Magnetic Moments of Odd Nuclei

H. MARGENAU,* *Yale University, New Haven, Connecticut*

AND

E. WIGNER, *Princeton University, Princeton, New Jersey*

(Received May 15, 1940)

To understand the grouping of the nuclear magnetic moments a generalization of Schmidt's single-particle model is considered. In the first approximation, the ground state of an odd nucleus is taken to be a doublet state with definite partition quantum numbers, but involving both possible values of the azimuthal quantum number L. Only very general assumptions are made concerning the detailed composition of the wave function. On this basis the *orbital* part of the magnetic moment turns out to be essentially that following from the liquid drop model; the *spin* part is identical with that of Schmidt. The limits of the magnetic moments, when plotted against J, are given by the solid lines on Fig. 1. Next, we have calculated the deviations from the magnetic moments thus obtained which arise from the admixture of states with higher partition quantum numbers to the original wave function. The result is that the magnetic moment is not greatly changed even when the partition quantum numbers are no longer sharp. (Cf. Fig. 2.) The present model fails to explain the near equality of the moments of isotopes which has been found experimentally.

MAGNETIC MOMENTS OF ODD NUCLEI

1. The rapid accumulation of experimental data on the magnetic moments of nuclei throughout the periodic table on the one hand and the great accuracy which can now be achieved in their measurement on the other, encourage theoretical consideration in this field. Previous calculations of magnetic moments may be divided into two groups: those proceeding with detailed assumptions about nuclear forces and the structure of the nucleus,[1] and others in which more general and qualitative ideas are used. The former calculations are restricted to light nuclei. They involve rather detailed assumptions concerning the nuclear wave functions such as the independent particle model or the α-particle model. Furthermore, Russell-Saunders coupling was assumed throughout until recently when Rose and especially Phillips emphasized the significance of the spin-spin interaction, particularly for the magnetic moments of even nuclei with odd number of protons and neutrons. These calculations were extended and critically reviewed recently by Inglis.[2]

The more schematic considerations of Schüler, Schmidt and Hund[3] as well as those of the present paper are based on the fact that the magnetic moments μ, plotted as function of the total angular momentum J, lie between two rather close curves. This regularity holds for odd nuclei, and the two curves for nuclei with an odd number of protons and those for nuclei with an odd number of neutrons are different. The experimental material at present available is illustrated in Fig. 1.

In order to explain this regularity, the above authors assumed tentatively that the whole magnetic moment is due to a single particle, which is a proton in the former and a neutron in the latter case. Its state can be described by a total and an orbital angular momentum. The former is the total J of the nucleus, the latter, l, is either $J+\frac{1}{2}$ or $J-\frac{1}{2}$. The magnetic moment is calculated by the customary Landé-Goudsmit formula and should lie either on the upper or on the lower broken line of Fig. 1, corresponding to the two possibilities $l=J\pm\frac{1}{2}$.

One would be tempted to interpret the fact that the observed magnetic moments lie *between* rather than *on* these curves by assuming a deviation from the L–S scheme. However, no

* Member of Institute for Advanced Study, Princeton, Autumn, 1939.

[1] M. E. Rose and H. Bethe, Phys. Rev. **51**, 205 (1937); H. Bethe, Phys. Rev. **53**, 842 (1938); L. R. Hafstad and E. Teller, Phys. Rev. **54**, 681 (1938); M. E. Rose, Phys. Rev. **56**, 1064 (1939); M. Phillips, Phys. Rev. **57**, 160 (1940); D. Inglis, Phys. Rev. **51**, 531 (1937); **53**, 882 (1938); **55**, 329 (1939); R. G. Sachs, Phys. Rev. **55**, 825 (1939).

[2] D. R. Inglis, Phys. Rev. **56**, 1175 (1939).

[3] Th. Schmidt, Zeits. f. Physik **106**, 358 (1937); H. Schüler, Zeits. f. Physik **107**, 12 (1937); F. Hund, Physik Zeits. **38**, 929 (1937).

such deviation exists in the nonrelativistic[4] one-body problem, on account of the parity rule. The nearest one could come to the above picture would be to assume for the correct wave function a linear combination of two wave functions of the above kind. These would correspond, respectively, to an even and an odd state of the core, both with angular momentum zero. The orbital angular momentum of the outside particle would be $J-\frac{1}{2}$ in one and $J+\frac{1}{2}$ in the other wave function. Such a model would naturally and directly lead to magnetic moments between the two Schmidt curves. The assumption concerning the angular momentum of the core finds support in the fact that all nuclei with an even number of both protons and neutrons have no angular momentum. The assumption that the whole orbital angular momentum is carried by one particle is to some extent supported by the remark of Hund that the total angular momentum is never greater than $\frac{1}{2}$ plus what the angular momentum of a single particle would be, assuming a reasonable shell structure.[5]

In spite of this, and in spite of the apparent success of the above model for the explanation of nuclear magnetic moments, it seems to us to be somewhat too specialized. In the first place, it is hard to understand that the core wave function should contain only states with zero angular momenta, considering that states with higher angular momenta of the core alone (an even-even nucleus) are known in many cases to be very close to the normal state. It appears indeed highly doubtful that the first excited state of even-even nuclei is a state with zero angular momentum in the majority of all cases. In addition to this, no simple two-particle operator has matrix elements between two wave functions of the above described character. The fact that both occur in an actual wave function would be, therefore, most probably due to the coupling of both to a third part in the wave function which corresponds to a core with a finite angular momentum. Although this third part could be very small, it appears improbable that it should couple only such states in which the core has no angular momentum. It may be mentioned, furthermore, that, as Bethe[6] has pointed out, there should be a marked difference between the Couloumb energies of nuclei with even and odd Z if a single particle model were valid for the last particle. Recent measurements indicate that such an effect, if it exists, is very small.[7] We have decided, therefore, to look for a more general model which is capable of explaining the grouping of the magnetic moments between the two Schmidt curves.

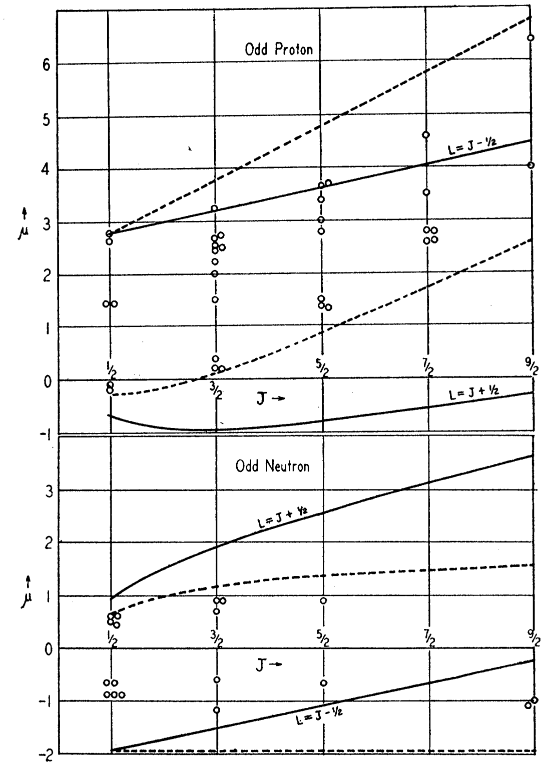

FIG. 1. Magnetic moments of odd nuclei plotted against J. Circles represent experimental values. Solid lines are upper and lower limits as obtained in this paper if only the lowest partition quantum numbers are taken into account. Dotted lines are limits obtained by Schmidt on the basis of a one-particle model.

[4] H. Margenau, Phys. Rev. **57**, 383 (1940). This calculation shows that the relativistic corrections to the magnetic momenta are too small to explain the deviation of the measured momenta from the curves of Fig. 1.

[5] W. Elsasser, J. de phys. et rad. **4**, 549 (1933); **5**, 389 635 (1934); H. Margenau, Phys. Rev. **46**, 613 (1934); F. Hund, reference 3.

[6] H. A. Bethe, Phys. Rev. **54**, 436 (1938).

[7] Recent measurements of W. H. Barkas, E. C. Creutz, L. A. Delsasso and M. G. White and of R. O. Haxby, W. E. Shoupp, W. E. Stephens and W. H. Wells (cf. also Bull. Am. Phys. Soc., New York meeting, February, 1940) show that the Coulomb energy difference is a much smoother function of the atomic mass than appeared at the time reference 6 was written. This applies, according to a kind personal communication of Dr. Stephens, particularly for nuclei with a mass number greater than 9.

This has been attempted before by K. Way[8] using an extreme form of the liquid drop model. She assumed that all particles participate about equally in the orbital motion so that the g factor ($2\ Mc/\hbar e$ times μ/J) for the orbital motion is $n_p/(n_p+n_N)$. This appears quite reasonable. Way assumes a similar g factor for the spin moment also. This would correspond to a very broad distribution of the spin angular momenta over all possible values—an assumption which is hardly acceptable. In the present paper, the g factor of the spin angular momentum will be calculated with a "symmetric Hamiltonian." This will give, essentially, the same g factor for the spin which obtains in Schmidt's model. This part of the model is, thus, to a large degree independent of the assumption that the moment is due to a single particle. For the g factor of the orbital moment, on the other hand, our result is identical with that of K. Way. Assuming Russell-Saunders coupling, we arrive at the conclusion that the magnetic moments should lie on the full lines of Fig. 1 while deviations from Russell-Saunders coupling (which are, of course, quite possible in our model) would shift the magnetic moments to the region between the full lines.

2. In describing the ground states of odd nuclei the assumption will be made that a spin-independent, symmetric Hamiltonian is a good starting point for the calculation of wave functions. In this approximation the states can be characterized by partition quantum numbers $(PP'P'')$ and the normal states of all odd nuclei are doublet states, and also the azimuthal quantum number L and $Y_\zeta=\frac{1}{2}\sum\tau_\zeta\sigma_Z$ have sharp values.[9] Considering now interactions which do not satisfy the above requirements, states with different S, L, etc. will mix. The mixing will be much stronger between states with different L and equal S and $(PP'P'')$, than for states which differ also in S and $(PP'P'')$. The reason for this is that states with different L can be very close to the normal state[10] while states with higher S or $(PP'P'')$ belong to a different "supermultiplet" and lie necessarily much higher in first approximation. Our assumption for the normal state will be, that the state has $S=\frac{1}{2}$ and the lowest possible partition quantum numbers (cf. further below), while for L only the assumption will be made that it can give, together with $S=\frac{1}{2}$, the total J of the nucleus. Thus, the wave function will contain both an $L=J-\frac{1}{2}$ and an $L=J+\frac{1}{2}$ part. These assumptions will make the spin moment equal to the value assumed by Schmidt.

In the following two sections, we shall carry out the calculations which were outlined above. Assuming a symmetric Hamiltonian, we could assume that the wave function has the lowest possible partition quantum numbers. This would allow us to calculate the magnetic moments, as outlined above. However, it would not permit us to estimate the deviation from this magnetic moment which is caused by an admixture of states with other partition quantum numbers to the original wave function. This, however, is quite necessary because the $(PP'P'')$ cannot be expected to be good quantum numbers for heavy nuclei. In spite of this, most of the heavy nuclei, also, fall between our two curves. This can be understood on the basis of the above picture only if moderate deviations from the assumed wave functions do not give very large changes in the magnetic moment, i.e., if the magnitude of μ is not changed very strongly even when $(PP'P'')$ loses its strict significance as a quantum number. This will indeed turn out to be the case. In view of this, it should be noted, even the fact that the moments of *all* light nuclei are between the two curves does not *prove* that $(PP'P'')$ is a very good quantum number for them.

3. The so-called magnetic moment of the nucleus, μ, is the component M_Z of the magnetic moment in a direction Z in which the component of the total angular momentum J_Z is equal to the total angular momentum J. The magnetic moment consists of two parts, the orbital magnetic

[8] K. Way, Phys. Rev. **55**, 963 (1939).

[9] E. Wigner, Phys. Rev. **51**, 106 (1937); F. Hund, Zeits. f. Physik **105**, 202 (1937). In explanation of the terminology here used it may be stated that $(PP'P'')$ is the name of a supermultiplet, just as S in atomic spectra labels a multiplet. P is the highest value which either S_z, T_ζ, or Y_ζ can take, just as, for ordinary multiplets, S is the highest value which S_z can take. P' is the highest value of one of the remaining pair (e.g. S_z or Y_ζ if $T_\zeta=P$); P'' is the highest possible value of the remaining operator (e.g., Y_ζ if $T_\zeta=P$, $S_z=P'$). Mathematically, S_z, T_ζ and Y_ζ play entirely identical roles. The normal state of a nucleus has as small values of P, P', P'' as possible. Since $P\geq P'\geq P''$, the best choice for P is the value of T_ζ. Thus, for the lowest state, $P=T_\zeta$, and P' and P'' are as small as possible. For the low excited states we still have $P=T_\zeta$, but P' and P'' are somewhat greater than for the ground state.

[10] E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778 (1938).

moment Λ_Z and the spin magnetic moment Σ_Z. It has been discussed repeatedly[11] whether it is permissible to consider the magnetic moment as the sum of the orbital and spin moments of the different particles. We shall assume here that this is permissible within the accuracy of our calculations.

We shall consider first the orbital magnetic moment which is, in units of the Bohr nuclear magneton, equal to the Z component of the orbital momenta of the protons, in units of $\hbar$. As mentioned before, it is not possible to carry out the calculation of this quantity rigorously but we shall adduce a plausibility argument for the correctness of the value assumed by Way.

Let us decompose the total wave function Ψ into parts with definite L and S

$$\Psi=\sum a_{LS}\Psi_{LS}. \tag{1}$$

If Russell-Saunders coupling could be assumed, (1) would consist of only one term. The Ψ_{LS} are assumed to be normalized, the sum of the squares of the $|a_{LS}|$ is 1. Since the total orbital angular momentum does not commute with the Z component of the orbital proton momentum, there will appear in the calculation of $(\Psi, \Lambda_Z\Psi)$ certain cross terms between Ψ_{LS} and $\Psi_{L'S}$. These are generally small in comparison with the diagonal terms and will here be disregarded. Justification for this procedure will be given at the end of this section. Thus the orbital magnetic moment for Ψ will be taken to be the sum of the magnetic moments of the different parts of Ψ in (1):

$$(\Psi, \Lambda_Z\Psi)=\sum_{LS} |a_{LS}|^2(\Psi_{LS}, \Lambda_Z\Psi_{LS}). \tag{2}$$

The Ψ_{LS} in (1) have a definite Z component of the total angular momentum J_Z but no definite Z components of L and S. We can denote the functions with definite Z components of L and S, out of which Ψ_{LS} is compounded, by $\Psi_{LSm\mu}$ where m and μ are the Z components L_Z and S_Z of L and S, respectively. We can then define a g factor for every one of these functions by the equation

$$(\Psi_{LSm\mu}, \Lambda_Z\Psi_{LSm\mu})=g_\Lambda(LS)m. \tag{3}$$

The g_Λ do not depend on m or μ but depend, of course, on all other quantum numbers. The matrix elements for Ψ_{LS} can be calculated from the g_Λ by the well-known Landé-Goudsmit formula

$$(\Psi_{LS}, \Lambda_Z\Psi_{LS}) =J_Z\frac{J(J+1)+L(L+1)-S(S+1)}{2J(J+1)}g_\Lambda(LS). \tag{4}$$

Our task is therefore to calculate the $g_\Lambda(LS)$. For this purpose we can further decompose each $\Psi_{LSm\mu}$ into parts in which the total orbital momenta of the protons and neutrons alone have definite values, l and l':

$$\Psi_{LS}=\sum_{ll'} b_{LSll'}\Psi_{LSll'}. \tag{5}$$

Again the sum of $|b_{LSll'}|^2$ over l and l' is 1, and the matrix element $(\Psi_{LS}, \Lambda_Z\Psi_{LS})$ can be calculated as the sum of the matrix elements of Λ_Z for the different $\Psi_{LSll'}$ separately. Formulas derivable from the vector addition principle yield immediately (since the g factor for protons is 1, for neutrons 0)

$$g_\Lambda(LS)=\sum_{ll'} |b_{LSll'}|^2\frac{L(L+1)+l(l+1)-l'(l'+1)}{2L(L+1)}. \tag{6}$$

The summation in (6) must be carried out, in general, over all values of l and l' which can give a total L according to the vector addition model. Eqs. (4) and (6) give the average value of Λ_Z for the wave function Ψ_{LS} in terms of the probabilities for this L being composed in the different possible ways out of orbital momenta of protons and neutrons, l and l'. At this point it is necessary to make some assumption concerning these probabilities. For a nucleus with an equal number of neutrons and protons, it follows from the symmetry of the Hamiltonian with respect to protons and neutrons that the probability for the angular momenta l and l' for protons and neutrons, is equal to the probability for the momenta l' and l. This means $|b_{LSll'}|^2=|b_{LSl'l}|^2$ and, under this condition, (6) can be summed up and gives $\frac{1}{2}$. This is in agreement with Way's value $n_P/(n_p+n_N)$. It can be shown, furthermore, that for $n_P\neq n_N$, the deviation of (6) from $\frac{1}{2}$ is proportional to the neutron excess. This is also in agreement with Way's assumption. It is not possible to show, in general, that the proportionality constant is $\frac{1}{2}(n_P+n_N)^{-1}$; in fact, it could be different for

[11] Cf. W. E. Lamb and L. I. Schiff, Phys. Rev. **53**, 651 (1938); D. R. Inglis, reference 2.

different isotopic spin multiplets. However, it clearly has this value if $n_P=0$ (in this case $l=0$ in (6)) and we shall assume it henceforth.

Schmidt's assumption, on the other hand, would amount to $b_{LSll'}=0$ unless $l'=0$ in case of an odd n_P, and $b_{LSll'}=0$ unless $l=0$ in case of odd n_N. Although, as mentioned before, this gives results which are in closer agreement with experimental data than our results are, we could not satisfy ourselves that it conforms with the picture of nuclear structure formed on the basis of other lines of evidence. It should be mentioned, however, that the nuclear moment does not depend critically on the validity of these assumptions since the whole orbital magnetic moment plays only a subordinate role in the total moment. Thus our curves do not differ greatly from Schmidt's.

Our final result for the orbital part of the magnetic moment is

$$(\Psi, \Lambda_Z\Psi)=\sum_{LS}|a_{LS}|^2\frac{n_P}{n_P+n_N}J_Z \times\frac{J(J+1)+L(L+1)-S(S+1)}{2J(J+1)}. \quad (7)$$

The $|a_{LS}|^2$ will be determined later.

We have neglected some cross terms in Eq. (2) and we wish now to give an estimate for them. Their exact calculation would, of course, necessitate the use of a specific nuclear model which would yield the coefficients $b_{LSll'}$, whose number is unlimited. Each cross term involves a summation of the type

$$\sum_{ll'} b_{LSll'}{}^* b_{L'Sll'}(\Psi_{LSll'}, \Lambda_Z\Psi_{L'Sll'}).$$

These cross terms vanish, of course, unless $L'=L\pm1$. For their evaluation it is important to note that the proton and neutron wave functions ψ_l and $\psi_{l'}$ out of which $\Psi_{LSll'}$ is composed, are in general different from the wave functions $\psi_l{}'$ and $\psi_{l'}{}'$ out of which $\Psi_{L'Sll'}$ is composed. However, the matrix elements of Λ_Z are largest if $\psi_l=\psi_l{}'$ and $\psi_{l'}=\psi_{l'}{}'$ and are otherwise proportional to the scalar products of these wave functions. These scalar products must be expected to be quite small. Furthermore, the b will vary quite irregularly as to sign and magnitude, and since the sum of their squares is unity, the result of the summation is further diminished. (The Hartree model would be quite unsuited for the calculation of the above sums since it gives only a small number of b's and since the above ψ_l, $\psi_{l'}$ are closely related to the $\psi_l{}'$, $\psi_{l'}{}'$.)

4. For the calculation of the spin magnetic moment we decompose the wave function into parts which have definite partition quantum numbers $(PP'P'')$ and definite angular momenta L

$$\Psi=\sum c_{PL}\Psi_{PL}. \quad (8)$$

P is an abbreviation for $(PP'P'')$ (the (STY) of reference 9). The Ψ_{PL} are assumed to be normalized, the sum of the squares of $|c_{PL}|$ is 1. If the "first approximation" of reference 9 could be assumed to be valid, (8) would contain only one term, corresponding to a single partition and a single L. Actually, it will be necessary to assume that terms corresponding to more than one value of L, at least, occur in (8). The operator for the magnetic moment is

$$\begin{aligned}\Sigma_Z&=\mu_P\sum_i \tfrac{1}{2}(1-\tau_{\zeta i})\sigma_{Zi}+\mu_N\sum_i \tfrac{1}{2}(1+\tau_{\zeta i})\sigma_{Zi}\\ &=\tfrac{1}{2}(\mu_P+\mu_N)\sum_i \sigma_{Zi}+\tfrac{1}{2}(\mu_N-\mu_P)\sum_i \tau_{\zeta i}\sigma_{Zi}.\end{aligned} \quad (9)$$

Here, $\tau_{\zeta i}$ is the isotopic spin coordinate for the ith particle. Its value is 1 for neutrons, -1 for protons. The σ_{Zi} are the usual matrices for the Z components of the spin angular momenta, with characteristic values $+1$ and -1. The μ_P and μ_N are the magnetic moments of proton and neutron. If we again express all magnetic moments in nuclear Bohr magnetons $e\hbar/2Mc$, (9) becomes

$$\Sigma_Z=0.855S_Z-4.715\,Y_\zeta, \quad (9a)$$

where

$$S_Z=\tfrac{1}{2}\sum_i \sigma_{Zi};\quad Y_\zeta=\tfrac{1}{2}\sum_i \tau_{\zeta i}\sigma_{Zi}.$$

Since (9a) does not depend on space coordinates, it has no matrix elements between parts of the wave function which belong to different L or to different partitions. Its mean value is, therefore

$$(\Psi, \Sigma_Z\Psi)=\sum_{PL}|c_{PL}|^2(\Psi_{PL}, \Sigma_Z\Psi_{PL}), \quad (10)$$

i.e., simply the sum of the magnetic moments of the different parts of Ψ. In atomic spectra, the partition quantum number completely determines the total spin angular momentum S, and conversely S determines the partition quantum number so that it is sufficient to specify one of these. In nuclear problems, on the other hand, because of the existence of the isotopic spin, we have "supermultiplets" and, except for special cases, several S values belong to one supermultiplet and have the same energy if we disregard spin dependent forces. Nevertheless, S can be specified simultaneously with $(PP'P'')$, J, J_Z and L. Hence, Ψ_{PL} can be decomposed again:

$$\Psi_{PL}=\sum_S c_S{}^{PL}\Psi_{PLS}, \tag{11}$$

where again

$$\sum_S |c_S{}^{PL}|^2=1.$$

TABLE I. *Magnetic moments of nuclei with odd proton and with odd neutron.*

	J	μ	SOURCE		J	μ	SOURCE
	Odd Proton				*Odd Proton*		
H^1	1/2	2.79	1	Eu^{151}	5/2	3.4	16
Li^7	3/2	3.25	2	Eu^{153}	5/2	1.5	16
B^{11}	3/2?	2.682	3	Lu^{175}	7/2	2.6	18
F^{19}	1/2	2.62	2	Au^{197}	3/2?	0.2	16
Na^{23}	3/2	2.216	4	Tl^{203}	1/2	1.45	19
Al^{37}	5/2	3.628	5	Tl^{205}	1/2	1.45	19
Cl^{35}	5/2?	1.365?	6	Bi^{209}	9/2	4.0	20
K^{39}	3/2	0.391	4				
K^{41}	3/2	0.22	7		*Odd Neutron*		
Sc^{45}	7/2	4.4–4.8	8	Be^9	3/2?	−1.175	21
Mn^{55}	5/2	3.0	9	Zn^{67}	5/2	0.9	22
Co^{59}	7/2	3.5	10	Kr^{83}	9/2?	−1.	23
Cu^{63}	3/2	2.43	11	Sr^{87}	9/2	∼−1.1	24
Cu^{65}	3/2	2.54	11	Cd^{111}	1/2	−0.65	25
Ga^{69}	3/2	2.11	12, 32	Cd^{113}	1/2	−0.65	25
Ga^{71}	3/2	2.69	12, 32	Sn^{117}	1/2	−0.89	26
As^{75}	3/2	1.5	13	Sn^{119}	1/2	−0.89	26
Rb^{85}	5/2	1.34	6	Xe^{129}	1/2	−0.9	23
Rb^{87}	3/2	2.74	6	Xe^{131}	3/2	0.7	27, 23
Ag^{107}	1/2	−0.10	14	Ba^{135}	3/2	0.9	28
Ag^{109}	1/2	−0.19	14	Ba^{137}	3/2	0.9	28
In^{115}	9/2	6.4	15	Yb^{171}	1/2	0.45	30
Sb^{121}	5/2	3.7	16	Yb^{173}	5/2	−0.65	30
Sb^{123}	7/2	2.8	16	Pt^{195}	1/2	0.6	31
I^{127}	5/2	2.8	17	Hg^{199}	1/2	0.5	29
Cs^{133}	7/2	2.57	4	Hg^{201}	3/2	−0.6	29
La^{139}	7/2	2.8	16	Pb^{207}	1/2	0.6	23

[1] Kellogg, Rabi, Ramsey, Zacharias, Phys. Rev. **56**, 728 (1939).
[2] Rabi, Millman, Kusch, Zacharias, Phys. Rev. **55**, 526 (1939).
[3] Millman, Kusch, Rabi, Phys. Rev. **56**, 165 (1939).
[4] Kusch, Millman, Rabi, Phys. Rev. **55**, 1176 (1939).
[5] S. Millman and P. Kusch, Phys. Rev. **56**, 303 (1939).
[6] P. Kusch and S. Millman, Phys. Rev. **56**, 527 (1939).
[7] J. H. Manley, Phys. Rev. **49**, 921 (1936).
[8] H. Kopfermann and H. Wittke, Zeits. f. Physik **105**, 16 (1937).
[9] R. A. Fisher and E. R. Peck, Phys. Rev. **55**, 270 (1939).
[10] K. R. More, Phys. Rev. **46**, 470 (1934).
[11] Tolanski and Forester, Proc. Phys. Soc. **50**, 826 (1938).
[12] Schüler and Korsching, Zeits. f. Physik **103**, 424 (1936).
[13] Schüler and Marketu, Zeits. f. Physik **102**, 703 (1936).
[14] D. A. Jackson and H. Kuhn, Proc. Roy. Soc. **158**, 372 (1937).
[15] Millman, Rabi, Zacharias, Phys. Rev. **53**, 384 (1938).
[16] Schmidt, Zeits. f. Physik **108**, 408 (1938).
[17] Schmidt, Zeits. f. Physik **112**, 199 (1939).
[18] Gollnow, Zeits. f. Physik **103**, 443 (1936).
[19] Schüler and Schmidt, Zeits. f. Physik **104**, 468 (1937).
[20] A. B. McLay and M. F. Crawford, Phys. Rev. **44**, 986 (1933).
[21] Kusch, Millman and Rabi, Phys. Rev. **55**, 666 (1939).
[22] Lyshede and Rasmussen, Zeits. f. Physik **104**, 434 (1937).
[23] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936).
[24] Heyden and Kopfermann, Zeits. f. Physik **108**, 232 (1938).
[25] Schüler and Keyston, Zeits. f. Physik **71**, 413 (1931).
[26] Schüler and Westmeyer, Naturwiss. **21**, 660 (1933).
[27] Kopfermann and Rindal, Zeits. f. Physik **87**, 460 (1934).
[28] A. Benson and R. A. Sawyer, Phys. Rev. **49**, 867 (1936).
[29] Schüler and Jones, Zeits. f. Physik **74**, 631 (1932).
[30] Schüler and Korsching, Zeits. f. Physik **111**, 386 (1939).
[31] Schmidt, Zeits. f. Physik **101**, 486 (1936).
[32] N. A. Renzetti, Phys. Rev. **57**, 753 (1940).

Introducing this into (10) we obtain

$$(\Psi,\Sigma_Z\Psi)=\sum_{PLS}|c_{PL}|^2|c_S{}^{PL}|^2(\Psi_{PLS},\Sigma_Z\Psi_{PLS}), \tag{12}$$

plus nondiagonal matrix elements in which Ψ_{PLS} is combined with a $\Psi_{PLS'}$. We shall neglect the latter, for reasons which will be apparent later. In order to calculate the $(\Psi_{PLS},\ \Sigma_Z\Psi_{PLS})$, we can go over from the wave functions Ψ_{PLS} in which J, J_Z, L, S are specified, to wave functions $\psi_{PLSm\mu}$ in which J, L, S, $L_Z=m$, $S_Z=\mu$ are specified. We define in analogy to (3)

$$(\Psi_{LSm\mu},\ \Sigma_Z\Psi_{LSm\mu})=g_S(PS)\mu. \tag{13}$$

Since the operator Σ_Z contains only spin quantities, its matrix elements are uniquely given if the dependence of $\Psi_{LSm\mu}$ on the ordinary and isotopic spin coordinates is known; that is, they are given by the partition quantum numbers $(PP'P'')$ and S. In particular, g_S is independent of L and the same for all wave functions with the same $(PP'P'')$ and S. For (12) we have according to the oft-quoted Landé-Goudsmit formula,

$$(\Psi_{PLS},\Sigma_Z\Psi_{PLS})$$

$$=J_Z\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}g_S(PS). \tag{14}$$

The calculation of the spin part of the magnetic moment is thus reduced to the calculation of the $|c_{PL}|^2$, the $|c_S{}^{PL}|^2$, and the $g_S(PS)$. The only approximation made so far is the neglect of the off-diagonal elements mentioned after Eq. (12).

The partition which has the lowest energy in the approximation in which the spin dependent forces are neglected is[12] $(PP'P'')=(T_\zeta\frac{1}{2}\pm\frac{1}{2})$ where $T_\zeta=\frac{1}{2}(n_N-n_P)$ is the isotopic spin. We thus obtain to this approximation for the elements with isotopic spin T_ζ only one S, namely $S=\frac{1}{2}$. Hence the L values in (10) which are compatible with a definite J are $J-\frac{1}{2}$ and $J+\frac{1}{2}$. If we assume that this partition alone is present in Ψ, (10) contains only two terms, P assuming only one and L only two values. For both of these, (11) contains only one term, corresponding to $S=\frac{1}{2}$, and there are, of course, no off-diagonal elements in addition to the terms (12) so that (14) holds in this case rigorously.

[12] E. Wigner, Phys. Rev. **51**, 947 (1937).

The value of $g_S(PS)$ can be obtained rigorously also. The first part of Σ_Z in (9a) gives simply 0.855μ in (13). The second part gives $-4715\cdot\frac{1}{2}$ if $\mu=S_Z=\frac{1}{2}$ and if we have a $(T_\zeta\frac{1}{2}\frac{1}{2})$ multiplet. It gives $+4715\cdot\frac{1}{2}$ if $\mu=\frac{1}{2}$ and we have a $(T_\zeta\frac{1}{2}-\frac{1}{2})$ multiplet. This is evident even from the definition of the $(PP'P'')$ symbols: P is the maximum possible value of the isotopic spin T_ζ for the supermultiplet. $P'=\frac{1}{2}$ is the maximum value of S_Z, compatible with the value P for T_ζ (it is because of this that S is uniquely given): P'', finally, is the maximum value of Y_ζ (in our case the only value) which is compatible with the maximum values of P and P'. Hence we obtain for

$$g_S((T_\zeta\tfrac{1}{2}\pm\tfrac{1}{2}),\, L,\, \tfrac{1}{2})=0.855\mp4.715. \qquad (15)$$

The sum $P+P'+P''$ has the form $2k-\frac{1}{2}(n_N+n_P)$ with an integer k. In our case, furthermore, $P=\frac{1}{2}(n_N-n_P)$ so that $P'+P''$ has the form $2k-n_N$. Since $P'=\frac{1}{2}$, we have $P''=+\frac{1}{2}$ if the number of neutrons is odd, $P''=-\frac{1}{2}$ if the number of neutrons is even, that of the protons therefore odd. In the former case the upper sign holds in (15), in the latter case the lower. The g_S which one obtains in this way, -3.86 and 5.57 are $2\mu_N$ and $2\mu_P$, i.e., the g factors for a single neutron and proton. This had to be expected as soon as it was established that the g_S can be calculated uniquely on the basis of our assumptions and are, thus, to a large degree independent of a special model. In consequence hereof, the g factor can be calculated using any special model compatible with the validity of the first approximation of reference 9. Such a model is, e.g., that consisting of one extra particle outside a closed, unmagnetic shell which contains an even number of protons and neutrons. The g factor for such a model is evidently that for the single particle outside the shell. Thus the mean value of Σ_Z is identical with the one postulated by Schmidt.

We can now easily calculate the mean value of Λ_Z also. If one assumes Russell-Saunders coupling, L can be either $J-\frac{1}{2}$ or $J+\frac{1}{2}$, and thus either $|a_{J-\frac{1}{2}\frac{1}{2}}|^2$ or $|a_{J+\frac{1}{2}\frac{1}{2}}|^2$ is one in (7), while all the other a_{LS} become zero. The whole magnetic moment should correspond, therefore, to one of these possibilities, represented by the full lines of Fig. 1. However, it was pointed out in the introduction that one hardly can expect strict

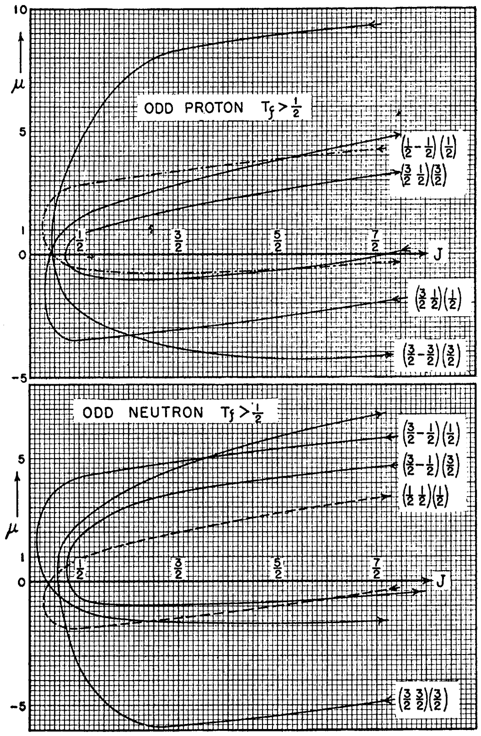

FIG. 2. Solid lines represent limits of magnetic moments for states of higher partition quantum numbers. Parentheses mean $(P'P'')(S)$. The two branches of the curves correspond to $L=|J-S|$ and $L=J+S$; the arrow points in the direction from $|J-S|$ to $J+S$. The broken lines are identical with the full lines of Fig. 1.

L–S coupling, and this makes the whole region between the full lines possible. Schmidt's curves are given as broken lines. Table I contains the experimental material used in preparing these figures.

5. Figure 1 shows that most magnetic moments lie in the region expected under the assumption that the partition of the wave function is $(T_\zeta\frac{1}{2}+\frac{1}{2})$.[13] This does not prove, of course, that this assumption is correct because it remains possible that the magnetic moments lie in this region even if other partitions are also present. In order to decide this point, it would be necessary to calculate the magnetic moment without the above assumption. This, however, is clearly impossible, as long as the coefficients in (10) are unknown. We have adopted, therefore, the following procedure.

[13] The elements whose magnetic moments do not fall between the full lines in Fig. 1 are: In^{115} (odd proton), Kr^{83} and Sr^{87} (odd neutron). They all have $J=9/2$.

Equation (12) shows that the total spin magnetic moment $(\Psi, \Sigma_Z\Psi)$ is the average of the spin magnetic moments of the different Ψ_{PLS} with weights $|c_{PL}c_S{}^{PL}|^2$ which are the probabilities for the corresponding partition, azimuthal and spin quantum numbers. In this, only the terms omitted in (12) are neglected which are always relatively small. The same thing holds for the orbital magnetic moment, given by (7). We have, therefore, calculated the total magnetic moment assuming that only one P, S and L occur in the total wave function but permitting various possibilities for these. The real magnetic moment would be a weighted average of the magnetic moments obtained in this way.

The results of these calculations are given in the Fig. 2. The broken lines are identical with the full lines of Fig. 1 and correspond to the partition $(T_\zeta\frac{1}{2}\pm\frac{1}{2})$. The other lines correspond to higher multiplets $(PP'P'')$, all with $P=T_\zeta$ while $(P'P'')(S)$ are given in brackets. The S is, naturally, always smaller than or equal to P'. The two branches of the curves correspond to $L=|J-S|$ and $L=J+S$, the arrow points in the direction from $|J-S|$ to $J+S$. The values of the magnetic moment for intermediate values of L are in all cases between the two branches of the curve.

For (7), $n_p/(n_p+n_N)=\frac{1}{2}$ was assumed. In order to calculate the spin magnetic moment by (12) and (14) the $g_S(PS)$ of (13) must be known. These consist of two parts, corresponding to the two parts of Σ_Z in (9a). The first part is evidently 0.855. The second part depends on the matrix elements of Y_ζ.

In all the partitions considered, every S occurs only once in every supermultiplet for $T_\zeta=P$. This means that although several terms are united to a supermultiplet, all these terms have different S. The g_S for the largest $S=P'$ is given by the fact that the matrix element of Y_ζ is P'' for the wave function with $T_\zeta=P$, $S_Z=S=P'$. For this S, therefore, the value of g_S is $4.715P''/P'$. The sum of the matrix elements of Y_ζ for $S_Z=P'-1$ can be obtained from Fig. 1 of reference 9 as the sum of the possible values of Y_ζ for this value of S_Z and T_ζ. If we subtract from this value the matrix element corresponding to the state $S_Z=S-1$, $S=P'$, which is $P''(S-1)/S$, we obtain the matrix element for the state $S_Z=S=P'-1$ and this gives the g_S for $S=P'-1$ if there is such a state. The matrix elements of Y_ζ and hence the g_S and hence all the matrix elements of the spin magnetic moment can be determined in this way. The calculation was carried out for those $(PP'P'')$ which give, according to Eqs. (10), (15) of reference 12 the lowest energies.

The fact that all moments of light nuclei and practically all moments of the other elements lie between the limits valid for the $(T_\zeta\frac{1}{2}+\frac{1}{2})$ multiplet indicates, at any rate, that this multiplet plays a preponderant role in the normal state of odd elements.

6. We wish to point in conclusion to an experimental finding which gives reason to more serious doubts in the considerations presented here than anything mentioned above. According to the preceding considerations, the magnetic moments are expected to be distributed between the two lines of Fig. 1 in a rather random fashion. We cannot see any reason, however, why two isotopes with equal total angular momenta should have exactly equal magnetic moments. In fact we fail to see any model which would explain such a result.[14] In spite of this, so nearly equal moments for isotopes seem to have been found in at least three cases (Cu, Re, In) that it does not appear possible to blame this on a chance coincidence. We are unable to give an explanation for this phenomenon.

[14] H. Schüler and H. Korsching, Zeits. f. Physik **105**, 168 (1937).