

This table, together with Table II in Margenau and Watson's paper,⁵ and Table II in Ny and Ch'en's⁶ paper gives a complete tabulation of all available results for testing Margenau's theory.

⁵ H. Margenau and W. W. Watson, Rev. Mod. Phys. **8**, 44 (1936).

⁶ Ny Tsi-Zé and Ch'en Shang-Yi, Phys. Rev. **52**, 1160 (1937).

The varying of the ratio $\Delta\nu_{1/2}/\Delta\nu$ between 1.2 and 8 is certainly not strong support for it.⁷

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⁷ Thanks to Dr. H. Kuhn of the Clarendon Laboratory, Oxford, for his private communication to one of the authors regarding the point.

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On the Theory of the Magnetic Resonance Method of Determining Nuclear Moments

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Two refinements of the theory of the resonance method of determining magnetic moments as given by Rabi are considered quantitatively: (1) A correction is included because of the fact that an oscillating, rather than a rotating, field is used, (2) the "end effect" pointed out by Millman is taken into account. The results caused by (1) agree with those found by Bloch and Siegert, but the method of calculation is simpler. In the case of (2), it is shown that, under assumptions which mean that the effect is small, and which are usually well satisfied in practice, the effect can be worked out without making more than very general assumptions as to the nature of the field in the end regions. Results are given for the displacement of the maximum, and the asymmetry in the curve, which depend on the sign of the magnetic moment. The sign of the displacement is in agreement with experiment, but that of the asymmetry is not; the discrepancy can, however, be explained when account is taken of the inhomogeneity of the velocities in the beam. The effects increase as the square of the amplitude of the oscillating field.

1.

THIS paper is concerned with two refinements of the theory of the molecular beam resonance method of determining nuclear magnetic moments as developed by Rabi and his collaborators.¹ In the first place, there is included a correction resulting from the fact that in the experiments an *oscillating* magnetic field is used, whereas Rabi's original theory² assumed a *rotating* field. Secondly, the "end effect" due to the passage of the beam into the region of the oscillating field is considered quantitatively. This effect was pointed out by Millman,³ who showed that it was of importance as enabling the *sign* of the magnetic moment to be determined experimentally.

¹ I. I. Rabi, S. Millman, P. Kusch and J. R. Zacharias, Phys. Rev. **55**, 526 (1939), and subsequent papers.

² I. I. Rabi, Phys. Rev. **51**, 652 (1937), and reference 1.

³ S. Millman, Phys. Rev. **55**, 628 (1939).

It is evident that both effects can be made small by choosing the proper experimental conditions, but it seems desirable, nevertheless, to have some quantitative estimate of them. The first ("nonrotating") effect has already been considered by Bloch and Siegert⁴ in a paper which appeared after the present calculations were begun; but inasmuch as the method of calculation used here is simpler than that of these authors, it is perhaps still of interest to include a brief account of it. Our results agree with theirs as far as this part of the calculation is concerned.

2.

We consider only the case where we deal, effectively, with a single atom for which $I = \frac{1}{2}$, $J = 0$. The case where I is arbitrary and $J = 0$ can be deduced from this simple one by means

⁴ F. Bloch and A. Siegert, Phys. Rev. **57**, 522 (1940).

of an elegant result due to Majorana.⁵ Let, then ξ, η denote the probability amplitudes for the states in which the z component of angular momentum is $+\frac{1}{2}\hbar, -\frac{1}{2}\hbar$, respectively, and let (H_x, H_y, H_z) be the components of magnetic field acting on the atom. The equations for ξ, η are then

$$\begin{aligned} i\hbar\dot{\xi} &= -\mu[H_z\xi + (H_x - iH_y)\eta], \\ i\hbar\dot{\eta} &= -\mu[-H_z\eta + (H_x + iH_y)\xi], \end{aligned} \quad (1)$$

where μ is the nuclear magnetic moment, reckoned positive if it is directed in the same sense as the nuclear spin.

In order to explain the "end effect," we reproduce in Fig. 1 a schematic diagram of the circuit producing the oscillating magnetic field, as given by Millman.³ An oscillating source of current is applied at A, H , and the circuit consists of the bent copper tubing $AB \cdots H, OO'$ being the direction of the beam. The portions AB, GH, CD, EF of the circuit are parallel and at right angles to BC, FG . We take a system of right-handed axes as shown, with the y axis in the direction of the beam. In addition to the oscillating field, a strong uniform field is applied in the direction Oz (or zO). Over most of the path of the beam where the oscillating field is effective (within OO'), this field is of constant amplitude in the x direction. But in the neighborhood of O or O' , there is an end effect which causes its amplitude and direction to change. It is evident that if the nonrotating and end effects are both small, we can (to the first order) consider them separately, and simply superpose the results.

We seek first a solution of Eqs. (1) for the region within OO' where the field is what we shall term "purely oscillatory" (i.e., of constant amplitude in the x direction). We put then in (1)

$$H_x = H_1 \cos(\omega t + \delta), \quad H_y = 0, \quad H_z = \pm H,$$

where H_1 is the amplitude of the oscillating field of frequency $\omega/2\pi$, H the magnitude of the uniform field, and the $+$ or $-$ sign is to be taken according as the direction of H is Oz or zO . We assume $H_1 \ll H$, since this is so in all the experi-

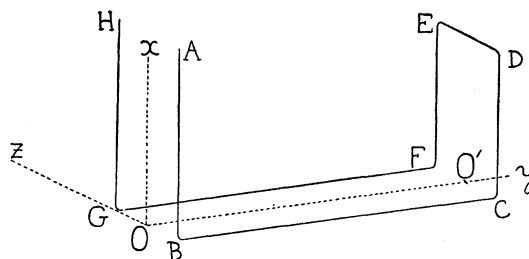


FIG. 1. Schematic diagram of the circuit producing the oscillating magnetic field.

ments. Equations (1) then become

$$\begin{aligned} ia\dot{\xi} &= \xi + \alpha[e^{-i(\omega t + \delta)} + e^{i(\omega t + \delta)}]\eta, \\ ia\dot{\eta} &= -\eta + \alpha[e^{i(\omega t + \delta)} + e^{-i(\omega t + \delta)}]\xi, \end{aligned} \quad (2)$$

where⁶

$$a = \pm \hbar/|\mu|H, \quad \alpha = H_1/2H (\alpha \ll 1). \quad (3)$$

The sign of a will be of importance when we come to discuss the end effect: $a > 0$ if $\mu > 0$ and H is in the direction zO . The direction of H can be conveniently related to the sense of rotation of the oscillating field caused by the end effect (cf. reference 3). It is easily seen from Fig. 1 that the end effects at O and O' are such as to rotate the oscillating field about the direction zO in the positive (right-hand screw) sense. We therefore have the rule: $a > 0$ if $\mu > 0$ and the end effect is such as to rotate the oscillating field about the direction of the uniform field in the positive sense; if either the sign of μ or the sense of rotation is reversed, the sign of a is reversed.

If (2) be solved by assuming ξ, η to be power series in α , the first approximation shows that, if $\eta = 0$ initially, then η remains permanently small, of order α , unless $|a\omega \pm 2|$ is itself small of the order α . In this case, secular terms, or terms of large amplitude, arise, which shows that this method of solution then breaks down when large time intervals are considered.⁷ For sufficiently short time intervals, however, the method of solution as power series in α is still applicable, and we shall use it in discussing the end effect.

⁶ We should also write $\alpha = \pm H_1/2H$. We can, however, still retain the definition (3) by suitably altering the phase δ ; the final results are independent of δ .

⁷ The higher approximations will in any case introduce secular terms, because of the method of solution. The first approximation is, however, a sufficient indication.

⁵ E. Majorana, *Nuovo Cimento* 9, 43 (1932).

The critical case $a\omega = \pm 2$, for which the transition probability need not be small even though α is small, arises when

$$H = H_0 = \hbar\omega/2|\mu|. \quad (4)$$

H_0 is the field for which the Larmor precession frequency is $\omega/2\pi$. When $H \sim H_0$, the important terms in α in (2), which give rise to the secular terms, are the first ones within the square bracket in each equation if $a > 0$, and the second ones if $a < 0$. Neglecting the unimportant terms is equivalent to replacing the oscillating field by a rotating field of half the amplitude (the sense of rotation depending on the sign of a); the presence of these terms gives rise to the nonrotating effect.

Let us consider the case $a > 0$. It is then convenient to put

$$\eta = e^{i(\omega t + \delta)} \eta'. \quad (5)$$

Equations (2) then become

$$\begin{aligned} ia\dot{\xi} &= \xi + \alpha[1 + e^{2i(\omega t + \delta)}]\eta', \\ ia\dot{\eta}' &= (\alpha\omega - 1)\eta' + \alpha[1 + e^{-2i(\omega t + \delta)}]\xi. \end{aligned} \quad (6)$$

The important terms in α are now the constant ones. We first recall the solution when only these terms are retained. It can be written

$$\begin{aligned} \xi &= Ae^{i\lambda_1 t} + Be^{i\lambda_2 t}, \\ \eta' &= -kAe^{i\lambda_1 t} + (1/k)Be^{i\lambda_2 t}, \end{aligned} \quad (7)$$

where A, B are arbitrary constants, and

$$\left. \begin{aligned} a\lambda_1 &= -1 + \alpha k, & a\lambda_2 &= -1 - \alpha/k, \\ k &= \beta + (1 + \beta^2)^{1/2}, \\ \beta &= (1/\alpha)(1 - |a|\omega/2) = 2(H - H_0)/H_1. \end{aligned} \right\} \quad (8)$$

With the initial conditions $\xi = 1, \eta = \eta' = 0$ when $t = 0$, we find for the probability of the transition $+\frac{1}{2}\hbar \rightarrow -\frac{1}{2}\hbar$, P , say ($P = |\eta|^2$),

$$P = (1 + \beta^2)^{-1} \sin^2 [\psi(1 + \beta^2)^{1/2}], \quad (9)$$

where

$$\psi = \alpha t/|a| = |\mu|H_1 l/2\hbar v, \quad (10)$$

l being the length of the purely oscillating field, and v the velocity of the atom. The result (9) for the transition probability, which is independent of the sign of a , is that obtained by Rabi.²

If $\psi < \pi$, P has a maximum for $\beta = 0$, or $H = H_0$;

the optimum value of ψ is $\pi/2$, i.e., the optimum value of H_1 is $\pi\hbar v/|\mu|l$. For larger values of ψ , P may have a *minimum* for $\beta = 0$, or maxima other than $\beta = 0$ may be of importance. The half-width of the P curve (width at half-maximum), regarding P as a function of β , or H ,⁸ depends on ψ . For $\psi = \pi/2$, the half-maximum values are at about $\beta = \pm 0.8$, so that the half-width ΔH is about $0.8H_1$.

It should be noted, however, that (9) and (10) apply for an atom of definite velocity v . Actually, the beam will contain atoms (or molecules) of different velocities, and an average over these velocities must be taken. This will have the effect of "smoothing out" the (\sin^2) -term in (9), making this factor more nearly a constant, so that P will probably *always* have a maximum for $\beta = 0$. The value of P , at this maximum, and the half-width, will also be affected by this averaging process. That this effect is operative can be seen from the experimental curves, which show no trace of the subsidiary maxima demanded by (9).

3.

We consider now the nonrotating effect, and sketch a method of approximating to the solution of the complete Eqs. (6) for small α . We first note, from (9) and (8), that within the important part of the P curve in the neighborhood of the maximum, which is all that we need be concerned with, $|\beta|$ varies between 0 and a quantity of the order of unity, and hence $||a|\omega - 2| \sim \alpha$, $k \sim 1$, $1/k \sim 1$.

Equations (6), being linear with periodic coefficients, possess an integral which is accurately of the form

$$\xi = e^{i\lambda' t} f(t), \quad \eta' = e^{i\lambda' t} g(t), \quad (11)$$

where f, g are periodic functions with the same period ($2\pi/\omega$). In general, there will be two independent integrals of this form. From (7), we have, for the zero-order solution corresponding

⁸ We shall always regard P in this manner, as in most of the experiments H is varied and the other parameters kept constant. In some of the latest experiments, however (P. Kusch, S. Millman and I. I. Rabi, Phys. Rev. 57, 765 (1940)), H is kept constant, and the frequency ω varied. The shape of the curve is exactly similar in this case, since β is a linear function of both H and ω .

to (11), $\lambda' = \lambda_1$, $g/f = -k$, or $\lambda' = \lambda_2$, $g/f = 1/k$. Considering the integral for which $\lambda' \sim \lambda_1$, we therefore assume expansions of the form

$$f = 1 + \alpha f_1 + \alpha^2 f_2 + \dots, \quad g = -k + \alpha g_1 + \alpha^2 g_2 + \dots, \\ \lambda' = \lambda_1 + c_1 \alpha + c_2 \alpha^2 + \dots$$

We then obtain a series of equations for the f 's and g 's which can be solved in succession. The condition that only periodic terms shall occur determines the constants c , and also effectively determines the arbitrary constants arising in the course of the work; any indeterminacy which remains is equivalent to multiplying the integral (11) by a constant factor. The method is sufficiently well known, and the calculations need not be reproduced in detail.⁹ In a similar manner, we can find an integral of the form (11) for which $\lambda' \sim \lambda_2$. With a convenient choice of the arbitrary constants, we thus obtain as the general solution of (2), for $a > 0$, and correct to the first order in α ,¹⁰

$$\xi = A e^{i\lambda_1' t} \left[1 + \frac{\alpha k}{4} \left(-\frac{2}{k^2 + 1} + e^{2i\phi} \right) \right] \\ + B e^{i\lambda_2' t} \left[1 + \frac{\alpha k}{4} \left(-\frac{2}{k^2 + 1} - \frac{1}{k^2} e^{2i\phi} \right) \right], \\ e^{-i\phi} \eta = -k A e^{i\lambda_1' t} \left[1 - \frac{\alpha k}{4} e^{-2i\phi} \right] \\ + \frac{1}{k} B e^{i\lambda_2' t} \left[1 + \frac{\alpha k}{4} e^{-2i\phi} \right],$$

where

$$a\lambda_1' = a\lambda_1 + \frac{\alpha^2}{4} \frac{k^2 - 1}{k^2 + 1}, \quad a\lambda_2' = a\lambda_2 - \frac{\alpha^2}{4} \frac{k^2 - 1}{k^2 + 1},$$

$$\phi = \omega t + \delta,$$

and A, B are arbitrary constants. If $a < 0$, we need only change the sign of ϕ throughout, if we retain the definitions (8) of k, λ_1, λ_2 .

With the initial conditions $\xi = 1, \eta = 0$ when $t = 0$, we obtain for the transition probability, for either sign of a , correct to the first order in α ,

$$P = (1 + \beta^2)^{-1} [1 - (\alpha/2)\beta(1 + \beta^2)^{-1}] \sin^2 [\psi(1 + \beta^2)^{\frac{1}{2}}(1 - (\alpha/4)\beta(1 + \beta^2)^{-1})].$$

We have omitted terms proportional to $\cos 2\delta, \sin 2\delta$, since we are dealing with a *beam* of particles, and must therefore average over the initial phases δ .⁴ To the same order, we may write

$$P = (1 + \beta'^2)^{-1} \sin^2 [\psi(1 + \beta'^2)^{\frac{1}{2}}],$$

where $\beta' = \beta + \alpha_0/4, \quad \alpha_0 = H_1/2H_0$.

Comparing with (9), we see that the shape of the curve is unaltered, but that the maximum is shifted to $\beta = -\alpha_0/4$, or

$$H = H_0 - H_1^2/16H_0$$

in agreement with the result of Bloch and Siegert.⁴ It will be seen that our method is more straightforward than theirs, which requires somewhat elaborate transformations.¹¹

⁹ See, for instance, the very full account in Moulton *et al.*, *Periodic Orbits* (Carnegie Institution, Washington, 1920), Chap. 1, Section 3. We can also proceed direct from Eqs. (2).

¹⁰ It can be shown that in order to calculate P to the first order, it is necessary to calculate λ' to the second order.

¹¹ The slightly more general case considered by them can be treated with equal facility by the present method, as can also the case (not considered by them) where the oscillating field has a component parallel to the uniform field. In this latter case, the effect is of the same order of magnitude as the "nonrotating" effect.

4.

We consider now the end effect. While the atom is under the influence of the end effects at O or O' (Fig. 1), there will be both x and y components to the oscillating field,¹² these components depending on the y coordinate of the atom as well as the current in the circuit. We therefore put in (1)

$$H_x = X(y) H_1 \cos(\omega t + \delta), \\ H_y = Y(y) H_1 \cos(\omega t + \delta), \quad H_z = \pm H,$$

where $X(y)$ denotes, at any moment, the ratio of the amplitude of H_x at the point whose coordinate is y to the amplitude at a point where the field is purely oscillatory, and $Y(y)$ denotes a similar ratio for H_y . Equations (1) then become, in the same notation,

$$i a \dot{\xi} = \xi + \alpha(X - iY)[e^{-i(\omega t + \delta)} + e^{i(\omega t + \delta)}]\eta, \\ i a \dot{\eta} = -\eta + \alpha(X + iY)[e^{i(\omega t + \delta)} + e^{-i(\omega t + \delta)}]\xi. \quad (12)$$

Evidently X and Y are functions with an upper bound of the order unity. Consideration of

¹² A possible z component would only give rise to a second-order effect. Cf. reference 11.

the field arising from the circuit of Fig. 1 shows that X and Y are positive in the neighborhood of O , while $X > 0$, $Y < 0$ in the neighborhood of O' . The end effects will be negligible except in regions in the neighborhood of O and O' whose length is of the order b , where b denotes the distance GB between the wires. The actual circuit may not conform very well to the idealized one of Fig. 1; but if the latter represents at all well the general features of the circuit, it is evident that the above statements are true.

We further define a parameter ϵ by

$$\epsilon = \alpha b / |a| v = |\mu| H_1 b / 2 \hbar v = \psi b / l, \quad (13)$$

and assume

$$\epsilon \ll 1, \quad v/b\omega \ll 1. \quad (14)$$

It will be shown that we can then integrate (12) approximately without making any further assumptions as to the functions X and Y beyond the very general ones already mentioned.

From (13), we see that the first of (14) is satisfied if $b/l \ll 1$, provided ψ has a value which does not differ much from the optimum one, namely $\pi/2$. The condition $b/l \ll 1$ is well satisfied in the actual apparatus. According to measurements given by Millman,³ the distance between the axes of the wires is about $\frac{1}{4}$ cm, and the length of the circuit about 6 cm, so that $b/l \sim 1/25$. In a later form of the apparatus described,¹³ $b/l \sim 1/50$. The second of (14) is also usually well satisfied; thus for $v = 10^5$

cm/sec., $\omega/2\pi = 3$ megacycles, $b = \frac{1}{4}$ cm, we have $v/b\omega \sim 1/45$.

The assumptions (14) ensure, as we shall see, that the end effect is *small*. The second assumption is not essential to the method of solution, but simplifies the analysis. It will be shown that the end effects are proportional to ϵH_1 , and therefore increase as H_1^2 . Hence H_1 must be increased beyond the optimum value in order to bring out the end effects prominently.

We suppose now that the end effect at O lasts from $t = -t_1$ to $t = 0$, the purely oscillating field from $t = 0$ to $t = T$, and the end effect at O' (which need not be identical with that of O) from $t = T$ to $t = T + t_2$. We suppose $a > 0$ for definiteness. For $0 < t < T$, the solution (neglecting the nonrotating effect) is given by (7). During the end effects, we solve (12) for ξ, η as power series in α .

For $-t_1 < t < 0$, we assume, with the initial conditions $\xi = 1, \eta = 0$ when $t = -t_1$,

$$\begin{aligned} \xi &= e^{-(i/a)(t+t_1)} + \alpha \xi_1 + \alpha^2 \xi_2 + \dots, \\ \eta &= \alpha \eta_1 + \alpha^2 \eta_2 + \dots, \end{aligned} \quad (15)$$

where the initial conditions require the vanishing of ξ_1, η_1, \dots for $t = -t_1$. The first approximation gives

$$\begin{aligned} ia\xi_1 - \xi_1 &= 0, \\ ia\eta_1 + \eta_1 &= (X + iY)[e^{i(\omega t + \delta)} + e^{-i(\omega t + \delta)}]. \end{aligned}$$

Hence, taking account of the initial conditions,

$$\begin{aligned} \xi_1 &= 0, \\ \eta_1 &= -(i/a)e^{(i/a)(t+t_1)} \left[e^{i(\delta - \omega t_1)} \int_{-t_1}^t (X + iY) e^{i(\omega - 2/a)(t+t_1)} dt + e^{-i(\delta - \omega t_1)} \int_{-t_1}^t (X + iY) e^{i(\omega + 2/a)(t+t_1)} dt \right]. \end{aligned}$$

Changing the variable of integration from t to y ($dy/dt = v$) in both integrals, and using (13), we therefore have

$$\begin{aligned} \alpha \eta_1(0) &= -i\epsilon e^{it_1/a} \left[e^{i(\delta - \omega t_1)} \int_{y_1}^{y_2} (X + iY) e^{i(\omega - 2/a)(y - y_1)/v} (dy/b) \right. \\ &\quad \left. + e^{-i(\delta - \omega t_1)} \int_{y_1}^{y_2} (X + iY) e^{i(\omega + 2/a)(y - y_1)/v} (dy/b) \right], \end{aligned} \quad (16)$$

where y_1, y_2 are the values of y at the beginning and end of the end effect at O .

Now, as remarked in Section 3, $|a\omega - 2|$ is (for $a > 0$) of the order α within the important part of the P curve, so that, since $(y_2 - y_1) \sim b$ according to our assumptions, the argument of the exponential

¹³ J. M. B. Kellogg, I. I. Rabi, N. F. Ramsey and J. R. Zacharias, Phys. Rev. **56**, 728 (1939).

in the first integral in (16) is of order $\alpha b/av = \epsilon$. Hence the first integral in (16) is, in absolute value, of the order unity, and we can put the exponential in this integral equal to 1 if we are only working to the first order in ϵ . In the second integral in (16), on the other hand, the argument of the exponential $\sim \omega(y - y_1)/v$. From the second of (14), we therefore see that the real and imaginary parts of this integral are rapidly oscillating functions with an upper bound of the order unity within the domain of integration, and the absolute value of the second integral will not exceed $v/b\omega$ in order of magnitude. Hence, to the first order in ϵ and $v/b\omega$ we can neglect the second integral in (16), and can write

$$\alpha\eta_1(0) = -i\epsilon e^{i(t_1/a + \delta - \omega t_1)}(I_1 + iI_2),$$

where

$$I_1 = \int_{y_1}^{y_2} X dy/b, \quad I_2 = \int_{y_1}^{y_2} Y dy/b. \quad (17)$$

I_1, I_2 are positive dimensionless quantities of the order unity.

Thus $|\alpha\eta_1(0)| \sim \epsilon$, and it is clear that the development (15) is justifiable, and that a good approximation is obtained by stopping at the first term. To the first order in ϵ , $v/b\omega$ we therefore have, from (15),

$$\xi(0) = e^{-i t_1/a}, \quad \eta(0) = -i\epsilon e^{i(t_1/a + \delta - \omega t_1)}(I_1 + iI_2). \quad (18)$$

Similarly the end effect at O' is dealt with. We assume, analogously to (15),

$$\xi = \xi(T)e^{-(i/a)(t-T)} + \alpha\xi_1 + \dots, \quad \eta = \eta(T)e^{(i/a)(t-T)} + \alpha\eta_1 + \dots,$$

and find, to the same order,

$$\eta(T+t_2) = e^{i t_2/a} [\eta(T) - i\epsilon e^{i(\delta + \omega T)}(I_1' + iI_2')\xi(T)], \quad (19)$$

where

$$I_1' = \int_{y_1'}^{y_2'} X dy/b, \quad I_2' = \int_{y_1'}^{y_2'} Y dy/b, \quad (20)$$

and y_1', y_2' are the values of y at the beginning and end of the end effect at O' . The constants I_1', I_2' are also of the order unity, and $I_1' > 0, I_2' < 0$. It is not necessary to give the value of $\xi(T+t_2)$.

We now use (18) to determine the arbitrary constants A, B in (7), calculate $\xi(T), \eta(T)$ from (7), and then use (19) to determine $\eta(T+t_2)$. To the first order we find

$$\exp i[(1/a)(t_1 - t_2) - (\omega T + \delta)] \cdot (k+1/k)\eta(T+t_2) = -e^{i\lambda_1 T} + e^{i\lambda_2 T} \\ - i\epsilon [(I_1 + iI_2)(ke^{i\lambda_1 T} + (1/k)e^{i\lambda_2 T}) + (I_1' + iI_2')((1/k)e^{i\lambda_1 T} + ke^{i\lambda_2 T})]. \quad (21)$$

Equation (21) applies for $a > 0$. In a similar manner the calculation for $a < 0$ is carried through, and we find that we need merely change the signs of ω, δ, ϵ where they occur explicitly in (21), if we retain the definition (13) of ϵ . With the help of (8), and using $T = l/v$, we find from (21) that the transition probability, $P = |\eta(T+t_2)|^2$, can be expressed, correct to the first order, in the form

$$P = (1 + \beta^2)^{-1} (1 - 2\epsilon C\beta) \sin^2 [\psi'(1 + \beta^2)^{\frac{1}{2}}], \quad (22)$$

where

$$\psi' = \psi + \epsilon C', \quad C = I_2 - I_2', \quad C' = I_1 + I_1'.$$

Equation (22) holds, of course, only within the

important part of the P curve ($|\beta| \lesssim 1$), and is for the case $a > 0$. If $a < 0$, we must change the sign of ϵ in the second factor of (22) (but *not* in the definition of ψ'). The constants C, C' are positive, and of order unity.¹⁴

The presence of the constant C' merely has the

¹⁴ A rough estimate of them may be made by supposing that the circuit of Fig. 1 consists of ideal linear circuits AB, BC, \dots , with right angles at B, \dots , and which can be treated as being of infinite length in comparison with b . For the field in the neighborhood of 0 we then find

$$2X = 1 + 2y(b^2 + 4y^2)^{-\frac{1}{2}}, \quad 2Y = b^2/(b^2 + 4y^2)$$

and can suppose that the end effect at 0 is confined (say) to the region $-3b/2 \leq y \leq 3b/2$. We then find

$$I_1 = I_1' = 1.5, \quad I_2 = -I_2' = 0.625,$$

whence $C = 1.25, C' = 3.0$.

effect of increasing the effective length l of the circuit, and is unimportant; but the presence of the second factor in (22) shifts the maximum of the P curve and introduces an asymmetry. Always working to the first order in ϵ , and taking the case $a > 0$ again, we find that the maximum occurs at

$$\beta = -C\epsilon/(1 - \psi' \cot \psi'),$$

or

$$H = H_0 - C\epsilon H_1/(1 - \psi' \cot \psi'). \quad (23)$$

For $\psi' = \pi/2$, this becomes

$$H = H_0 - C\epsilon H_1/2. \quad (24)$$

The shift of the maximum given by (23) is toward the low field side if $\psi' < \pi$, and it is probable that when an average over molecular velocities is taken, the shift is always in this direction (cf. the discussion in Section 2). Thus, *the maximum is shifted to the low field or high field side according as a is positive or negative*. As already discussed, the sign of a depends on the sign of μ and the direction of the uniform field relative to the circuit. This result is in accordance with the results of Millman.³

As regards the asymmetry, (22) makes the curve steeper on the low field or high field side of the maximum according as a is positive or negative, which is precisely the *opposite* of the result found experimentally. The discrepancy can, however, be explained when the already mentioned averaging over molecular velocities is taken into account. For this would make the (\sin^2) -terms in (22) more nearly a constant, and would probably have the effect of reversing the asymmetry. This is certainly the case if we assume as a rough approximation that the (\sin^2) -terms in (22) can be put equal to a constant. The values of β for which P has half the maximum value are then given by

$$\beta' = 1 - \epsilon C, \quad -1 - \epsilon C,$$

where $\beta' = \beta + \epsilon C$, so that the origin for β is

transferred to the maximum. Hence the *difference* in the half-widths on either side of the maximum is $2\epsilon C$, or, in terms of H ,

$$\Delta H_1 - \Delta H_2 = C\epsilon H_1, \quad (25)$$

the curve being wider on the low field or high field side according as a is positive or negative. The displacement of the maximum is again given by (24).

A quantitative comparison of our results with experiment is scarcely possible at present, since the value of H_1 is not usually made clear in the experiments, and most of the curves are given for one direction only of the uniform field, so that only the asymmetry can be judged, for which, as pointed out, the averaging over molecular velocities is of importance. A possible source of comparison is furnished by Fig. 3 of Millman's paper,³ where curves for LiCl molecules are shown with both directions of the uniform field. If we assume that (24) and (25) hold for this case, we see that the difference in the half-widths for either curve should be equal to the distance between the intensity minima (P maxima) of the two curves. Actually, the difference in the half-widths is only about one-half of this. The discrepancy can be accounted for by the fact that (25) is obtained by quite a crude assumption as to velocity averaging.

The calculations of this section all relate to the case where the end effect is *small* ($\epsilon \ll 1$). If the amplitude of the oscillating field H_1 is so large that our analysis becomes inapplicable, it is not surprising that the P curve becomes more complicated and may show additional maxima.³ But for most purposes it is, of course, *desirable* to make the end effect small.¹⁵

I wish to acknowledge some helpful suggestions made by the referee.

¹⁵ *Note added in proof.*—The end effect could be reduced by rotating the loop $CDEF$ of the circuit (Fig. 1) through 180° about CF , should such a change be feasible, for this would change the sign of I_2' , and hence reduce C .