that $(S^{-1}pS)_{1j}=\delta_{1j}$. Since $S^{-1}pS$ is symmetric $(S^{-1}pS)_{j1}=\delta_{j1}$. The only nonvanishing element in the first row and column of $S^{-1}pS$ is the diagonal element which is unity. The elements $(S^{-1}pS)_{ij}$ with i, j running from 2 to n must be, therefore, those of an n-1-dimensional symmetric unitary matrix U. Aside from these requirements the conditions used do not restrict U. Conversely for any symmetric unitary U one can form

$$\|p_{ij}\|=S\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}S^{-1} \tag{6.8}$$

and (6.3) as well as the other conditions for $\|p_{ij}\|$ are then satisfied.

Examples

One mode

The scattering matrix consists of one element. The notation is simplified by dropping the suffixes. One has $p=1$, $q=ie^{-2i\theta}$, $C=\rho e^{-i\theta}$, $E_c-E_c^*=-i\rho^2$ and

$$a=b=\frac{C^2}{E-E_c}+ie^{-2i\theta}=ie^{2i(K-\theta)}, \tag{7}$$

where

$$e^{2iK}=(E-E_c^*)/(E-E_c), \tag{7.1}$$

so that K is real and varies from 0 to π as E passes through $(E_c+E_c^*)/2$ from smaller to larger values. The asymptotic form of $r\varphi$ is

$$r\varphi\sim e^{-ikr}+e^{i(kr+2K-2\theta)}$$

and

$$-\frac{1}{2i}e^{iL\pi/2}r\varphi\sim\sin(kr-L\pi/2)-\frac{1}{2i}e^{i(kr-L\pi/2)}[1+ie^{i(2K-2\theta+L\pi)}]. \tag{7.2}$$

If the channel represents disintegration with relative angular momentum L, the second term represents the addition to the function usually called F_L. The square of the absolute value of this term is

$$\sin^2[K-\theta+(2L-1)\pi/4]$$

and the scattering cross section is

$$\sigma_s=(2L+1)(\lambda^2/\pi)\sin^2[K-\theta+(2L-1)\pi/4]. \tag{7.3}$$

According to (7.1) the rate of variation of K with E is a maximum for $E=(E_c+E_c^*)/2$ and there is a correspondingly rapid variation of the argument of $\sin^2$ in (7.3) at this energy.

Two modes

The matrix U in Eq. (6.8) has one element and its most general form is $e^{2i\xi}$ with a real ξ. The general form of $\|p_{ij}\|$ is thus easily obtained. Substituting it into (4.1) one has

$$\|b_{ij}\|=ie^{2iK}\begin{pmatrix}c^2e^{-2i\theta_1}, & cse^{-i(\theta_1+\theta_2)}\\ cse^{-i(\theta_1+\theta_2)}, & s^2e^{-2i\theta_2}\end{pmatrix}+ie^{2i\xi}\begin{pmatrix}s^2e^{-2i\theta_1}, & -cse^{-i(\theta_1+\theta_2)}\\ -cse^{-i(\theta_1+\theta_2)}, & c^2e^{-2i\theta_2}\end{pmatrix} \tag{8}$$

with

$$c=\rho_1/(\rho_1^2+\rho_2^2)^{\frac{1}{2}},\quad s=\rho_2/(\rho_1+\rho_2)^{\frac{1}{2}} \tag{8.1}$$

and K as in Eq. (7.1). To the modes of disintegration 1 and 2 there correspond angular momenta of relative motion L_1, L_2. By means of the above matrix one forms the asymptotic solutions

$$-\frac{1}{2i}e^{iL_1\pi/2}r_1\varphi_1'\sim\sin(k_1r_1-L_1\pi/2)-\frac{1}{2i}\{e^{i(k_1r_1-L_1\pi/2)}+i[c^2e^{2iK}+s^2e^{2i\xi}]e^{i(k_1r_1+L_1\pi/2-2\theta_1)}\};$$

$$-\frac{1}{2i}e^{iL_1\pi/2}r_2\varphi_2'\sim-\tfrac{1}{2}(v_1/v_2)^{\frac{1}{2}}cs[e^{2iK}-e^{2i\xi}]e^{i(k_2r_2-\theta_1-\theta_2+L_1\pi/2)}.$$

These formulas give the collision cross sections $\sigma_{1\to1}$, $\sigma_{1\to2}$ corresponding to incidence in channel 1

giving rise to scattering and disintegration via channel 2, respectively. One obtains in this way

$$\frac{\sigma_{1\to 2}}{(2L_1+1)\Lambda_1^2}=\frac{\sigma_{2\to 1}}{(2L_2+1)\Lambda_2^2}=\frac{\rho_1^2\rho_2^2}{\pi(\rho_1^2+\rho_2^2)^2}\sin^2(K-\xi) \tag{8.3}$$

and

$$\begin{aligned}\sigma_{1\to 1}+\sigma_{1\to 2}&=\frac{2L_1+1}{\pi}\Lambda_1^2[\rho_1^2\cos^2(K-\Theta_1)+\rho_2^2\cos^2(\xi-\Theta_1)]/(\rho_1^2+\rho_2^2),\\ \sigma_{2\to 2}+\sigma_{2\to 1}&=\frac{2L_2+1}{\pi}\Lambda_2^2[\rho_2^2\cos^2(K-\Theta_2)+\rho_1^2\cos^2(\xi-\Theta_2)]/(\rho_1^2+\rho_2^2).\end{aligned} \tag{8.4}$$

Here

$$2\Theta_i=2\theta_i-(2L_i+1)\pi/2. \tag{8.5}$$

Equation (8.3) shows the well-known reciprocity of cross sections which may be derived by considerations of detailed balance. *Resonance* is again defined most simply as that energy at which dK/dE is a maximum. According to (8.3) this energy can be determined, in principle, from a graph of $\sigma_{1\to 2}$ against E which may then be used also for a determination of ξ as well as ρ_1/ρ_2. The latter quantity cannot be determined uniquely from $\sigma_{1\to 2}$ and $\sigma_{2\to 1}$ since ρ_1/ρ_2 and ρ_2/ρ_1 can be interchanged. The width of resonance determines $\rho_1^2+\rho_2^2$ and hence in principle one has from the experimental $\sigma_{1\to 2}$ two possibilities for ρ_1^2, ρ_2^2 as well as values of ξ and $(E_c+E_c^*)/2$. It is obviously possible to use these values in data on $\sigma_{1\to 1}+\sigma_{1\to 2}$ and to test by means of the first equation (8.4) the validity of this formula.

Equation (8) gives a unitary symmetric matrix independently of whether Eq. (7.1) is obeyed or not. The only essential point is that K, θ_1, θ_2, c, s be real and that $c^2+s^2=1$. Discarding Eq. (7.1) and adopting (8) *ad hoc* one still obtains (8.3) and (8.4) but without an explicit connection between $\rho_1^2+\rho_2^2$ and the width of resonance. Such a connection is not necessary, however, in such a formulation because (8.3), (8.4) are homogeneous of degree zero in ρ_1, ρ_2. Only c, s enter into the observable cross sections. Such a formulation can also be tested experimentally by using (8.3) to determine $K-\xi$ as a function of the energy. On the assumption that the maximum of (8.3) corresponds to $\sin^2(K-\xi)=1$ one obtains cs and hence the first Eq. (8.4) contains only one adjustable constant $\xi-\Theta_1$.

The contribution due to $K-\Theta_1$ in (8.4) varies with E while the term in $\cos^2(\xi-\Theta_1)$ is independent of the energy and behaves as a background. If $\rho_2^2\gg\rho_1^2$ the background may be large in comparison with the maximum fluctuation in $\sigma_{1\to 1}+\sigma_{1\to 2}$ near resonance which is $(2L_1+1)\Lambda_1^2\rho_1^2/\pi(\rho_1^2+\rho_2^2)$. Thus, for example, Eq. (8.4) does not exclude the possibility of a scattering + disintegration cross section of slow neutrons of the order Λ_n^2, the resonance contribution on the other hand is limited to $\sim\Lambda_n^2\rho_n^2/(\rho_n^2+\rho_\gamma^2)$ which is of the same order as the capture cross section. The maximum value of $\sigma_{1\to 1}+\sigma_{1\to 2}$ for best values of $K-\Theta_1$, $\xi-\Theta_1$ is $(2L_1+1)\Lambda_1^2/\pi$.