Relation Between Actual and Virtual Ionospheric Height

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Translation of virtual height of the ionosphere into actual height is achieved by fitting a parabolic maximum of electron density to observations of variation of virtual height with wave frequency. Methods for measuring the thickness of a region and the height of maximum electron density are described and applied. The simplest measure of the actual height of maximum electron density is the virtual height at five-sixths of the penetration frequency. This measure is found to be remarkably reliable for F region at night. During the daytime it is often necessary to correct F_1 observations for presence of E region, and F_2 observations for presence of F_1 region. A simple technique of correction is described and applied. The disadvantages of minimum virtual height for quantitative study of diurnal variation of ionospheric height are emphasized.

I. INTRODUCTION

THE object of this paper is to describe a method for translating virtual height of the ionosphere into actual height. The method is quite simple, and yet sufficiently accurate for many practical purposes. It applies primarily to the F region, but difficulty arises when this region is separated into F_1 and F_2 regions. It is assumed that the observational material consists of variation with frequency of virtual height of the ionosphere for radio waves reflected vertically. Virtual height is proportional to time delay. Observations of this type over practically the entire range of wave frequency for which echoes are normally received from the F region are made automatically every 15 minutes at the observatories maintained at Huancayo, Peru, and Watheroo, Western Australia, by the Department of Terrestrial Magnetism of the Carnegie Institution of Washington. The problem of translating virtual height into actual height is one to which thought has already been given.[1–3] One method is to fit to the observations a parabolic maximum of electron density (see Fig. 1). It is this method which will be considered here.

In Section II we treat an idealized problem. It is supposed that: (i) The ionosphere consists of a parabolic maximum of electron density; (ii) propagation of radio waves is unaffected by presence of the earth's magnetic field; (iii) there is no Lorentz polarization correction;[4, 5] (iv) observations have been made of variation of virtual height with wave frequency from zero frequency up to the penetration frequency. The problem is to find convenient methods for deducing from the observations the constants which define the parabolic maximum of electron density. The extent to which the results of Section II are applicable in practice is discussed in Section III, and the actual distribution of electron density with height is deduced for two sample ionospheric records. In Section IV the results of Sections II and III are considered in connection with the

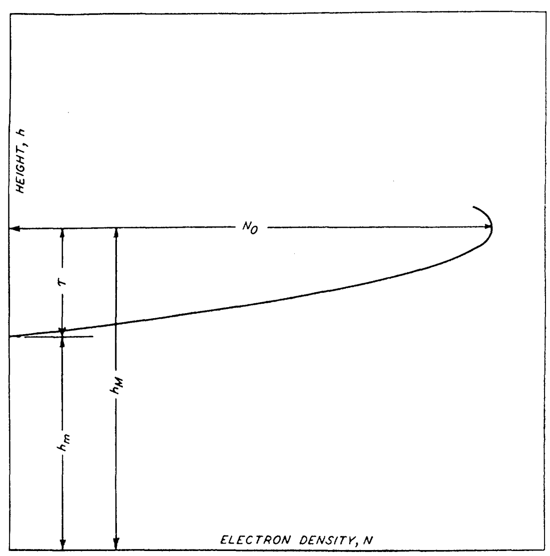

FIG. 1. Parabolic maximum of electron density.

[1] E. V. Appleton, Proc. Phys. Soc. **41**, 43–59 (1928).
[2] E. V. Appleton, Proc. Roy. Soc. **A162**, 451–479 (1937).
[3] F. H. Murray and J. B. Hoag, Phys. Rev. **51**, 333–341 (1937).
[4] C. G. Darwin, Proc. Roy. Soc. **A146**, 17–46 (1934).
[5] H. G. Booker and L. V. Berkner, Terr. Mag. **43**, 427–450 (1938).

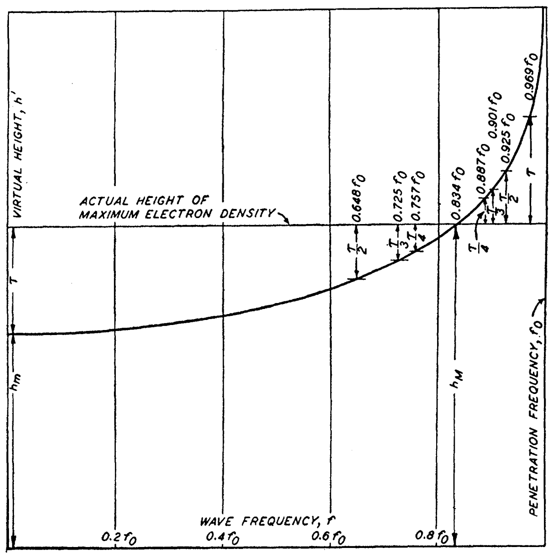

FIG. 2. Relation between virtual height and wave frequency for parabolic maximum of electron-density.

problem of routine scaling of a mass of ionospheric records.

II. Parabolic Maximum of Electron Density

An ionosphere consisting of a parabolic maximum of electron density is depicted in Fig. 1. The distribution of electron-density N with height h is defined by three quantities: (i) The maximum electron density, N_0; (ii) the height h_M at which the electron density is maximum; (iii) the height h_m at which the electron density begins to increase from zero.

The relation between N and h is

$$\frac{N_0-N}{N_0}=\left(\frac{h_M-h}{h_M-h_m}\right)^2. \tag{1}$$

The quantity

$$\tau=h_M-h_m \tag{2}$$

will be called the thickness of the parabolic maximum of electron density.

An ionosphere of the type shown in Fig. 1 has a penetration-frequency f_0. Only radio waves of frequency less than f_0 are reflected at vertical incidence. It is well known [compare reference 2] that, on the assumptions mentioned in Section I, the variation of virtual height h' with wave frequency f for the parabolic maximum (1) of electron density is given by

$$\frac{h'-h_m}{h_M-h_m}=\frac{f}{2f_0}\log_e\frac{f_0+f}{f_0-f}. \tag{3}$$

This variation is depicted in Fig. 2.

The problem is: Given the variation of h' with f shown in Fig. 2, to determine the corresponding variation of N with h shown in Fig. 1. This is equivalent to determining the three quantities N_0, h_M, and h_m, or alternatively the

TABLE I. *Tabulation of* $\varphi(x)=\frac{x}{2}\log_e\left|\frac{1+x}{1-x}\right|-1$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	−1.000	−1.000	−1.000	−0.999	−0.998	−0.997	−0.996	−0.995	−0.994	−0.992
0.1	−0.990	−0.988	−0.986	−0.983	−0.980	−0.977	−0.974	−0.971	−0.967	−0.963
0.2	−0.959	−0.955	−0.951	−0.946	−0.941	−0.936	−0.931	−0.925	−0.919	−0.913
0.3	−0.907	−0.901	−0.894	−0.887	−0.880	−0.872	−0.864	−0.856	−0.848	−0.839
0.4	−0.830	−0.821	−0.812	−0.802	−0.792	−0.782	−0.771	−0.760	−0.749	−0.737
0.5	−0.725	−0.713	−0.700	−0.687	−0.674	−0.660	−0.646	−0.631	−0.616	−0.600
0.6	−0.584	−0.568	−0.551	−0.533	−0.515	−0.496	−0.477	−0.457	−0.436	−0.415
0.7	−0.393	−0.370	−0.347	−0.322	−0.297	−0.270	−0.243	−0.214	−0.185	−0.154
0.8	−0.121	−0.087	−0.051	−0.014	+0.026	+0.068	+0.112	+0.160	+0.211	+0.265
0.9	+0.325	+0.390	+0.462	+0.542	+0.634	+0.740	+0.868	+1.029	+1.252	+1.620
1.0	+∞	+1.678	+1.354	+1.170	+1.045	+0.950	+0.874	+0.812	+0.759	+0.714
1.1	+0.674	+0.639	+0.608	+0.580	+0.554	+0.531	+0.510	+0.490	+0.471	+0.455
1.2	+0.439	+0.424	+0.410	+0.397	+0.385	+0.373	+0.362	+0.352	+0.342	+0.333
1.3	+0.324	+0.316	+0.307	+0.300	+0.292	+0.285	+0.279	+0.272	+0.266	+0.260
1.4	+0.254	+0.249	+0.243	+0.238	+0.233	+0.229	+0.224	+0.220	+0.215	+0.211
1.5	+0.207	+0.203	+0.199	+0.196	+0.192	+0.189	+0.186	+0.182	+0.179	+0.176
1.6	+0.173	+0.170	+0.167	+0.165	+0.162	+0.159	+0.157	+0.155	+0.152	+0.150
1.7	+0.148	+0.145	+0.143	+0.141	+0.139	+0.137	+0.135	+0.133	+0.131	+0.129
1.8	+0.127	+0.126	+0.124	+0.122	+0.120	+0.119	+0.118	+0.116	+0.114	+0.113
1.9	+0.112	+0.110	+0.109	+0.107	+0.106	+0.105	+0.104	+0.102	+0.101	+0.100

three quantities N_0, h_M, and τ. N is derived directly from the observed penetration frequency f_0. If N_0 is measured per cc and f_0 in mc/sec., then

$$N = kf_0^2, \tag{4}$$

where

$$k = 1.24 \times 10^4. \tag{5}$$

The problem is therefore reduced to the following: Given the variation of h' with f shown in Fig. 2, to deduce h_M and h_m, or h_M and τ.

Define a function

$$\varphi(x) = \frac{x}{2} \log_e \left|\frac{1+x}{1-x}\right| - 1. \tag{6}$$

$\varphi(x)$ is plotted in Fig. 3 and tabulated in Table I. Using (2) and (6), one may write (3)

$$h' = h_M + \tau \cdot \varphi(f/f_0). \tag{7}$$

Given the variation of h' with f, we may deduce h_M and τ as follows: (i) Measure the values of h' corresponding to any two values of f/f_0 lying between 0 and 1; (ii) from Table I find the values of φ corresponding to these two values of f/f_0; (iii) substitute the two pairs of corresponding values of h' and φ into (7), and solve for h_M and τ.

In practice it is convenient to choose the two values of f/f_0, not at random, but so that the corresponding values of φ are numerically simple. Values of f/f_0 corresponding to a series of useful values of φ are tabulated in Table II and indicated in Fig. 2. The values in Table II are deduced from Table I.

Consider the wave frequency

$$f = 0.834 f_0. \tag{8}$$

From Table II the corresponding value of φ is zero. Hence, at the wave frequency (8), (7) becomes

$$h' = h_M. \tag{9}$$

Eqs. (8) and (9) state that, at the wave frequency equal to 0.834 times the penetration frequency, the virtual height is the actual height of maximum electron density. Thus, if h'_{834} is the virtual height at the wave frequency equal to 0.834 times the penetration frequency, then

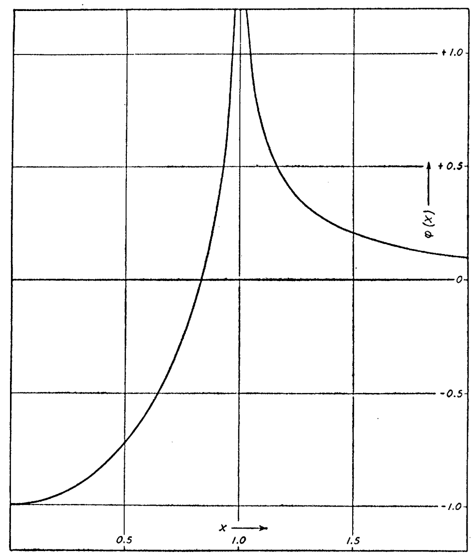

FIG. 3. Graph of $\varphi(x) = x/2\left(\log_e\left|\frac{1+x}{1-x}\right| - 1\right)$.

$$h_M = h'_{834} \tag{10}$$

(see Fig. 2). This remarkable result is valid no matter what may be the values of the thickness τ and the penetration frequency f_0. The peculiar property of the wave frequency (8) is demonstrated in Fig. 4, which shows the effect upon the (h', f) curve of varying τ keeping h_M constant. We may note that 0.834 (or more accurately 0.8336) is practically five-sixths.

Measurement of h' at any two of the wave frequencies mentioned in Table II gives immediately the values of both h_M and τ. For example, let h'_{648} and h'_{925} be the virtual heights at the wave frequencies equal to 0.648 and 0.925 times the penetration frequency. From Table II the values of φ corresponding to these wave frequencies are $-\frac{1}{2}$ and $+\frac{1}{2}$, respectively. Hence (7) gives

$$\begin{cases} h'_{648} = h_M - (1/2)\tau, & (11) \\ h'_{925} = h_M + (1/2)\tau. & (12) \end{cases}$$

TABLE II. *Useful wave frequencies.*

$\varphi =$	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1
$f/f_0 =$	0	0.648	0.725	0.757	0.834	0.887	0.901	0.925	0.969

Solving (11) and (12) for h_M and τ we obtain

$$\begin{cases} h_M=(1/2)(h'_{925}+h'_{648}), & (13)\\ \tau=h'_{925}-h'_{648}. & (14)\end{cases}$$

The validity of (13) and (14) is also obvious from Fig. 2.

Formulas for h_M and τ involving values of h' at more than two wave frequencies are easily deduced. For example, it is obvious from Fig. 2 that

$$\begin{cases} h_M=(1/5)\{(h'_{925}+h'_{648}) \\ \qquad +(h'_{887}+h'_{757})+h'_{834}\}, & (15)\\ \tau=(4/5)(h'_{925}-h'_{648}) \\ \qquad +(2/5)(h'_{887}-h'_{757}). & (16)\end{cases}$$

III. Distribution of Electron Density with Height

Figure 5 shows a typical example of an ionospheric record obtained at night by the automatic multifrequency technique.[6] Between sunset and sunrise the ionosphere consists normally of a single main ionized region—the F region. We shall show that the results of Section II may be applied directly to records such as this to quite a high degree of accuracy. To do this we must examine the validity of the assumptions upon which Section II is based.

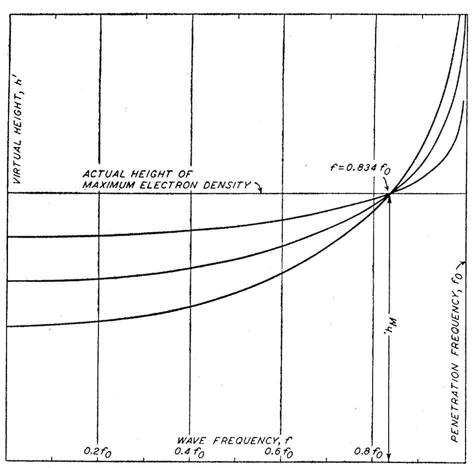

Fig. 4. Effect upon (h',f) curve of varying τ keeping h_M constant.

[6] L. V. Berkner, H. W. Wells and S. L. Seaton, Trans. Edinburgh Meeting 1936; Internat. Union Geod. Geophys., Ass. Terr. Mag. Electr., Bull. No. 10, pp. 340–357 (1937).

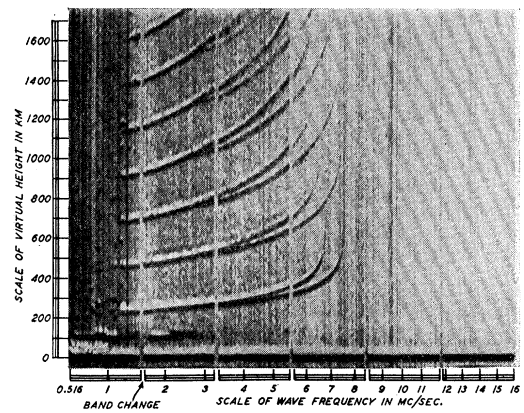

Fig. 5. Observed variation of virtual height with wave frequency, Watheroo magnetic observatory, March 14, 1939, $20^h\,15^m$ to $20^h\,30^m$ local time (120° East Meridian).

It is assumed that the ionosphere consists of a parabolic maximum of electron density. According to Section II h_M and τ may be deduced from the observed values of h' at any two wave-frequencies. Table III shows the results obtained for the record reproduced in Fig. 5 with pairs of wave frequencies mentioned in Table II. Heights were scaled by use of the second multiple echo ($2F$). The consistency of results obtained with the ordinary wave is quite remarkable. As might be expected, results were not quite so consistent for the extraordinary wave. There is little doubt that, for the record reproduced in Fig. 5, the distribution of electron density with height is substantially parabolic as shown in Fig. 1. This distribution is maintained from the maximum down to an electron density, say, 20 percent of the maximum. However, as might be expected, it is not maintained down to zero electron density. This is demonstrated by the fact that the minimum virtual height of the F region for the record reproduced in Fig. 5 is only 225 km, whereas h_m is 247 km.

It is assumed in Section II that propagation of radio waves in the ionosphere is unaffected by the

Table III. *Consistency of deduced heights.*

Ratio of Frequencies Used to Penetration Frequency	Ordinary Wave			Extraordinary Wave		
	h_M	h_m	τ	h_M	h_m	τ
0.648, 0.925	359	247	112	350	240	110
0.725, 0.901	358	247	111	350	230	120
0.757, 0.887	358	248	110	355	235	120
0.834, 0.969	358	246	112	355	220	135

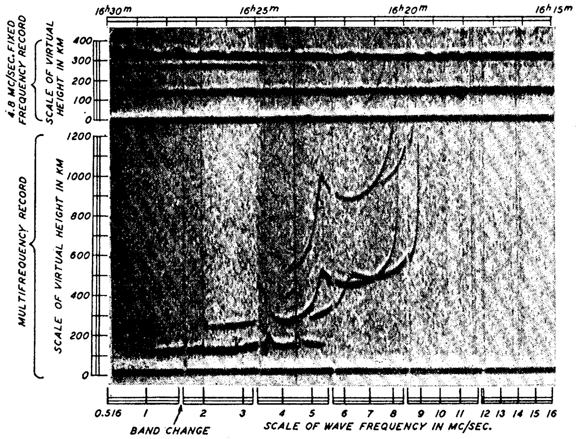

FIG. 6. Observed variation of virtual height with wave frequency, Watheroo magnetic observatory, December 18, 1939, $16^h\,15^m$ to $16^h\,30^m$ local time (120° East Meridian).

earth's magnetic field. At Huancayo, close to the geomagnetic equator, this assumption may be regarded as fully satisfied for the ordinary wave. But at Watheroo, where the magnetic dip is about $-64°$, the earth's magnetic field affects propagation of both magneto-ionic components. However, the work of Goubau[7] shows that the correction required on this account is at least twice as great for the extraordinary wave as for the ordinary. It follows that, first, it is better to apply the methods of Section II to the ordinary wave rather than the extraordinary wave and, second, the error in so doing should be less than the difference in results obtained from the two magneto-ionic components. The difference between the virtual heights for the two components at the wave frequencies equal to 0.834 times their respective penetration frequencies is usually between five and ten km when the ionosphere is substantially a parabolic maximum of electron density. We conclude that effect of the earth's magnetic field upon results obtained from the ordinary wave is small. Nevertheless it is systematic.

Similar remarks apply to the assumption in Section II that there is no Lorentz polarization correction. The only important effect of including this correction would be to replace (5) by

$$k = 1.86 \times 10^4. \qquad (17)$$

[7] G. Goubau, Hoch: tech. u. Elek: akus. **44**, 17–23, 138–139 (1934).

We may sum up by saying that, for records of the type reproduced in Fig. 5, the technique of Section II applied to the ordinary wave gives results sufficiently accurate for most practical purposes.

Between sunrise and sunset, records are not usually as simple as that reproduced in Fig. 5. Fig. 6 shows a record (the trace at the top of this record shows variation of virtual height with time for a fixed wave frequency of 4.8 mc/sec.) more typical of conditions existing during daylight. Three main regions are apparent—E, F_1, and F_2. In these circumstances it is clearly impossible to represent the ionosphere by a single parabolic maximum of electron density. It is possible, however, to represent it by a triplet of parabolas as shown in Fig. 7. Fig. 7 was deduced from the trace corresponding to the ordinary wave on the record reproduced in Fig. 6 in the following way:

(i) The technique of Section II was first applied to E region.

(ii) The E region was assumed to consist of a complete parabolic maximum of electron density and observations of virtual height for F_1 region were corrected for presence of E region. This was done by subtracting from the virtual height of F_1 region at wave frequency f an amount $2\tau \cdot \varphi(f/f_0)$, where τ and f_0 are, respectively, the thickness and penetration frequency deduced for E region, and the function φ is given by Table I.

(iii) The technique of Section II was applied to F_1 region corrected as above.

(iv) The F_1 region was assumed to consist of half of a parabolic maximum of electron density

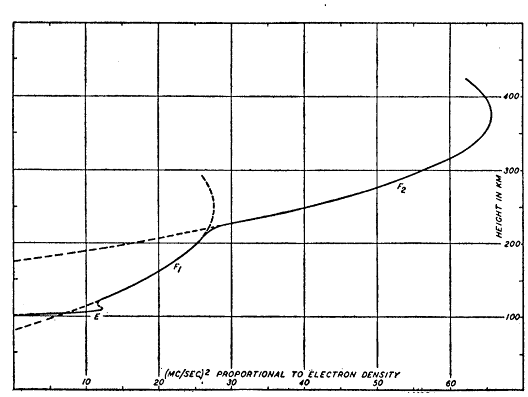

FIG. 7. Distribution of electron density with height corresponding to record reproduced in Fig. 6.

FIG. 8

FIG. 9.

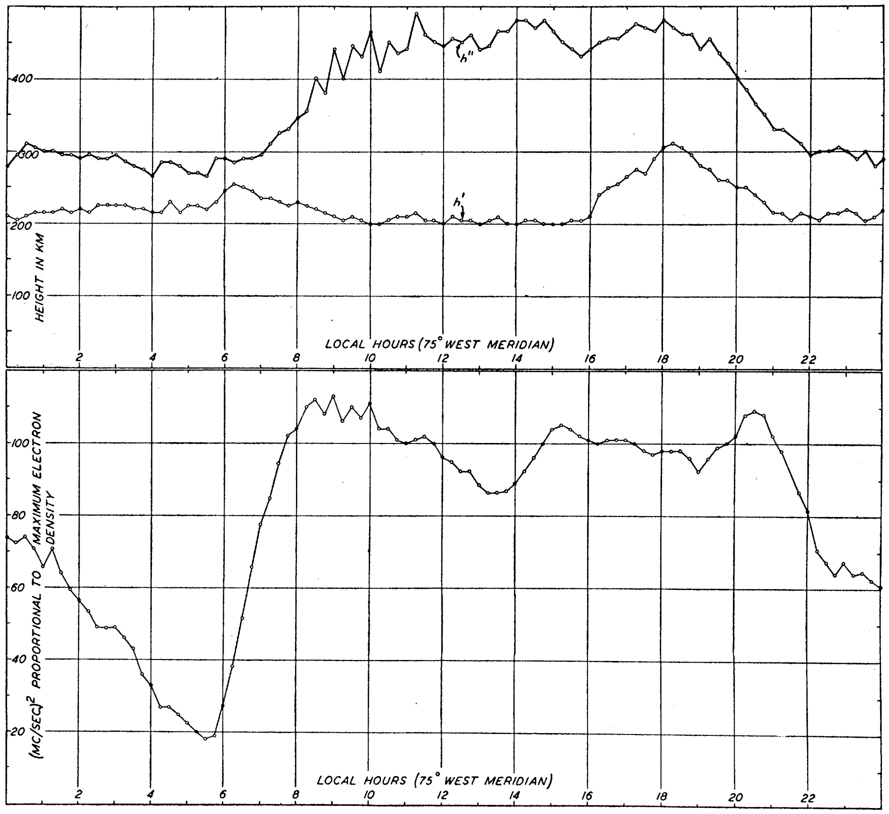

FIG. 8. Diurnal variation of h' and h'' for F region, Huancayo magnetic observatory, May 31, 1939.
FIG. 9. Diurnal variation of square of penetration frequency of F region for ordinary wave, Huancayo magnetic observatory, May 31, 1939.

and observations of virtual height for F_2 region were corrected for presence of F_1 region. This was done by subtracting from the virtual height of F_2 region at wave frequency f an amount $\tau\cdot\varphi(f/f_0)$, where τ and f_0 are, respectively, the thickness and penetration frequency deduced for F_1 region. Correction of F_2 observations for presence of E region was found to be negligible.

(v) The technique of Section II was applied to F_2 region corrected as above.

It is believed that Fig. 7 is quite a good representation of the distribution of electron density with height corresponding to the record reproduced in Fig. 6. Note the low actual heights associated with F_1 region.

IV. ROUTINE SCALING OF IONOSPHERIC RECORDS

In mass analysis of ionospheric records it has been the practice to make scalings for E, F_1, and F_2 (or F) regions as follows. Records made at hourly intervals are scaled for penetration frequency f_0 of the ordinary wave and for minimum virtual height. It would now be desirable to replace scaling of minimum virtual height by scaling of the height h_M of maximum electron density and the thickness τ by the technique of Sections II and III. This would provide practically a complete record of the distribution of electron density with height in the ionosphere up to the level of maximum electron density. Moreover, it would then be possible to study variation of the quantity $(2/3)\tau k f_0^2$ which is representative of the total number of electrons in a region below the level of maximum electron density. Such a program may not be practical because of the labor involved. In the meantime it is desirable that a less ambitious program be drawn up.

Consider the following program. Scale for the F_1 and F_2 (or F) regions: (i) The penetration

FIG. 10.

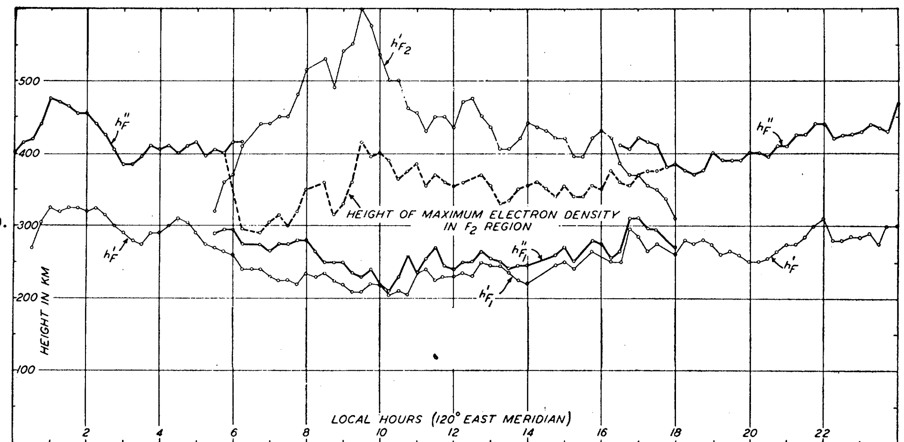

FIG. 11.

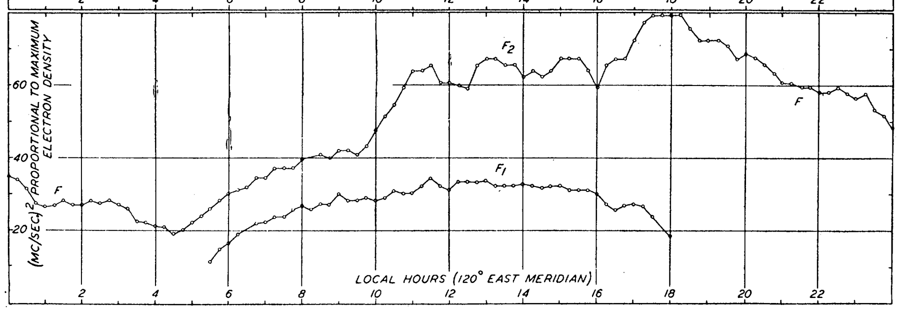

FIG. 10. Diurnal variation of h' and h'' for F_1 and F_2 regions, Watheroo magnetic observatory, December 18, 1938.
FIG. 11. Diurnal variation of squares of penetration frequencies of F_1 and F_2 regions for ordinary wave, Watheroo magnetic observatory, December 18, 1938.

frequency f_0 for the ordinary wave; (ii) the virtual height h'' for the ordinary wave at the wave frequency $0.834f_0$; (iii) the minimum virtual height h'. The result of scaling in this way the records from Huancayo for a single day is shown in Figs. 8 and 9. Separation of F region during daylight into F_1 and F_2 regions was not sufficiently marked to take account of. At night h' does not differ greatly from h_m (Fig. 1), and gives a useful indication of what may be called the bottom of F region. During the day, however, h' is appreciably greater than h_m owing to presence of E region. h'' is quite a satisfactory measure of the height h_M of maximum electron density in F region throughout this particular day. A check on this was obtained by comparing two values for the intensity of the earth's magnetic field. One is the intensity deduced for height h'' from the intensity at ground level on the assumption that the strength of the earth's magnetic field varies inversely as the cube of distance from the center of the earth. The other is the intensity deduced for the actual level of maximum electron density from the penetration frequencies of the region for the ordinary and extraordinary waves.[8] The average of the 96 ratios of the second intensity to the first obtained for the day represented in Figs. 8 and 9 is 0.99 ± 0.01. This is to be regarded as a satisfactory check. It may be mentioned that it is difficult to obtain a similar check at Watheroo for the following reason. Because of the inclination of the earth's magnetic field at Watheroo, the ordinary wave is deviated in the magnetic meridian towards the south pole and the extraordinary wave towards the equator. Consequently, the two magneto-ionic components do not penetrate the ionosphere at exactly the same points. There is a small systematic difference between the maximum electron densities at the two points of penetration, and this is sufficient to ruin the check unless a reliable correction can be applied.

[8] E. V. Appleton, Nature **133**, 793 (1934).

Figures 10 and 11 show a more complicated state of affairs than that represented in Figs. 8 and 9. Figs. 10 and 11 refer to a magnetically disturbed day at Watheroo. During the night h_F' gives a useful indication of the bottom of F region, and h_F'' is quite a satisfactory measure of the height of maximum electron density. During the day, however, the situation is very different. At sunrise F region separates into F_1 and F_2 regions. h_{F2}' increases rapidly to enormous virtual heights. Between $06^h\,15^m$ and $06^h\,30^m$, h_{F2}' becomes equal to h_{F2}''. This is because the wave frequency at which virtual height of F_2 region is minimum has increased to 0.834 times the penetration frequency of F_2 region. In these circumstances h_{F2}'' clearly exceeds the actual height of maximum electron density in F_2 region due to presence of F_1 region. Between $06^h\,30^m$ and $11^h\,00^m$ the wave frequency at which virtual height of F_2 region is minimum actually exceeds 0.834 times the penetration frequency of F_2 region. Consequently even the minimum virtual height of F_2 region exceeds the height of maximum electron density. During such an interval it is useless to scale h_{F2}'' unless it is corrected for presence of F_1 region. Between $11^h\,00^m$ and $16^h\,15^m$ the wave frequency at which virtual height of F_2 region is minimum is roughly equal to 0.834 times the penetration frequency of F_2 region. Consequently, h_{F2}' is approximately equal to h_{F2}'', and only one curve is shown. In this interval the common value of h_{F2}' and h_{F2}'' is necessarily greater than the height of maximum electron density, but not to the enormous extent prevailing between $06^h\,30^m$ and $11^h\,00^m$. The broken curve shows the actual height of maximum electron density in F_2 region between $05^h\,45^m$ and $17^h\,45^m$ deduced by correcting for presence of F_1 region by the method outlined at the end of Section III.

The height h_{F1}'' in Fig. 10, is appreciably greater than h_{F1}' throughout most of the daytime and gives a useful measure of the height of maximum electron density in F_1 region so far as this region has individuality (see Fig. 5). Correction of h_{F1}'' for presence of E region would decrease values by only about ten km. h_{F1}', on the other hand, does not give a useful indication of the bottom of F_1 region due to presence of E region; see Fig. 5, which refers to epoch $16^h\,15^m$ in Figs. 10 and 11.

It is worth emphasizing that minimum virtual height is ill adapted for quantitative study of solar diurnal variation of the height of F region. At night minimum virtual height is a measure of the height of the bottom of the region. But during the day minimum virtual height usually measures the height of a level appreciably above the bottom of the region, and not uncommonly above even the level of maximum electron density. This arises partly from variable retardation in a lower region, but mainly from variation in the ratio of the wave frequency at which minimum virtual height is measured to the penetration frequency of the region under examination.

Such applications as have so far been made of the technique described in this paper indicate that it should prove a powerful method of approaching a number of ionospheric problems. Berkner, Wells, and Seaton have already used the technique in connection with an intense magnetic storm.[9] In studying regular diurnal variation of the ionosphere it is desirable to use observations averaged for a number of days. Monthly averages will shortly be available.

Throughout the conduct of this work we have had the benefit of helpful discussion with L. V. Berkner, to whom we desire to express our thanks.

[9] L. V. Berkner, H. W. Wells and S. L. Seaton, Terr. Mag. **44**, 283–311 (1939).

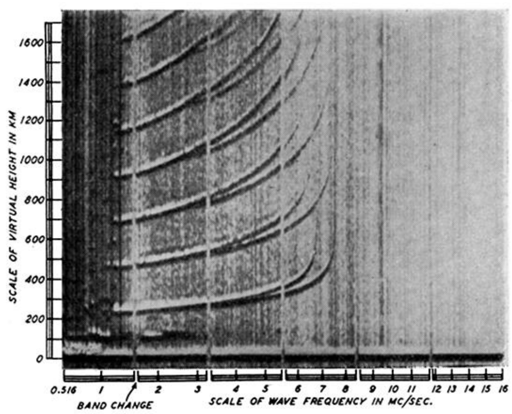

FIG. 5. Observed variation of virtual height with wave frequency, Watheroo magnetic observatory, March 14, 1939, $20^h\,15^m$ to $20^h\,30^m$ local time (120° East Meridian).

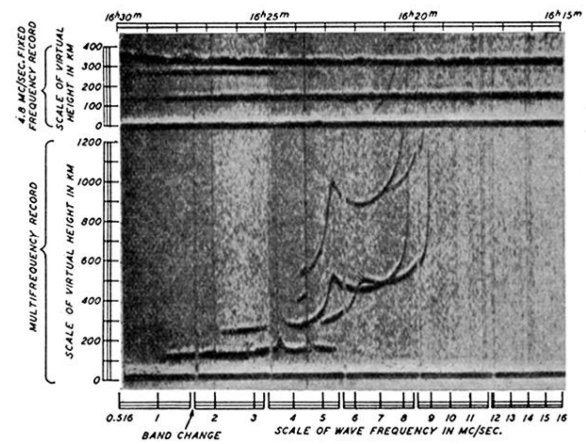

FIG. 6. Observed variation of virtual height with wave frequency, Watheroo magnetic observatory, December 18, 1939, $16^h\ 15^m$ to $16^h\ 30^m$ local time (120° East Meridian).