though this identity was to be expected *a priori*.

The same argument holds also for the last and simplest of the three problems mentioned above: if x is any fixed or randomly chosen point in $0<x<D-T$, the conditional probability that the next event will occur between $x+\tau$ and $x+\tau+d\tau$ is obviously

$$e^{-f\tau}fd\tau\cdot e^{-f(D-\tau)}\{f(D-\tau)\}^{n-1}/(n-1)!P(B) = \frac{n}{D}\left(1-\frac{\tau}{D}\right)^{n-1}$$

and hence we get by a single integration again the correct answer (3).

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On the Deviations from Ohm's Law at High Current Densities

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(Received March 2, 1940)

The deviations from Ohm's law at high current densities are calculated on the basis of the wave-mechanical theory of conductivity. A current density of 10^9 amp./cm^2 causes a 1 percent deviation only. No observable deviations are to be expected at the experimental current densities available at present (10^6 amp./cm^2). This is in agreement with the experiments of Barlow, neither does it contradict those of Bridgman, if the effects found by him are due to secondary factors. The method used consists in the actual solving of the fundamental equation for conductivity in a higher approximation. It is proved that in the Lorentz model (fixed metal ions) the fundamental equation is not soluble in the second approximation in the field strength. A solution in this approximation can be obtained only by assuming inelastic collisions between the electrons and the metal ions. The analogy between the distribution function containing the influence of the electric field and a distribution function found by Pidduck for the motion of ions in gases is pointed out. A generalization of the present theory is indicated by taking into account the influence of the external field on the lattice waves.

Introduction

Ohm's law has been extraordinarily successful for many decades; but our present knowledge makes it clear that this law can be, not a fundamental law (such as Coulomb's law), but a derived law, which describes reality only to a first approximation. Therefore, attempts have been made to fix its limits experimentally. As early as 1876, Maxwell made observations up to a current density of 5×10^4 amp./cm^2, but obtained only negative results. Later, measurements by Lecher and Rausch von Traubenberg also gave negative results up to current densities of 10^7 amp./cm^2 within the limits of error of their experiments. Bridgman[1] performed more accurate experiments using Ag and Au foils of 10^{-5} cm thickness, and his measurements seemed to indicate that deviations from Ohm's law existed at current densities of 10^6 amp./cm^2. His results, however, are contested by Barlow,[2] who considers Ohm's law valid up to 2×10^6 amp./cm^2.

Until the present time all theories of metals consider only the first approximation of the dependence of the current upon the electric field, and it is supposed that deviations from Ohm's law can be explained without further assumptions by simply computing higher approximations. This has been carried out in this paper, and the deviations from Ohm's law which are automatically obtained at high current densities show

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* The joint investigations of the authors, on which this paper is based took place in 1936–38. The present article has been prepared by the senior author (E. G.).

[1] P. W. Bridgman, Phys. Rev. **19**, 387 (1922); Proc. Am. Acad. Arts **57**, 131 (1922). Professor Bridgman, in a conversation, pointed out that the effects he observed were certainly real, but at that time the question had not been settled as to whether the effects could be accounted for by some minute phenomena not yet studied in detail. Such phenomena are, for instance, (a) time lag in the thermal conductivity, (b) some kind of electromotive forces connected with a change in temperature (cf. Bridgman, second reference above, p. 145).

[2] H. M. Barlow, Phil. Mag. **9**, 1041 (1931).

the possible limitations of this law. While detailed treatment is given only to the region of high temperatures, the only region in which the whole theory of metallic conduction has a safe foundation, indications will be made as to how the theory could be extended to the region of low temperatures.

Our result is that deviations should be observable at a current density of $\sim 10^9$ amp./cm^2 only. Thus, Ohm's law should hold for all present available experimental current densities. In order to obtain this result, a general method of solving the wave-mechanical equation for conductivity which is applicable for the case of a strong electric field was developed. The distribution function showing the influence of the field is compared with Pidduck's distribution function obtained for the problem of motions of ions in gases.

THE VALIDITY OF LORENTZ' APPROXIMATION

H. A. Lorentz[3] first established a systematic theory of metals in terms of the "electron-gas" picture. He considered the metal to consist of a mixture of two gases, an electron gas and a gas of metal ions, and applied the kinetic theory of gases to this model. The problem was simplified by Lorentz in making the assumption that the mass of an ion is infinitely large in comparison with the mass of an electron, thus allowing one to consider the ions as being practically immovable in electron-ion collisions.

The following notations used in describing the motion of the electrons are due to Lorentz:

(x, y, z) = a point in position space,
(ξ, η, ζ) = a point in velocity space before collision,
$r = (\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}$,
(ξ', η', ζ') = a point in velocity space after collision,
n = number of atoms per unit volume,
R = radius of sphere of action of an electron for a collision with an atom,
φ = azimuth around the ξ axis,
$X = eF/m$ = force acting on electron,
F = electric field strength assumed to act in the x direction.

The motion of the electrons is describable by a distribution function $f(x, y, z, \xi, \eta, \zeta)$. If there is no temperature gradient, f reduces to a function of the velocity coordinates alone, and the following equation of Lorentz[3] holds:

$$nR^2r\int_0^{\pi/2}\int_0^{2\pi}\{f(\xi',\eta',\zeta')-f(\xi,\eta,\zeta)\}\times\cos\vartheta\sin\vartheta d\vartheta d\varphi=X\frac{\partial f}{\partial\xi}. \tag{1}$$

The left side represents the change of the distribution function by the impacts of electrons with the ions. The right side is the change due to the electric field. To integrate over total solid angle, a new variable $\theta=2\vartheta$ is introduced:

$$\frac{nR^2r}{4}\int_0^{\pi}\int_0^{2\pi}\{f(\xi',\eta',\zeta')-f(\xi,\eta,\zeta)\}\times\sin\theta d\theta d\varphi=X\frac{\partial f}{\partial\xi}. \tag{2}$$

Lorentz obtains an approximate solution of this equation by substituting

$$f=f_0(r)+g;\quad g\ll f_0.$$

He shows that $g\sim X$; therefore, f may be expanded in powers of the parameter X

$$f=f_0+Xf_1+X^2f_2+\cdots. \tag{3}$$

Substituting this in (2) and comparing the coefficients of X, we obtain the following equations:

$$\frac{nR^2r}{4}\int_0^{\pi}\int_0^{2\pi}(f_1'-f_1)\sin\theta d\theta d\varphi=X\frac{\partial f_0}{\partial\xi}, \tag{4}$$

$$\frac{nR^2r}{4}\int_0^{\pi}\int_0^{2\pi}(f_2'-f_2)\sin\theta d\theta d\varphi=X\frac{\partial f_1}{\partial\xi}. \tag{5}$$

. . . .

If we put $X=0$, any function of r alone will solve Eq. (2), so that f_0 is not determined by this equation. It was this very fact that allowed Sommerfeld to reconstruct the theory of metals by assuming the Fermi distribution instead of the Maxwell distribution for the electrons without any change in the Lorentz equation itself.

If f_0 is given, the necessary and sufficient

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[3] H. A. Lorentz, *Theory of Electrons* (Leipzig, 1909), p. 266.

condition for the solubility of the equation

$$\int\int(f'-f)\sin\theta d\theta d\varphi=\Phi$$

is that the right-hand side Φ be orthogonal to the solutions of the homogeneous equation. It can be shown that (4) has only solutions which are functions of r alone. The condition for solubility is, therefore,

$$\int_0^{\pi}\int_0^{2\pi}\Phi f(r)\sin\psi d\psi d\epsilon=0 \qquad (6)$$

and applying this condition to (4)

$$\int_0^{\pi}\int_0^{2\pi}\frac{\partial f_0}{\partial\xi}\sin\psi d\psi d\epsilon = \frac{df_0}{dr}\int_0^{\pi}\int_0^{2\pi}\sin\psi\cos\epsilon\sin\psi d\psi d\epsilon=0.$$

Hence (4) has a solution. The result of the first approximation, as obtained by Lorentz,[4] is:

$$f=f_0-\frac{eF}{m}\frac{\xi}{r}\frac{l}{r}\frac{df_0}{dr} \quad (l=1/nR^2r). \qquad (7)$$

To obtain a higher approximation, we calculate the right side of (5)

$$\frac{\partial f_1}{\partial\xi}=-\frac{l}{r^2}\frac{df_0}{dr}-\frac{\xi^2}{r}\frac{d}{dr}\left(\frac{l}{r^2}\frac{df_0}{dr}\right). \qquad (8)$$

From the condition of solubility (6) follows a restriction for f_0 which cannot possibly be fulfilled. Therefore, Eq. (5) cannot be solved, and consequently (1) has physical sense only in the first approximation.[5]

The reason for the insolubility of (1) in the second approximation for the field strength may be ascertained by an examination of the fundamental restriction, which Lorentz took as a basis for his calculations. If the ratio of the mass of the metal ion to the mass of the conduction electron does not have a finite value, then there can occur no transfer of energy during collision. In the first approximation this does not matter. However, in the second approximation a stationary state does not exist because the electrons continually absorb energy from the electric field, the absorption being proportional to F^2, and are therefore accelerated more and more. This absorption of energy is easily seen to be simply the Joulean heat. According to the Lorentz model, all this heat would be stored in kinetic energy of the electrons. Actually, this is not the case. The electron-ion collisions are not elastic. Consequently, in virtue of the small specific heat of the electrons as compared to that of the ions, in thermal equilibrium practically all Joulean heat will be received by the ions.

Because of these limitations, Lorentz' equation cannot be used for our case, and one must look for a generalization of it in which the above assumption is not made. In the theory of metals, only those generalizations are useful which lead to the Fermi distribution in the zero-order approximation: either a generalization of Nordheim's equation[6] for the mixture of two Fermi gases or the wave-mechanical one of Bloch. On account of the uncertainty of the limits of the electron-gas picture, the wave-mechanical theory will be considered.

THE WAVE-MECHANICAL THEORY OF CONDUCTIVITY

If one considers the interaction of the electrons with the lattice, the following equation for the distribution function is valid:

$$(\partial f/\partial t)_{\text{impact}}+(\partial f/\partial t)_{\text{field}}=0.$$

According to Sommerfeld and Bethe,[7] the first term is

$$\left(\frac{\partial f}{\partial t}\right)_{\text{impact}}=\frac{2C^2G^{-3}}{9M\hbar}\frac{\partial}{\partial t}\sum_{\mathbf{q}}\frac{q^2}{\omega_q}\{\Omega(E_{\mathbf{k}+\mathbf{q}}-E_{\mathbf{k}}-h\nu)[(1-f(\mathbf{k}))f(\mathbf{k}+\mathbf{q})(N_{\mathbf{q}}+1)-f(\mathbf{k})(1-f(\mathbf{k}+\mathbf{q}))N_{\mathbf{q}}]$$
$$+\Omega(E_{\mathbf{k}+\mathbf{q}}-E_{\mathbf{k}}+h\nu)[1-f(\mathbf{k})f(\mathbf{k}+\mathbf{q})N_{\mathbf{q}}-f(\mathbf{k})(1-f(\mathbf{k}+\mathbf{q}))(N_{\mathbf{q}}+1)]\},$$

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[4] Reference 3, p. 269.

[5] B. Davydow, Physik. Zeits. Sowjetunion **8**, 59 (1935) is not correct in stating that the expansion into powers of the field diverges and that the second approximation becomes infinite.

[6] L. W. Nordheim, Proc. Roy. Soc. **A119**, 689 (1928).

[7] Sommerfeld-Bethe, *Handbuch der Physik*, Vol. 24/2, formula (34,40).

where $f(\mathbf{k})$ is the number of electrons having the wave vector $\mathbf{k}$. N is the number of quanta of the elastic vibrations of the lattice in the state

$$h\nu=\hbar\omega_{\mathbf{q}}\,;\quad \text{i.e.,}^{8}\quad N=\frac{1}{e^{h\nu/kT}-1}.$$

$\mathbf{q}$ is the vector by which $\mathbf{k}$ is changed in a transition: $\mathbf{k}'=\mathbf{k}+\mathbf{q}$. The prime designates, as previously, the state after an impact. For other abbreviations we refer to Sommerfeld-Bethe.[9] Furthermore, it is always assumed that: $E=\hbar^2k^2/2m$.

Instead of summing, one can integrate, and there follows:

$$\sum_{\mathbf{q}}=\left(\frac{G}{2\pi}\right)^{3}\cdot\frac{1}{n}\int_0^{q_0}q^2dq\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\vartheta d\vartheta.$$

Only when energy is conserved will the function Ω not vanish, and therefore the integral can be evaluated:

$$\frac{\partial}{\partial t}\int_0^{\pi}\Omega\sin\vartheta d\vartheta=\frac{2\pi\hbar}{qdE/dk}.$$

If $\omega=u_0q$, there results

$$\left(\frac{\partial f}{\partial t}\right)_{\text{impact}}=\frac{2C^2}{9M}\frac{2\pi}{u_0dE/dk}\frac{1}{2\pi^3}\frac{1}{n}\int_0^{q_0}q^2dq\int_0^{2\pi}d\varphi J,$$

where

$$\begin{aligned}J=&(1-f(\mathbf{k}))f(\mathbf{k}+\mathbf{q},E+h\nu)(N_{\mathbf{q}}+1)-f(\mathbf{k})(1-f(\mathbf{k}+\mathbf{q},E+h\nu))N_{\mathbf{q}}\\&+(1-f(\mathbf{k}))f(\mathbf{k}+\mathbf{q},E-h\nu)N_{\mathbf{q}}-f(\mathbf{k})(1-f(\mathbf{k}+\mathbf{q},E-h\nu))(N_{\mathbf{q}}+1)\\=&\frac{1}{e^{h\nu/kT}-1}\{(1-f_k)f^{+}{}_{k+q}e^{h\nu/kT}-f_k(1-f^{+}{}_{k+q})-(1-f_k)f^{-}{}_{k+q})-f_k(1-f^{-}{}_{k+q})e^{h\nu/kT}\}.\end{aligned}$$

Here

$$f^{+}{}_{k+q}\text{ denotes }f(\mathbf{k}+\mathbf{q})\Big|_{E_{\mathbf{k}+\mathbf{q}}=E_{\mathbf{k}}+h\nu}.$$

Finally, we obtain

$$\frac{2C^2}{18Mu_0(dE/dk)\pi^2}\frac{1}{n}\int_0^{q_0}q^2dq\int_0^{2\pi}d\varphi J=+\frac{eF}{\hbar}\frac{\partial f}{\partial k_x},\tag{9}$$

where the right-hand side represents the field term. This is a functional equation, for the unknown function f not only appears for the energy E, but also for $E\pm h\nu$. To solve this equation, the distribution functions are developed into spherical harmonics:

$$f(\mathbf{k})=\psi_0+P_1\psi_1+P_2\psi_2+\cdots;\quad f(\mathbf{k}+\mathbf{q})=\psi_0+\mathcal{P}_1\psi_1+\mathcal{P}_2\psi_2+\cdots,$$

where

$$P_1=\frac{k_x}{k};\quad P_2=\tfrac{1}{2}(3P_1^2-1);\quad \mathcal{P}_1=\frac{k_x+q_x}{|\mathbf{k}+\mathbf{q}|};\quad \mathcal{P}_2=\tfrac{1}{2}(3\mathcal{P}_1^2-1).$$

Since the values of $\mathcal{P}_1,\mathcal{P}_2$ are different for emission and absorption of sound quanta a distinction will be made between these two cases by upper indices $\mathcal{P}_1^{+}$; $\mathcal{P}_1^{-}$. The same notation will be used for the ψ: ψ_0^{+}, ψ_0^{-} $\cdots$. Collecting the coefficients of spherical harmonics on the left-hand side, the coefficient of P_0 becomes:

$$(1-\psi_0)\psi_0^{+}e^{h\nu/kT}-\psi_0(1-\psi_0^{+})+(1-\psi_0)\psi_0^{-}-\psi_0(1-\psi_0^{-})e^{h\nu/kT}.\tag{10}$$

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[8] Boltzmann's constant k should not be mistaken for $k=|\mathbf{k}|$ because it is nearly always with T: kT.

[9] Sommerfeld-Bethe, reference 7.

Further, we have

$$e^{h\nu/kT}[-P_1\psi_1\psi_0^{+}+\mathcal{P}_1^{+}(1-\psi_0)\psi_1^{+}]-P_1\psi_1(1-\psi_0^{+})+\mathcal{P}_1^{+}\psi_0\psi_1^{+}-P_1\psi_1\psi_0^{-}$$
$$\times\mathcal{P}_1^{-}(1-\psi_0)\psi_1^{-}-[P_1\psi_1(1-\psi_0^{-})-\mathcal{P}_1^{-}\psi_0\psi_1^{-}]e^{h\nu/kT}. \tag{11}$$

On the right-hand side we obtain

$$\frac{eF}{\hbar}\frac{\partial f}{\partial k_x}=\frac{eF}{\hbar}\left(\frac{\partial\psi_0}{\partial k_x}+\psi_1\left[\frac{1}{k}-\frac{k_x}{k^2}\frac{k_x}{k}\right]+\frac{k_x}{k}\frac{\partial\psi_1}{\partial k_x}+\cdots\right)$$
$$=\frac{eF}{\hbar}\left\{\frac{2}{3}\frac{E}{k}P_0\left(\frac{\partial\psi_1}{\partial E}+\frac{1}{E}\psi_1\right)+\frac{\hbar^2}{m}P_1k\frac{\partial\psi_0}{\partial E}+\cdots\right\}. \tag{12}$$

THE CASE FOR ZERO ELECTRIC FIELD

When the electric field is zero, the right-hand side of (9) vanishes, so that the integrand of the left-hand side must vanish. If the coefficient of P_0 (10) is set equal to zero, we obtain the functional equation for ψ_0:

$$(1-\psi_0(E))\psi_0(E+h\nu)e^{h\nu/kT}-\psi_0(E)(1-\psi_0(E+h\nu))$$
$$+(1-\psi_0(E))\psi_0(E-h\nu)-\psi_0(E)(1-\psi_0(E-h\nu))e^{h\nu/kT}=0,$$

which may be solved by expanding into a Taylor series

$$\psi_0(E+h\nu)=\psi_0+h\nu\frac{d\psi_0}{dE}+\frac{(h\nu)^2}{2!}\frac{d^2\psi_0}{dE^2}+\cdots\frac{h\nu}{E}\approx\frac{1}{1000}.$$

It is advantageous to change the variables by introducing

$$x=E/kT;\quad \alpha=h\nu/kT;\quad \psi_0(E+h\nu)=\psi_0+\alpha\frac{d\psi_0}{dx}+\frac{\alpha^2}{2}\frac{d^2\psi_0}{dx^2}+\cdots. \tag{13}$$

When the coefficients of the various powers of α are set equal to zero, each gives an equation for ψ_0, provided they do not vanish identically. The first and second coefficients vanish; the third coefficient gives

$$(1-\psi_0)\left(\frac{\psi_0}{2}+\psi_0'+\frac{\psi_0''}{2}\right)+\frac{\psi_0\psi_0''}{2}+\frac{(1-\psi_0)\psi_0''}{2}-\psi_0\left(-\frac{\psi_0''}{2}+\psi_0'+\frac{1-\psi_0}{2}\right)=0$$

or

$$\psi_0''+\psi_0'(1-2\psi_0)=0. \tag{14}$$

This equation is the derivative of

$$\psi_0'+\psi_0-\psi_0^2=0. \tag{14a}$$

The constant of integration is equal to zero. Making use of the transformation $(\psi_0\neq 0)z=1/\psi_0$ we obtain

$$z'-z+1=0.$$

The general solution is

$$z=Ce^{x}+1;\quad \text{or}\quad \psi_0=1/Ce^{E/kT}+1.$$

All solutions of (14) are Fermi distributions and differ only in the values of the constant C which is usually written as $1/A$. The other coefficients of α give equations of the Fermi distributions also, but of higher order. The Fermi distribution is the result of the interaction between the electron-waves and the impacts alone. It is not necessary to consider the whole impact term (10) in order to obtain the Fermi distribution; we may use one-half of the expression on the left and equate to zero. The same method of expansion into a power series of α may be employed, and we obtain a differential equation of every order of the Fermi distribution.

The distribution function of the electrons is determined by (10) alone. This is also valid if Pauli's principle is neglected, but then one obtains the Maxwell distribution:

$$\psi_0{}^+e^{h\nu/kT}-\psi_0+\psi_0{}^--\psi_0e^{h\nu/kT}=0\,;\quad \psi_0''+\psi_0'=0\,;\quad \psi_0=Ae^{-x}=Ae^{-E/kT}. \tag{15}$$

The essential point is that an exchange of energy between the electrons and the lattice must take place. If the impacts are elastic and the lattice is rigid ($h\nu=0$), then in the case of zero field, (10) vanishes identically, and the approximation of order zero remains indefinite. Again we arrive at Lorentz' approximation, where the ratio of the masses of metal ions to electrons is infinite.

Thus, one can say that in any kinetic theory of gases and metals the distribution function of order zero is determined only if the impacts are not elastic.

General Case

Considering (11) and expanding, we obtain:

$$\psi_0{}^\pm=\psi_0\pm\alpha\frac{d\psi_0}{dx}+\frac{\alpha^2}{2}\frac{d^2\psi_0}{dx^2}\pm\cdots;\quad \alpha=h\nu/kT$$

$$\begin{aligned}&\left(1+\alpha+\frac{\alpha^2}{2}+\cdots\right)\left\{-P_1\psi_1\left(\psi_0+\alpha\psi_0'+\frac{\alpha^2}{2}\psi_0''+\cdots\right)+\mathcal{P}_1{}^+(1-\psi_0)\left(\psi_1+\alpha\psi_0'+\frac{\alpha^2}{2}\psi_1''+\cdots\right)\right\}\\ &\quad-P_1\psi_1\left(1-\psi_0-\alpha\psi_0'-\frac{\alpha^2}{2}\psi_0''+\cdots\right)+\mathcal{P}_1{}^+\psi_0\left(\psi_1+\alpha\psi_1'+\frac{\alpha^2}{2}\psi_1''+\cdots\right)\\ &\quad-P_1\psi_1\left(\psi_0-\alpha\psi_0'+\frac{\alpha^2}{2}\psi_0''+\cdots\right)+\mathcal{P}_1{}^-(1-\psi_0)\left(\psi_1-\alpha\psi_1'+\frac{\alpha^2}{2}\psi_1''+\cdots\right)\\ &\quad-\left(1+\alpha+\frac{\alpha^2}{2}+\cdots\right)\left\{P_1\psi_1\left(1-\psi_0+\alpha\psi_0'-\frac{\alpha^2}{2}\psi_0''+\cdots\right)-\mathcal{P}_1{}^-\psi_0\left(\psi_1-\alpha\psi_1'+\frac{\alpha^2}{1}\psi_1'''+\cdots\right)\right\}.\end{aligned}$$

Collecting the coefficients of the powers of α:
coefficient of

$$\alpha^0:\ -2P_1\psi_1+(\mathcal{P}_1{}^++\mathcal{P}_1{}^-)\psi_1,$$

coefficient of

$$\alpha^1:\ \psi_1'(\mathcal{P}_1{}^+-\mathcal{P}_1{}^-)+\psi_1(\mathcal{P}_1{}^+-\mathcal{P}_1{}^-)+\psi_0\psi_1(\mathcal{P}_1{}^--\mathcal{P}_1{}^+).$$

Inserting this into the expression for J and having expanded also the Bose-Planck distribution of lattice vibrations, we have:

$$J=\frac{kT}{h\nu}[(\psi_0''+\psi_0'(1-2\psi_0))\alpha^2P_0+\psi_1(-2P_1+\mathcal{P}_1{}^++\mathcal{P}_1{}^-+((\alpha))+\cdots].$$

The symbol $((\alpha))$ means that there are terms of order α^1 still present, but that they can be neglected. We may now integrate over φ and obtain approximately

$$\int_0^{2\pi}\mathcal{P}_1{}^+d\varphi\approx\frac{1}{k}\left(1-\frac{h\nu}{2E}\right)2\pi k_x\left(1+\left[-\frac{q^2}{2k^2}+\frac{h\nu}{2E}\right]\right);$$

$$\int_0^{2\pi}\mathcal{P}_1{}^-d\varphi\approx\frac{1}{k}\left(1+\frac{h\nu}{2E}\right)2\pi k_x\left(1+\left[-\frac{q^2}{2k^2}-\frac{h\nu}{2E}\right]\right).^{10}$$

If terms of higher order are neglected, there results

$$\int_0^{2\pi}(\mathcal{P}_1{}^++\mathcal{P}_1{}^-)d\varphi=\frac{2\pi k_x}{k}\cdot 2.$$

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[10] Reference 7 (35,15).

Using the abbreviation

$$R = C^2\Omega_0 m/18\pi^2 M u_0 \hbar^2,$$

we have

$$\left(\frac{\partial f}{\partial t}\right)_{\text{impact}} = \frac{R}{k}\int_0^{q_0} q^2 dq \left[P_0(\psi_0'' + \psi_0'(1-2\psi_0))\alpha^2\cdot 2\pi + P_1\frac{-q^2}{2k^2}4\pi\psi_1\right]\frac{kT}{h\nu}$$

and since

$$\alpha = h\nu/kT = \hbar\omega_0/kT = \hbar u_0 q/kT,$$

$$\left(\frac{\partial f}{\partial t}\right)_{\text{impact}} = \frac{R}{k}\left\{P_0\cdot 2\pi\frac{\hbar u_0}{kT}\frac{q_0^4}{4}(\psi_0'' + \psi_0'(1-2\psi_0)) - P_1\frac{4\pi}{2k^2}\frac{kT}{\hbar u_0}\frac{q_0^4}{4}\psi_1\right\}.$$

Finally, comparing the coefficients of α of the impact term and the field term

$$-\left(\frac{\partial f}{\partial t}\right)_{\text{field}} = \frac{eF}{\hbar}\left\{P_0\cdot\frac{2}{3}\cdot\frac{1}{|\mathbf{k}|}(x\psi_1' + \psi_1) + P_1\frac{\hbar^2}{m}|\mathbf{k}|\frac{1}{kT}\psi_0'\right\},$$

$$\psi_0'' + \psi_0'(1-2\psi_0) = eF\cdot A(x\psi_1' + \psi_1); \quad A = 4\cdot(2/3)kT/\hbar R\cdot 2\pi q_0^4\hbar u_0,$$

$$\psi_1 = -eFBx^2\psi_0'; \quad B = 4m\cdot 2u_0/\hbar^2 R\pi q_0^4,$$

we obtain

$$\psi_0''(1+\beta x^3) + \psi_0'(1-2\psi_0+3\beta x^2) = 0; \quad \beta = (eF)^2 AB. \tag{16}$$

This equation determines that part of the distribution function of the electrons which does not depend on direction. (16) differs from the differential equation of the Fermi distribution (14) by an additional term proportional to $\beta \approx F^2$. For weak electric fields it is permissible to equate this term to zero and then the Fermi distribution is obtained.

Equation (16) is the derivative of the following:

$$\psi_0'(1+\beta x^3) + \psi_0 - \psi_0^2 = 0. \tag{16a}$$

Let us compare this equation with the differential equation of first order, which the distribution function of Pidduck[11] obeys:

$$f = Ae^{-E/kT}(E/kT+b)^b; \quad f'(1+b/x) + f = 0;$$

$$x = E/kT.$$

Both equations refer to relatively strong electric fields, but one from the classical point of view, the other from the quantum point of view. We know from the preceding section that in the case of $\beta = 0$, by omitting $-\psi_0^2$, the differential equation of first order for the Fermi distribution reduces to that of the Maxwell distribution. The only difference, therefore, lies in the form of the dependence of the additional term upon the energy. According to Pidduck b is proportional to the square of the mean free path of an electron, but according to wave mechanics l depends on the square of energy. So b is proportional to the fourth power of x, $b/x \sim x^3$ and the conformity is clear.

To obtain the solution of (16a) we proceed as in the case of zero field. The general solution is:

$$\psi_0 = 1/C \exp\left[\int \frac{dx}{1+\beta x^3}\right] + 1.$$

Practically, βx^3 is always very small compared to 1, and the denominator may be expanded, obtaining

$$\psi_0 \approx 1/Ce^{x-(\beta x^4/4)} + 1.$$

To compute the current, we start from the Sommerfeld-Bethe equation (36.9) and replace f

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[11] F. B. Pidduck, Proc. Lond. Math. Soc. (11) **15**, 89 (1915). Pidduck treats the mixture of two gases according to the classical kinetic theory of gases applying Boltzmann's equations. He starts with the equation

$$\int\int\int_{-\infty}^{+\infty} d\omega \int_0^{\infty} b\,db \int_0^{2\pi} d\varphi\cdot r(f'F' - fF) = X\frac{\partial f}{\partial \xi},$$

taking for F the Maxwellian distribution and assuming a finite ratio of the masses. After transforming this equation into one with a symmetric kernel, he deduces a series of equations by expansion into spherical harmonics. These equations are solved then by another expansion into the ratio of the masses. In this way the following distribution function as modified by the influence of the electric field is obtained:

$$f_0 = Ae^{-E/kT}(E/kT+b)^b$$
$$b = (eF/kT)^2\cdot l/6;\ l\text{: mean free path.}$$

The same distribution function was rediscovered twenty years later by B. Davydow (Physik. Zeits. Sowjetunion **9**, 433 (1936)) who started from a different equation. The identity of the results of Pidduck and Davydow was first pointed out by Mayerhöfer (Thesis, Vienna, June 1937). Pidduck himself noticed it (Phys. Rev. **53**, 197 (1938)).

by ψ_0:

$$J=-(2/3)e^2F\int_0^\infty \frac{1}{\hbar}\frac{1}{(2\pi)^3}4\pi\frac{2m}{\hbar^2}E\frac{d\psi_0}{dE}l(E)dE,$$

where

$$\frac{d\psi_0}{dE}=-\psi_0{}^2C\exp\left[\frac{E}{kT}-\frac{\beta}{4}\left(\frac{E}{kT}\right)^4\right]\times\frac{1}{kT}\left(1-\beta\left(\frac{E}{kT}\right)^3\right).$$

It is not necessary to compute the integral exactly. If we neglect the terms in the exponential function and denote by f_0 the Fermi function, we have

$$\frac{d\psi_0}{dE}\approx\frac{df_0}{dE}[1-\beta(E/kT)^3].$$

Furthermore, the bracket and just as well E and $l(E)$ may be factored out of the integral, if their value is taken at the point ζ_0, and the current becomes

$$J=J_0[1-\beta(E_0/kT)^3],$$

where J_0 is the current for very small fields, or explicitly

$$J=\sigma_0F\left(1-\frac{ABe^2F^2}{(kT)^3}E_0{}^3\right). \tag{17}$$

This formula gives the dependence of the current upon higher powers of the electric field in the case where the second term is relatively small compared to the first.

If the second term of (17) is small, then Ohm's law is valid. If it is not small, then the limits of Ohm's law have been reached. To evaluate it numerically we take[12]

$$\theta=217;\quad C=6.7\text{ ev};\quad E_0=5.5\text{ ev}$$

and calculate the field strength at which the additional term amounts to 1 percent. We obtain about 5 abs. e.s.u. corresponding to a current density of[13]

$$J=\sigma F=6\times7\times10^{17}\times5\text{ e.s.u.}\approx10^9\text{ amp./cm.}^2$$

This then is the current density at which the deviation from Ohm's law is of the order of 1 percent. Since this current density is almost 1000 times greater than those applied by Bridgman and Barlow, no observable deviations from Ohm's law should occur in their experiments, a fact which is in agreement with Barlow who states that Ohm's law is valid for current densities up to 2×10^6 amp./cm².

INFLUENCE OF THE ELECTRIC FIELD ON THE LATTICE WAVES

In the present theory of conductivity as used in this paper, the influence of the electric field only upon the distribution function of the electrons is considered. In a more consistent theory the equilibrium of the electron waves and the lattice waves must be considered. This is analogous to the case of a mixture of two gases where (according to Boltzmann) a system of two simultaneous integro-differential equations holds. If we denote the distribution functions of electrons and lattice waves by f and N, respectively, then the fundamental equations written symbolically would be as follows:

$$(f,f)+(f,N)=(\partial f/\partial t)_{\text{field}},$$
$$(N,N)+(N,f)=(\partial N/\partial t)_{\text{dissipation}}.$$

In the first approximation the perturbation of the lattice waves is small. Consequently, in the first equation f may be assumed to be constant and in the second equation, N. The two equations are not coupled then. The first equation reduces to the fundamental equation of the present theory, the second being omitted.

In the second approximation, however, the two equations are simultaneous. For a solution we may start from the first approximation considering the coupling as a perturbation. The influence of the external field on the lattice may be connected with the problem of the so-called "Umklapp-Prozesse."[14]

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[12] According to E. L. Peterson and L. W. Nordheim, Phys. Rev. **51**, 355 (1937). Cf. also J. Bardeen, Phys. Rev. **52**, 688 (1937).

[13] A very crude estimate can be obtained as follows: For Lorentz' development: $f=f_0+f_1$, the condition $f_1\ll f_0$ must hold, because f_1 on the right-hand side of his equation can be neglected only in this case. Assuming

$$\frac{f_1}{f_0}=-\frac{1}{f_0}\frac{l}{r}\frac{eF}{m}\frac{df_0}{dr}\frac{\xi}{r}=(1-f_0)\frac{leF}{3kT}=10^{-2},$$

we obtain $J=\sigma F\sim10^8$ amp./cm². Such an estimate was also made by Dr. Nordheim.

[14] Cf. also W. V. Houston, Phys. Rev. **55**, 1255 (1939).