On the Resonance Scattering of Alpha-Particles

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The elastic scattering of alpha-particles is considered from the point of view of the many-body theory. The restriction is made to scattering by nuclei of zero spin since in this case a unique determination of the angular momentum of the resonance level (of the compound nucleus) responsible for the anomalous scattering is possible. The procedures which have been used for the determination of the resonance energies and the assignments of angular momenta are discussed and criticized. These methods are based on a comparison of the experimental data with the dispersion formula for mono-energetic alpha-particles and thus do not take into account the straggling of the incident beam which is of primary importance under the actual experimental conditions. In addition, in the determinations of resonance energies and the widths of resonance levels the interference scattering should be considered. An alternative method is suggested for obtaining the angular momenta directly from measurements of the *angular* distribution of the scattering at fixed energies. This method is essentially independent of the straggling of the alpha-particles and of the details of the theory as well. The only requirement is fairly good angular resolution, *viz.*, a range of scattering angles of about 10° would be sufficient and this resolution is already attained in the experiments. A procedure for a more accurate determination of the energies and widths of resonance levels is given. Here again the straggling is eliminated insofar as the straggling parameter (width of the primary energy distribution) does not appear explicitly. Instead, the resonance energy and width depend only on readily measurable quantities: mean alpha-particle energy, scattering angle, angular momentum of the resonance level and the strengths of the resonance extrema as observed from the energy distribution of the scattering. The result for the resonance energy, in particular, is rather insensitive to errors in measurement and to approximations made in the derivation of the formulas.

Introduction

The elastic scattering of alpha-particles has been used by several investigators to determine energies and angular momenta of resonance levels in various light nuclei. The earliest indications of a departure from the classical Coulombian scattering were found in the experiments of Rutherford and others.[1] More extensive investigations which demonstrated the resonance character of the scattering were made by Riezler[2] who found large deviations from the classical scattering for 7.5-Mev alpha-particles in Be, B, C, O, Ne and Al. The scattering of Ra C′ and Th C′ alpha-particles in O, N, Ne and A was investigated by Brubaker[3] who found the characteristic resonance extrema for the ratio of observed to classical scattering as a function of energy, in the case of the first three nuclei whereas only a monotonic deviation was observed for A up to 8 Mev. Devons[4] observed the scattering of alpha-particles from an active deposit of Ra (B+C), energies up to 6.5 Mev. The scattering substances investigated were C, O, N, F and Ne at a scattering angle of 90°. Resonance extrema were observed which were attributed to resonance levels as follows: C, 5.7 Mev alpha-particle energy; N, 4.6 and 5.2 Mev; O, 5.8 Mev and F, 3.5 and 4.7 Mev. In Ne no evidence for resonances was observed in contrast to Brubaker's results. In addition, for the nuclei of zero spin the following assignments of angular momentum J (in units $\hbar$) were made: for C, $J=1$; for O, $J \leqslant 2$. In contrast to these results Walker[5] finds evidence for two resonance levels from the scattering in C in the energy range investigated by Devons, *viz.*, at 4.3 and 5.5 Mev alpha-particle energy and the J values assigned were 1 and 2, respectively. Ferguson[5] finds two resonances in the scattering in O, at 5.4 and 6.7 Mev. Both are interpreted as P levels, $J=1$. Possible reasons for these apparent conflicts in the data are discussed below.

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[1] E. Rutherford, Phil. Mag. **37**, 537 (1919); J. Chadwick and E. S. Bieler, Phil. Mag. **42**, 923 (1921); E. S. Bieler, Proc. Roy. Soc. **A105**, 434 (1924); E. Rutherford and J. Chadwick, Phil. Mag. **50**, 889 (1925); Phil. Mag. **4**, 605 (1927); J. Chadwick, Proc. Roy. Soc. **A128**, 114 (1930).

[2] W. Riezler, Proc. Roy. Soc. **A134**, 154 (1932); Ann. d. Physik **23**, 198 (1935).

[3] G. Brubaker, Phys. Rev. **54**, 1011 (1938); **56**, 1181 (1939).

[4] S. Devons, Proc. Roy. Soc. **A172**, 127 (1939).

[5] I am indebted to Drs. Walker and Ferguson for making these results available to me before publication.

No attempt is made in the experiments to determine level widths with any great degree of accuracy. Aside from the difficulty of obtaining sufficiently good energy resolution, the determination of the width is hampered by a lack of knowledge of the penetrability of the potential barrier, which, of course, depends on the angular momentum of the resonance level and is therefore decidedly uncertain, and by the effect of interference between nuclear and Coulombian scattering as well as the scattering arising from partial overlapping of neighboring levels.

In the following sections a discussion of the above assignments of angular momenta is given and an alternative procedure is suggested. In addition, the question of determining resonance energies and level widths is considered.

Angular Momenta of Resonance Levels

The procedures which have been adopted in assigning values for the angular momenta may be best understood from a consideration of the scattering cross section as given by the many-body theory.[6] In the following we shall be interested in the case of scattering by nuclei of zero spin. For the alpha-particle energies used in the experiments, the particle wave-length is not small compared to the nuclear radius so that alpha-particles with nonvanishing orbital momentum (l up to about 4) may be involved in the scattering. For the special case under consideration the angular momentum of the resonance level in the compound nucleus is then identical with the orbital momentum of the alpha-particle. It will be sufficient to restrict our considerations to the case of scattering from a single resonance level. As will be apparent from the following, the scattering angle may usually be chosen so as to make the scattering from at least one neighboring level rather small and since, at a particular energy, there is appreciable interference between only two levels in general, the restriction to scattering from a single level is not so serious. With these restrictions the ratio R of total to Coulomb scattering is

$$R=1+(\rho^2+2\rho\sin\zeta+2\rho x\cos\zeta)/(1+x^2). \quad (1)$$

where

$$\rho=(2J+1)(2/\gamma)(\Gamma_{r\alpha}/\Gamma_r)\sin^2\tfrac{1}{2}\vartheta P_J(\cos\vartheta), \quad (1a)$$

$$e^{i\zeta}=e^{i\zeta_0}\prod_1^J\frac{(n+i\gamma)^2}{n^2+\gamma^2}, \quad \zeta_0=\gamma\log\sin^2\tfrac{1}{2}\vartheta, \quad (1b)$$

$$x=2(E-E_r)/\Gamma_r, \quad \gamma=2Ze^2/\hbar v. \quad (1c)$$

Here J, E_r, Γ_r and $\Gamma_{r\alpha}$ are, respectively, the angular momentum, energy, total width and partial width for alpha-particle emission from the resonance level responsible for the scattering. E is the kinetic energy associated with the relative motion and v is the velocity of the incident alpha-particle in the laboratory reference system. Z is the atomic number of the scattering substance and ϑ is the deflection angle in the center of gravity reference system. The deflection Θ in the laboratory reference system is given by

$$\tan\Theta=A\sin\vartheta/(4+A\cos\vartheta), \quad (2)$$

where A is the mass number of the scattering nucleus. In (1) the term in ρ^2 is the direct nuclear scattering while the terms linear in ρ are due to interference between nuclear and Coulombian scattering.

The assignments of angular momenta made by Devons are based on a comparison of the observed scattering ratio with an upper limit for the theoretical ratio (1). This upper limit is taken as the resonance value of the ratio and interference scattering is neglected. That is

$$R_{\max}=1+\rho^2. \quad (3)$$

Since the application is made to scattering in C^{12} and O^{16} the quantity ρ will contain only one unknown parameter, J: The only energetically possible process other than the scattering of the alpha-particle is gamma-ray emission and since the radiation widths are negligible compared to the particle widths, especially for light nuclei, for all practical purposes one may take[7]

$$\Gamma_{r\alpha}=\Gamma_r. \quad (4)$$

[6] H. A. Bethe, Rev. Mod. Phys. 9, 69 (1937). See Chapter XII.

[7] Other (abundant) light nuclei of zero spin emit either neutrons or protons upon alpha-particle bombardment. Although the Q values for all these reactions are negative, the threshold energy is so small ($\sim 1-2$ Mev) that it would be impossible to observe only alpha-particle scattering due to the small barrier penetrability at these low energies.

Since the upper limit given in (3) generally increases with the angular momentum J, a lower limit for J is obtained. Large values of J are very improbable because the barrier penetrability for the energies used in Devons' experiments would then be very small.

In criticism of this procedure for determining angular momenta it may be noted first, that the neglect of the interference scattering is not always justifiable. This is obviously the case when the scattering angle $\Theta=90°$ ($\vartheta=109.5°$ and $104.5°$ for C and O, respectively) as in Devons' experiments. For levels of odd (but not large) angular momentum such as $J=1$, ρ will be small due to the factor $P_J(\cos\vartheta)$, cf. (1a). In this case the interference scattering may be as important as the direct nuclear scattering.

A more important objection is the fact that straggling has been neglected. Obviously the theoretical scattering ratio (1) refers to mono-energetic alpha-particles whereas in the experiments there is a rather large spread in energy. Under the conditions of observation straggling of the alpha-particles arises mainly from the fluctuations in energy loss due to stopping in air or mica and from the finite thickness of the source and target (or scattering volume). In all the experiments the energy resolution has been rather poor and straggling widths were of the order of a few hundred kilovolts. This is comparable with the actual widths of resonance levels even for alpha-particle energies above the top of the potential barrier. Moreover, a considerable portion of the energy range used in the experiments is below the top of the barrier (even for angular momentum zero) and in these cases the straggling width may be much larger than the natural width (that is, width above the top of the barrier times barrier penetrability). Therefore the broadening effect of straggling will be quite important and the height (and depth) of the resonance extrema will be considerably less than would be expected from the scattering ratio as given by (1). The angular momenta deduced from a comparison of the observations with the uncorrected formula (1) would thus tend to be too small.

Other methods which have been used for the determination of the angular momenta are subject to the same criticism, i.e., neglect of straggling. These methods are based on a comparison of the experimental data with certain simple relationships derived from the unstraggled scattering ratio (1). In these relationships the unknown resonance energy and level width are eliminated and the angular momentum is given in terms of the observed maximum and minimum scattering ratios, alpha-particle energy and scattering angle. Writing (1) as

$$R=1+(a+bx)/(1+x^2), \tag{5}$$

$$a=\rho^2+2\rho\sin\zeta,\quad b=2\rho\cos\zeta, \tag{5a}$$

one can readily verify the fact that a and b are comparatively insensitive with energy. Therefore the positions of the extrema are given by

$$x_{1,2}=-(a/b)\mp(1+a^2/b^2)^{\frac{1}{2}} \tag{6}$$

and the extrema themselves by[8]

$$R_{1,2}=1+b/(2x_{1,2}). \tag{7}$$

Either (7) combined with (6) or more simply the relation

$$\rho=\pm(R_1^{\frac{1}{2}}-R_2^{\frac{1}{2}}) \tag{8}$$

may be used to obtain J from the observed values of E, ϑ and R_1, R_2 (cf. 1a). However, when straggling is taken into account the positions and magnitudes of the extrema depend on the straggling width and the simple relations (7) and (8) are no longer valid.

Since it is rather inconvenient to determine the straggling width by direct measurement or to regard it as an additional parameter to be determined by a comparison of observations with a theoretically straggled formula, it would be very desirable to have a method for the determination of the angular momenta which is to a large degree independent of straggling. In the following such a method is discussed.

The following procedure for the determination of angular momenta of resonance levels may be

[8] Obviously for all a and b, $x_1<0$ and $x_2>0$. Therefore if $b>0$, R_1 is the minimum and R_2 the maximum whereas the reverse is true when $b<0$. Essentially the same situation applies when straggling is taken into account (cf. (12)). This fact may be used in some cases to obtain some information about the angular momentum by inspection of the observed scattering ratio as a function of energy. Thus for energies up to at least 7 Mev and for J up to at least 5 and for scattering in O^{16} at 90°, b is positive for even J and negative for odd J. Therefore if the maximum ratio lies at higher energy J is even and for the positions of the extrema interchanged J is odd.

applied to any scattering nucleus of spin zero.[9] For such nuclei it is evident that, regardless of the presence of straggling, the nuclear scattering (direct and interference scattering) vanishes when $P_J(\cos\vartheta)$ vanishes. Therefore if one measures the scattering at a fixed energy as a function of *angle* the positions of the nodes ($R=1$) give a direct empirical determination of the angular momentum. In applying this method there are, however, two things to be considered. First, in addition to the nodes due to $P_J(\cos\vartheta)$ vanishing there may be nodes due to the cancellation of the direct and interference scattering and these may accidentally coincide or lie near one of the $P_J(\cos\vartheta)$ nodes. However, the position of such "interference" nodes will obviously depend on the energy of the alpha-particles and if this energy is changed these nodes will shift. Therefore, it is only necessary to measure the angular distribution at two different energies and to disregard the nonstationary nodes. Second, although the energy spread is inessential, insofar as the nuclear scattering vanishes for vanishing $P_J(\cos\vartheta)$ regardless of the energy, a certain amount of energy resolution is necessary in order to avoid difficulties due to partial overlap of the scattering due to neighboring resonance levels. The two fixed energies at which the angular distribution is to be investigated should be chosen so that the scattering is due almost entirely to a single level. In order to accomplish this it is necessary that the straggling width be small compared to the spacing between neighboring levels ($\sim$1–2 Mev).[10]

The condition that the straggling width be small compared to the spacing between levels is also of importance for the elimination of inelastic scattering when the scattering nucleus has an excitation level very near the ground state. If there are no very low lying levels there will be no need to take precautions against inelastic scattering since inelastically scattered alpha-particles will be too slow to penetrate the potential barrier to any appreciable extent as long as the initial energy is of the order of the energies used in the experiments ($\lesssim$8 Mev). If there is a level close to the ground state the corresponding inelastic scattering can be eliminated by using absorption foils if the straggling width is smaller than the excitation energy of the level in question. This will usually be fulfilled since straggling widths are rarely larger than 500 kev.

Aside from the fact that the procedure here suggested eliminates the large uncertainties arising from an undetermined straggling, a further advantage is the directness of the method since it is independent of the details of the theory and depends only on the fact that the nuclear scattering amplitude for a level of angular momentum J is proportional to $P_J(\cos\vartheta)$. Moreover, as will be apparent from the following, the angular resolution required is not greater than that actually attained in the experiments, *viz.*, an angular spread of 10° is sufficient. It is of course not necessary to measure at all angles but only in the neighborhood of a few angles—the nodes of $P_J(\cos\vartheta)$ for the first few J values. These nodes, in terms of the laboratory scattering angle Θ (cf. (2)), are given in Table I for several scattering substances and for values of $J\leqslant 4$. Levels with larger values of J are very unlikely to affect the scattering appreciably.[11] Only the most abundant nuclei of spin zero have been given in the table.

For $J=2$, 3 and 4 only the largest angles have been given in the table. If these nodes are located by measuring above and below the angle for zero nuclear scattering and interpolating, the interpolation is more accurate at the larger scattering angles since the amount of nuclear scattering is

[9] For nuclei with nonvanishing spin the angular momenta of the resonance levels may be determined only in the trivial case that the alpha-particles are sufficiently slow (wave-length large compared to the nuclear radius) so that only particles of orbital momentum zero can be scattered appreciably. That is, with the nuclear radius $=\frac{1}{2}(e^2/mc^2)A^{\frac{1}{3}}$ where A is the mass number, we have as a condition on the energy $E\ll\frac{1}{4}(137^2/1840)(4+A)/A^{5/3}$ Mev which is considerably smaller than the barrier height except for very light nuclei (H and He). In addition, the Coulomb scattering is large at small energies and will mask the nuclear scattering unless the nuclear charge is small. Therefore the nuclear scattering of such slow alpha-particles could be observed only in very light nuclei for which there are usually no low lying levels so that no resonance effects would be observed.

[10] If there is a slight amount of overlap the observed scattering ratio at the angle for which $P_J(\cos\vartheta)$ vanishes will be slightly different from unity and the node will be displaced by a small amount which does depend on the energy. However, in many cases this displacement of the node may not even be detectable and in any case an accurate location of the node is unnecessary. It is only necessary to find the angle for unit scattering ratio *near* a node of $P_J(\cos\vartheta)$.

[11] With the nuclear radius as in reference 9, the barrier height for an orbital momentum of 5 units varies from 25.4 Mev for C^{12} to 19.5 Mev for Ca^{40} with a minimum barrier of 18.3 Mev for A^{40}.

greater in this case. It may be seen that the largest node for $J=3$ and the larger of the two nodes given for $J=4$ are rather close together and if a node were actually found in this range of angles the angular resolution may be insufficient to distinguish between $J=3$ and $J=4$. In this case measurements in the neighborhood of the smaller node for $J=4$ will permit a decision to be made. Finally it may be noted that the nodes listed for $J=1$ obviously apply to all odd J so that preliminary measurements at these angles would eliminate either all even J or all odd J.[11a]

The only investigation of the angular distribution of the scattered alpha-particles has been made by Riezler.[2] Our considerations can be applied only to Riezler's results for C since all the other angular distributions investigated refer to nuclei which have nonvanishing spin. For C Riezler finds a node at Θ slightly less than 127° and no nodes at larger angles. This would correspond to $J=3$ which is greater than the J values assigned by both Devons and Walker. However, since no measurements were made at other energies, so that the possibility of an interference node cannot be excluded, nor at smaller angles for which a node would appear if J were smaller than 3, this value for the angular momentum is by no means certain.

It is of interest to discuss the case of scattering in O^{16} in some detail since it is probable that all or most of the low lying levels in the compound nucleus Ne^{20} are known. From the neutron groups observed by Bonner[12] in the reaction

$$F^{19}+H^2=Ne^{20}+n,$$

evidence is found for levels in Ne^{20} at excitation energies of 1.5, 4.2, 5.4, 7.3, 9.0 and 10.1 Mev. In addition, from the capture of protons by F^{19} with gamma-ray emission the existence of at least ten

TABLE I. *Scattering angles Θ, laboratory reference system, for which the nuclear scattering from a resonance level of angular momentum J, vanishes. The largest angles are listed in each case.*

J	1	2	3	4	
C^{12}	71.6°	106.7°	124.9°	136.1°	90.6°
O^{16}	76.0	111.8	129.7	140.2	95.6
Ne^{20}	78.7	114.8	132.3	142.4	98.6
Mg^{24}	80.5	116.7	133.9	143.8	100.6
S^{32}	82.9	119.0	135.7	145.4	103.0
A^{40}, Ca^{40}	84.3	120.3	136.8	146.3	104.5

closely spaced levels lying between 13 and 15 Mev may be inferred.[13] The two lowest states found in the (d,n) reaction are stable against alpha-particle emission and the highly excited states around 14 Mev (corresponding to 11.5 Mev alpha-particle energy in the laboratory frame of reference) are too narrow and too far removed from the energy range investigated in the scattering experiments to participate in the scattering of the alpha-particles. The level at 5.4 Mev excitation, corresponding to 0.8 Mev alpha-particle energy, is very probably to be neglected in a consideration of the scattering of particles of more than 3 Mev energy. This leaves three levels which may contribute appreciably to the scattering between 3 and 7 Mev. The 10.1 Mev level (6.7 Mev alpha-particle energy) and possibly any levels which may lie between this energy and 13 Mev excitation no doubt accounts for the rise in the scattering ratio for energies above 6 Mev as observed by Brubaker. There is reason to believe that the scattering below 6 Mev is mainly due to the 9.0-Mev level. From the appearance of Devons' scattering curve for O we may conclude that this level has even J.[8] Assuming a resonance energy consistent with Bonner's result and with any reasonable width and straggling (cf. (11), (12) below) it is found that Devons' curve can be reasonably well fitted (within the experimental error for all but the lower energies) only for $J=2$, in agreement with Devons' lower limit. Other values of J up to $J=4$ give a nuclear scattering which is only 10 percent (or less) of the observed amount. However, no curve can be adjusted to fit the experimental results at the lower energies (3 to 4 Mev) where the observed scattering would appear to be

[11a] If there is evidence for the existence of an appreciable amount of potential scattering, for example minimum scattering ratio greater than unity or in general, scattering ratios larger than can be accounted for by resonance scattering alone, the above procedure can be applied if the energy dependence of the potential scattering across the resonance region can be neglected. This will be the case if a background scattering ratio can be observed which is essentially the same above and below resonance. Then the node of the nuclear scattering is determined with respect to the background scattering ratio. In the above $R-1$ would be replaced by the difference between the observed ratio and the observed background.

[12] T. W. Bonner, Phys. Rev. **56**, 207 (1939); Proc. Roy. Soc. **A174**, 339 (1940).

[13] E. J. Bernet, R. G. Herb and D. B. Parkinson, Phys. Rev. **54**, 398 (1938).

abnormally low. But this may be readily understood when the scattering by the 7.3-Mev level (3.2 Mev alpha-particle energy) is taken into account. In order to obtain quantitative agreement with Devons' results the 3.2-Mev (alpha-particle energy) level must diminish the scattering at 3–4 Mev so that the scattering ratio for this level alone should have a minimum at higher energies and a (slight) maximum at lower energies. This would indicate an odd angular momentum for this level,[8] probably $J=1$ since at these energies the barrier penetrability for $J=3$ is negligible. It is interesting to note that, although Devons' scattering curve superficially appears to give evidence for only one resonance level, a closer analysis indicates two resonance levels in the energy region investigated whose positions are in reasonable agreement with Bonner's results. From our assignment of angular momentum for the lower lying level it would follow that the scattering from this level at other angles, particularly $\Theta=112°$ (see Table I), would be more prominent since the scattering from such an odd level is somewhat suppressed at the 90° scattering angle used in Devons' experiments.

This choice of scattering angle may explain the appearance of only one resonance level in Devons' results for C whereas two levels are found by Walker. The missing level has $J=1$ according to the latter, and hence the scattering from this level at 90° might be easily masked by the scattering from the nearby 5.5-Mev level for which J is apparently 2. The same argument may be advanced as a possible explanation for the absence of resonance peaks in Devons' results for Ne whereas such effects were found by Brubaker. While it is true that one of the scattering angles investigated by Brubaker was very close to 90°, namely, 88.5°, the angular spread in these experiments was considerably larger than in Devons' measurements. Even then the deviations from classical Coulombian scattering observed by Brubaker were comparatively small.

For the scattering in O it will be noted that the positions of the resonance levels as deduced from the scattering data of Ferguson and Devons are in reasonably good agreement with the disintegration data. The 3.2-Mev level was not observed by the former inasmuch as the energy region investigated did not extend below 4.0 Mev. However, the angular momentum of the level observed by both investigators (alpha-particle energy 5.4 Mev) is assigned the value 1 by Ferguson and 2 according to the data of Devons. A possible source of the discrepancy is the fact that the former assignment is based on the strengths of the resonance extrema (cf. (8)) and aside from the objection that straggling is not taken into account, a relatively small number of points were obtained, necessitating a rather uncertain interpolation in obtaining the height and depth of the extrema.

In connection with the resonance levels found in the scattering by C^{12} an investigation of the excited levels in O^{16} by disintegration experiments would be desirable. Such an experiment would be the cloud-chamber study of recoils produced by the neutrons from

$$N^{15}+H^2=O^{16}+n$$

or

$$C^{13}+He^4=O^{16}+n.$$

With O^{16} formed in the ground state, both reactions are exoergic with energy releases of $Q=10.0$ and 2.4 Mev, respectively. Since there are presumably two excited states in O^{16} which are supposed to lie at 10.6 and 11.5 Mev excitation, deuterons of more than 1.7 Mev or alpha-particles of more than 12.1 Mev would give rise to two neutron groups in addition to the group corresponding to the formation of O^{16} in the ground state. These experiments might best be carried out by first observing the neutrons from normal C or N samples and then comparing these results with the neutron groups observed when C or N samples enriched in the heavy isotope are used. Of the two reactions the one with N is perhaps preferable. The emission of neutrons from N^{14} has been observed by Stephens, Djanab and Bonner[14] and two neutron groups were found, with Q values of 5.1 and 1.1 Mev corresponding to the formation of O^{15} in the ground state and in an excited state at 4.0 Mev:

$$N^{14}+H^2=O^{15}+n+Q.$$

Thus there should be no overlap between the neutron groups from normal N (N^{14}) and those from N^{15}, the spacing between groups being at

[14] W. E. Stephens, K. Djanab and T. W. Bonner, Phys. Rev. **52**, 1079 (1937).

least 1.6 Mev and the groups from the two isotopes should be easily distinguishable. On the other hand, the observation of the neutron groups from the C reaction would require much greater resolution since the neutron energies corresponding to O^{16} in either excited state or O^{15} in the ground state differ by 0.5 Mev or less. Obviously, information concerning the levels of O^{16} would be of great interest for the interpretation of the experiments on proton capture by F.[15]

Energy and Width of Resonance Levels

It is apparent that even without straggling the energy for which the scattering ratio is a maximum (or a minimum) does not coincide with the resonance energy because of interference scattering. However, if this were the only factor to be considered, the resonance energy could almost always be determined within a few hundred kilovolts from the positions of the extreme scattering ratios. From (6) and (1c) it follows that the resonance energy always lies between the extrema.[8] The maximum error which could be made in the resonance energy, taking it to lie midway between the extrema, would be one-half the separation between extrema or about one-quarter of the level width. The effect of straggling in shifting the higher energy extremum towards higher energies and the lower energy extremum towards lower energies, thus increasing their separation by about the straggling width, increases the uncertainty in the resonance energy by a like amount.

A more accurate method for determining the resonance energy, and the level width as well, is the following. We shall assume a special form for the energy distribution of the incident alpha-particles and then show that the straggling may be essentially eliminated by observing the heights and positions of the resonance extrema. We assume a (normalized) straggling function of the form

$$\varphi(E-\bar{E})dE=\frac{2}{\pi\Gamma_s}\frac{dE}{1+4(E-\bar{E})^2/\Gamma_s{}^2}, \qquad (9)$$

where $\bar{E}$ is the most probable ($\sim$average) energy and Γ_s is the straggling width. The scattering ratio now becomes

$$\bar{R}(\bar{E})=\int R\sigma_0\varphi dE\Big/\int\sigma_0\varphi dE\cong\int\varphi R dE, \qquad (10)$$

where R is given in (1) and σ_0 is the Coulomb scattering cross section. The integral may be easily evaluated if the widths are assumed constant. The result for the scattering ratio with straggling is then

$$\bar{R}=1+\frac{\Gamma_r}{\Gamma_r+\Gamma_s}\frac{a+b\bar{x}}{1+\bar{x}^2}, \qquad (11)$$

where

$$\bar{x}=2(\bar{E}-E_r)/(\Gamma_r+\Gamma_s), \qquad (11a)$$

so that the widths are additive. The effect of straggling may then be visualized as a stretching of the scattering curve along the energy axis (and about the resonance energy) in the ratio of the straggled to unstraggled widths and a shrinking of the scattering ratio about $R=1$ in the inverse ratio. The approximations made in deriving (11) should not lead to a serious error.

The maximum and minimum scattering ratios are approximately

$$\bar{R}_{1,2}=1+\frac{\Gamma_r}{\Gamma_r+\Gamma_s}\frac{b}{2\bar{x}_{1,2}}, \qquad (12)$$

where $\bar{x}_{1,2}$ are the same as $x_{1,2}$ given in (6). If the values of $\bar{x}_{1,2}$ from (11a) are inserted in (12) this becomes

$$\bar{R}_{1,2}=1+\frac{1}{2}\frac{\Gamma_r(\bar{E}_{1,2})b}{\bar{E}_{1,2}-E_r}, \qquad (13)$$

in which the straggling width no longer appears explicitly. It may be noted that if additivity of the squares of the widths rather than of the widths themselves were assumed, the same result would follow. The straggling affects the extrema only insofar as their positions $\bar{E}_{1,2}$ which appear in (13) are affected. The level width Γ_r at energy $\bar{E}_n$ is written as[16]

$$\Gamma_r(\bar{E}_n)=G_rP_n, \qquad (14)$$

[15] See, e.g., J. R. Oppenheimer and J. S. Schwinger, Phys. Rev. **56**, 1066 (1939); D. M. Dennison, Phys. Rev. **57**, 454 (1940).

[16] Strictly speaking, this is valid only if the total width and the partial width for alpha-particle emission are about equal; that is, in the case of C^{12} and O^{16} (and for alpha-particle energies below 6.5 Mev for C^{12} and below 10.2 Mev for O^{16}—above these energies proton emission is energetically permissible). However, (14) can no doubt be extended somewhat above these limits.

where P_n is the penetrability of the potential barrier for alpha-particles of energy $\bar{E}_n$ and G_r, the width without barrier, is to be taken independent of the energy. Eqs. (13) are two relations from which the two unknowns, the resonance energy E_r and the level width G_r, can be determined from the measured values of $\bar{R}_{1,2}$ and $\bar{E}_{1,2}$. It is assumed, of course, that the angular momentum of the level is known, (cf. preceding section), so that the penetrabilities P_1 and P_2 can be calculated from the well-known formulas.[17] From (13) we find for the resonance energy

$$E_r=\frac{\kappa_1\bar{E}_1+\kappa_2\bar{E}_2}{\kappa_1+\kappa_2} \tag{15}$$

and for the level width

$$G_r=\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}(\bar{E}_2-\bar{E}_1), \tag{16}$$

where[11a]

$$\kappa_n=\frac{4}{P_n}\left|\frac{R_n-1}{b}\right|. \tag{17}$$

It will be noted that the parameter b occurring in (17) involves the usually unknown ratio of partial width to total width. To the extent that b may be assumed to be independent of the energy (cf. remarks preceding (6)), the resonance energy (but not the width) can be determined since it is independent of b. Of course, for C^{12} and O^{16} there is no difficulty since the ratio in question is essentially unity and both the resonance energy and width may be obtained from readily observable quantities. An accurate determination of the resonance energy and level width by the application of these formulas evidently requires an accurate location of the resonance extrema which would in turn require improved energy resolution. However, it is seen that the resonance energy as given by (15) is more or less independent of systematic errors; the use of (16) would seem to be the only quantitative procedure available for the determination of the width, unless the straggling is measured.

While the above method involves a certain amount of labor, certain special cases which may occur in practice allow a sufficiently accurate interpolation for the resonance energy in a simple way. In the following it should be noted that the interval between extrema is always greater than the straggling width: from (11a) and (6) we have

$$\bar{E}_2-\bar{E}_1=\tfrac{1}{2}[\Gamma_r(\bar{E}_2)\bar{x}_2+\Gamma_r(\bar{E}_1)/\bar{x}_2] \\ +\tfrac{1}{2}\Gamma_s(\bar{x}_2+1/\bar{x}_2)>\tfrac{1}{2}\Gamma_s(\bar{x}_2+1/\bar{x}_2)\geqslant\Gamma_s. \tag{18}$$

Therefore the separation between extrema will in general be sufficient to make $P_2\gg P_1$ if at least $\bar{E}_1$ lies appreciably below the top of the potential barrier. Then the relative magnitude of κ_1 and κ_2 and the approximate position of the resonance energy follow from a comparison of the strengths of the extrema $|\bar{R}_1-1|$ and $|\bar{R}_2-1|$ as follows:

(1) $|\bar{R}_1-1|\gg|\bar{R}_2-1|$. If both extrema are above the top of the barrier $P_1=P_2$ and therefore $\kappa_1\gg\kappa_2$ so that from (15) the resonance energy lies near the low energy extremum, $E_r\approx\bar{E}_1$. Below the top of the barrier $P_2\gg P_1$ and the same conclusion follows *a fortiori*.

(2) $|\bar{R}_1-1|\sim|\bar{R}_2-1|$. Above the top of the barrier we have $\kappa_1\sim\kappa_2$ and thus the resonance energy lies about midway between the extrema, $E_r\approx\frac{1}{2}(\bar{E}_1+\bar{E}_2)$. This will be more accurate the closer are $\bar{E}_1$ and $\bar{E}_2$, i.e., for narrow levels which are the rule above the top of the barrier. Below the top of the barrier we again have $\kappa_1\gg\kappa_2$ because of the smaller penetrability at $\bar{E}_1$. Thus $E_r\approx\bar{E}_1$.

(3) $|\bar{R}_1-1|\ll|\bar{R}_2-1|$. Above the top of the barrier $\kappa_1\ll\kappa_2$ and the resonance energy is almost coincident with the position of the higher energy extremum, $E_r\approx\bar{E}_2$. If both the extrema lie below the top of the barrier a definite conclusion may be drawn only if the ratio of the strengths of the extrema is much greater than the ratio of the corresponding penetrabilities P_2/P_1. In this case $\kappa_2\gg\kappa_1$ and $E_r\approx\bar{E}_2$.

This shows that the identification of the resonance energy with the position of the strongest extremum is usually justified. When the extremum at higher energy is the stronger, however, this procedure may give rise to an appreciable error, especially in the case of wide levels. Finally it may again be emphasized that these considerations are valid only if the overlap between neighboring levels is small.

[17] Reference 6, p. 178.