# On the Time Distribution of So-Called Random Events

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(Received February 10, 1940)

This paper contains a discussion and complete solution of a problem treated recently by A. Ruark.

## 1

UNDER the same title, A. Ruark[1] recently discussed a statistical problem which seems frequently encountered in analyzing the results of some counting experiments and which "appears to have an interest far transcending its immediate applications." In mathematical terms Ruark's problem is the following one.

Consider a sequence of "random events," that is to say, suppose that the probability $W_n(t)$ of the occurrence, in a run of duration $t$, of exactly $n$ events is given by the Poisson formula:[2]

$$W_n(t)=(ft)^n e^{-ft}/n! \quad W_0(t)=e^{-ft}, \tag{1}$$

where $f>0$ is a constant characteristic for the process. Now, out of a series of observations, pick out at random[3] some time-interval of length $D$; let $n$ be the number of actually observed events during that time. Pick out, again at random, two consecutive events among these. What is the probability that the time elapsed between their occurrence will exceed a given number $T<D$? In other words: we wish to calculate the conditional probability $P(T; n, D)$ that the interval between two consecutive events will exceed $T$, under the hypothesis that it is known that these events occurred in a run of duration $D$ during which there occurred exactly $n\geq 2$ events.

Using Bayes' theorem, Ruark finds for this probability the expression

$$\frac{1}{1-e^{-fD}}\frac{n}{fD}\left(1-\frac{T}{D}\right)^{n-1}. \tag{2}$$

But by definition $P(0; n, D)=1$, for all possible $n$, $D$, whereas the expression (2) may take on any value between 0 and $\infty$. In an actual counting experiment, the probability that (2) will largely exceed 1 is small, because the probability that $n\sim fD$ is overwhelming. Still it is clear that (2) is not the correct expression for any probability distribution. We propose to show that the correct solution of our problem is

$$P(T; n, D)=(1-T/D)^n. \tag{3}$$

Now Ruark considers $P(T; n, D)$, or (2), only as an approximation to another conditional probability $P_1(T; n, D)$, with which he was primarily concerned: the $n$ events divide the whole interval $D$ into $n+1$ sub-intervals (the probability of an event occurring exactly in an end-point being 0). In the above formulation we considered only $n-1$ of them, namely, those between two consecutive events. In the original problem, as formulated by Ruark, we are to pick out at random any of the $n+1$ intervals; the probability of its length exceeding $T$ is $P_1(T; n, D)$. Ruark considers $P(T; n, D)$ as an approximative value of $P_1(T; n, D)$. It will be shown, however, that

$$P(T; n, D)\equiv P_1(T; n, D). \tag{4}$$

More precisely, our calculations will show that it is not necessary to choose the interval at random: $P(T; n, D)$ can be interpreted also as the conditional probability that, for any fixed $k\leq n$, the length of the $k$th among the said intervals will

[1] A. Ruark, Phys. Rev. **56**, 1165 (1939).

[2] Sometimes also called after Bateman. The assumptions leading to (1) are: (i) the probability of the occurrence of an event in any time-interval $(t, t+\Delta t>t)$ is, independently of the previous events, $f\cdot\Delta t+o(\Delta t)$, where $o(\Delta t)$ stands for terms of order negligible as compared with $\Delta t$; (ii) the probability of the occurrence of more than one event during $(t, t+\Delta t)$ is $o(\Delta t)$. Obviously the second assumption cannot be deduced from the first one and is essential for the deduction of (1).

[3] This assumption is essential for Ruark's problem: the calculation would be different if we were to consider e.g. time intervals of duration $D$ starting with some actually observed event (in which case $n$ would be replaced by $n-1$). For the reader's convenience we follow Ruark's notations as far as possible. $D$ is used both to denote our time-interval and its length.

exceed $T$, under assumption that the interval $(0, D)$ contains exactly $n$ events.

There is a third interpretation of $P(T; n, D)$, which is the simplest and, perhaps, the most natural one: $P(T; n, D)$ is also the conditional probability that if we pick out at random any point $t$ of our interval $D$, there is no event between $t$ and $t+T$. The interpretation given above for (3) can be considered as a consequence of this. Similarly, in Poisson's original formula (1) $W_0(t)$ gives the probability that the interval between two consecutive events will exceed $t$, but that is only a consequence of the primary definition. We proceed now to the actual computations.

## 2

The conditional probability $P_B(A)$ of an event $A$ under the hypothesis of another event $B$ is computed directly from the definition

$$P_B(A) = P(AB)/P(B), \tag{5}$$

where $P(B)$ is the absolute probability of the event $B$, $P(AB)$ the probability of the combination of both events $A$ and $B$. For all conditional probabilities which we require, the hypothesis $B$ consists in the occurrence of exactly $n$ events in a run of duration $D$; hence by (1)

$$P(B) = (fD)^n e^{-fD}/n! \tag{6}$$

Now let $(0, D)$ be the given interval and denote by $E_1, \cdots, E_n$ the $n \geq 2$ events which are to occur in it. Let us first fix some $k$ $(1 \leq k \leq n-1)$, and denote by $A$ the event that the time interval between $E_k$ and $E_{k+1}$ exceeds a given number $T$ with $0 \leq T \leq D$. Suppose the times of occurrence of $E_k$ and $E_{k+1}$ to be $t$ and $\tau$. Then the realization of the event $A$ requires that $0 < t < D-T$ and $t+T < \tau < D$. In addition, the simultaneous realization of $A$ and $B$ means: (i) during $(0, t)$ there are exactly $k-1$ events; (ii) an event occurs at the moment $t$; (iii) no events between $t$ and $\tau$; (iv), an event at the moment $\tau$; (v) exactly $n-k-1$ events during $(\tau, D)$. The probability of this event is:

$$e^{-ft}\frac{(ft)^{k-1}}{(k-1)!}fdt \cdot e^{-f(\tau-t)}fd\tau \cdot e^{-f(D-\tau)}\frac{\{f(D-\tau)\}^{n-k-1}}{(n-k-1)!}$$
$$= e^{-fD}f^n\frac{t^{k-1}(D-\tau)^{n-k-1}}{(k-1)!(n-k-1)!}dtd\tau.$$

Thus, summing over the possible values of $t$ and $\tau$

$$P(AB) = e^{-fD}\frac{f^n}{(k-1)!(n-k-1)!}$$
$$\times \int_0^{D-T} t^{k-1}dt \int_{t+T}^{D} (D-\tau)^{n-k-1}d\tau$$
$$= e^{-fD}\frac{\{f(D-T)\}^n}{n!}.$$

Finally by (5) and (6) the Eq. (3) results. Thus the conditional probability distribution for the interval sizes is the same for the time-intervals between any $E$ and the next. Hence, we may fix $k$ arbitrarily or choose $E_k$ at random among $E_1, \cdots, E_{n-1}$: the conditional probability distribution of the time to wait for the next event is in both cases given by (7).

So far, the assertion (3) has been proved. In Ruark's original problem, we have also to consider the intervals from the $E_1$ to $D$, and from the beginning to the occurrence of $E_1$ and from $E_n$ to $D$. Consider, e.g., the first one, and let $E_1$ occur at the time $\tau$. The realization of the event $A$ means $\tau \geq T$. The realization of $AB$ requires: (i) no event occurs during $(0, \tau)$; (ii) an event occurs at $\tau$; (iii) $(n-1)$ events occur during $(\tau, D)$. The elementary probability of this event is

$$e^{-f\tau}fd\tau \cdot e^{-f(D-\tau)}\frac{\{f(D-\tau)\}^{n-1}}{(n-1)!},$$

and hence

$$P(AB) = e^{-fD}\frac{f^n}{(n-1)!}\int_T^D (D-\tau)^{n-1}d\tau$$
$$= e^{-fD}\frac{\{f(D-T)\}^n}{n!},$$

which again coincides with the common probability for the intervals between any $E_k$ and $E_{k+1}$. The same argument applies for the interval between $E_n$ and $D$. Hence, if we are to choose at random any of the $n+1$ sub-intervals of $D$, we still fall back on (3). This again proves (4), though this identity was to be expected *a priori*.

The same argument holds also for the last and simplest of the three problems mentioned above: if $x$ is any fixed or randomly chosen point in $0<x<D-T$, the conditional probability that the next event will occur between $x+\tau$ and $x+\tau+d\tau$ is obviously

$$e^{-f\tau}fd\tau\cdot e^{-f(D-\tau)}\{f(D-\tau)\}^{n-1}/(n-1)!P(B) = \frac{n}{D}\left(1-\frac{\tau}{D}\right)^{n-1}$$

and hence we get by a single integration again the correct answer (3).

MAY 15, 1940 PHYSICAL REVIEW VOLUME 57

# On the Deviations from Ohm's Law at High Current Densities

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(Received March 2, 1940)

The deviations from Ohm's law at high current densities are calculated on the basis of the wave-mechanical theory of conductivity. A current density of $10^9$ amp./cm$^2$ causes a 1 percent deviation only. No observable deviations are to be expected at the experimental current densities available at present ($10^6$ amp./cm$^2$). This is in agreement with the experiments of Barlow, neither does it contradict those of Bridgman, if the effects found by him are due to secondary factors. The method used consists in the actual solving of the fundamental equation for conductivity in a higher approximation. It is proved that in the Lorentz model (fixed metal ions) the fundamental equation is not soluble in the second approximation in the field strength. A solution in this approximation can be obtained only by assuming inelastic collisions between the electrons and the metal ions. The analogy between the distribution function containing the influence of the electric field and a distribution function found by Pidduck for the motion of ions in gases is pointed out. A generalization of the present theory is indicated by taking into account the influence of the external field on the lattice waves.

## INTRODUCTION

OHM'S law has been extraordinarily successful for many decades; but our present knowledge makes it clear that this law can be, not a fundamental law (such as Coulomb's law), but a derived law, which describes reality only to a first approximation. Therefore, attempts have been made to fix its limits experimentally. As early as 1876, Maxwell made observations up to a current density of $5\times10^4$ amp./cm$^2$, but obtained only negative results. Later, measurements by Lecher and Rausch von Traubenberg also gave negative results up to current densities of $10^7$ amp./cm$^2$ within the limits of error of their experiments. Bridgman[1] performed more accurate experiments using Ag and Au foils of $10^{-5}$ cm thickness, and his measurements seemed to indicate that deviations from Ohm's law existed at current densities of $10^6$ amp./cm$^2$. His results, however, are contested by Barlow,[2] who considers Ohm's law valid up to $2\times10^6$ amp./cm$^2$.

Until the present time all theories of metals consider only the first approximation of the dependence of the current upon the electric field, and it is supposed that deviations from Ohm's law can be explained without further assumptions by simply computing higher approximations. This has been carried out in this paper, and the deviations from Ohm's law which are automatically obtained at high current densities show

* The joint investigations of the authors, on which this paper is based took place in 1936–38. The present article has been prepared by the senior author (E. G.).

[1] P. W. Bridgman, Phys. Rev. **19**, 387 (1922); Proc. Am. Acad. Arts **57**, 131 (1922). Professor Bridgman, in a conversation, pointed out that the effects he observed were certainly real, but at that time the question had not been settled as to whether the effects could be accounted for by some minute phenomena not yet studied in detail. Such phenomena are, for instance, (a) time lag in the thermal conductivity, (b) some kind of electromotive forces connected with a change in temperature (cf. Bridgman, second reference above, p. 145).

[2] H. M. Barlow, Phil. Mag. **9**, 1041 (1931).