ν, κ and E are energies of the initial quantum, final quantum, and electron in units of mc^2 and α is the fine structure constant 1/137. We divide this by κ to get the number of scattered quanta per second. This is also, of course, the number of recoil electrons per second. It is given in terms of D which is unknown. But the total number of electrons internally converted in the K shell for a dipole transition is also given in terms of D. It is[6]

$$N_K=\frac{4}{3}\left(\frac{\nu+2}{\nu}\right)^{\frac{1}{2}}(\nu^2+2)D^2Z^3\alpha^4.$$

Dividing the above expression for the number of electrons scattered per second in the energy range dE by this gives:

$$\frac{N_E dE}{N_K}=\frac{3\alpha}{2\pi}\left(\frac{\nu}{\nu+2}\right)^{\frac{1}{2}}\frac{W_E}{\nu^2+2}dE,$$

i.e., the number of electrons scattered in dE per electron internally converted. The above asymptotic expression for W_E gives a number about twice as large as that calculated from the more exact formula in the energy range 2.39–2.52 Mev for the 2.62-Mev gamma-ray. The average magnitude of the exact form of W_E in this range is about 800. This gives about 1/60 for the number of electrons scattered in this range per electron internally converted.

The authors are grateful to Professor J. R. Oppenheimer for suggesting this problem and for his kind guidance throughout the work.

[6] S. Dancoff and P. Morrison, Phys. Rev. **55**, 122 (1939).

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The Oppenheimer-Phillips Process

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A critical study of the previous theoretical treatments of the process of neutron capture by heavy nuclei bombarded with deuterons shows that while the dependence of the transmutation function on the incident deuteron energy, W, has been given correctly, the energy distribution of the outgoing protons has not been satisfactorily estimated. In I, the application to the Oppenheimer-Phillips process of the usual formula for the cross section is shown to be justified for restricted values of the atomic number Z and of the deuteron energy. Bethe's method is used to express the partial cross section as a product of three factors: the penetrability of the potential barrier, the sticking probability of the neutron, and the energy transfer factor. In II, methods of obtaining the deuteron wave-function are discussed. Calculations of the transmutation function are extended to higher values of Z and W, and results obtained by using the O-P-Bethe and the Kapur methods are compared. In III, the proton energy distribution is discussed. A re-evaluation of the dependence of the transfer factor on the proton energy, K, leads to a result differing from Bethe's. The transfer factor is found to have a fairly sharp maximum, and to determine essentially the proton energy distribution. For high Z and low W the position and half-breadth of this maximum is given roughly by $K_0\sim W$ and $\Delta K_0\sim 3.3\ WZ^{-\frac{1}{2}}$. Lifshitz' and Kapur's treatments of proton energies are examined, and found to be unsatisfactory.

INTRODUCTION

A GENERAL theoretical treatment of the nuclear reactions induced by deuteron bombardment is made difficult by the fact that they are many-body processes to which ordinary perturbation methods are not applicable because of the strong short-range forces involved. An attempt to simplify the problem may be made by considering first the probabilities of the proton

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and the neutron penetrating to the surface of the nucleus, and then treating the interaction of these particles with the nucleus schematically in terms of a "sticking probability." Such a method is useful in discussing a mechanism proposed by Oppenheimer and Phillips[1] to describe the process of neutron capture by heavy nuclei bombarded with deuterons. According to this mechanism the deuteron is polarized by the Coulomb field, the neutron penetrates to the surface of the nucleus where it is captured, while the proton is repelled by the Coulomb field without penetrating deeply into the potential barrier. This mechanism undoubtedly is important for describing neutron capture reactions in light as well as in heavy nuclei, and over a wide range of deuteron bombarding energies. However, the theoretical treatment of this process based on considering separately the penetrabilities and the "sticking probabilities" of the particles is shown to be justified only for deuteron energies considerably smaller than the Coulomb barrier of the nucleus involved in the reaction.

In this paper we consider only the neutron capture process occurring under bombardment with deuterons whose energy satisfies the above restriction. Oppenheimer and Phillips[1] first found the dependence of the transmutation function on the incident deuteron energy. Kapur[2] repeated the calculation using a different approximation for the deuteron wave-function obtained by a generalization of the W.K.B. method developed by Kapur and Peierls.[3] Lifshitz,[4] although starting from a point of view different from that of Kapur, arrives at results identical with those of Kapur for the dependence of the transmutation function on the incident deuteron energy. Finally, Bethe[5] expresses the cross section for this process in terms of the "sticking probability" of the neutron, and the penetrability of the potential barrier. Lifshitz,[4] Bethe[5] and Kapur[6] give discussions of the outgoing proton energies, but their methods of treating this question turn out to be unsatisfactory. However, Bethe's treatment needs modification in only one essential point, and is taken as the basis of the present discussion of proton energies.

The purpose of this paper is threefold: (a) to show under what restrictions the usual perturbation-theoretic formula for the cross section in terms of the square of the matrix element of the perturbing interaction is applicable to this problem where the perturbation is not small; (b) to extend the calculations of the transmutation function to higher Z and higher bombarding energies, comparing the results of the O-P-Bethe, and the Kapur methods; and (c) to give a qualitative discussion of the proton energy distribution with reference to previous theoretical treatments of this topic.

I. Derivation of the Cross Section for the O-P Process

Bethe's[5] derivation of the cross section for the O-P process is based on the application to this problem of the formula

$$\sigma=(2\pi/\hbar)\left|\int V_A(N,n)\chi_A^*(N)\varphi_W^*(p,n)\times\chi_B(N,n)\psi_K(p)dNdndp\right|^2, \quad (1)$$

where N, p, n denote the coordinates of the particles in the initial nucleus, A, and in the incoming deuteron, φ_W and ψ_K are properly normalized Coulomb-field wave functions of the incoming deuteron of energy, W, and the outgoing proton of energy, K, χ_A and χ_B are the internal wave functions for the initial and final nuclei, and V_A is the schematic interaction between the original nucleus and the neutron. The usual perturbation-theoretic derivation of (1) is based on considering transitions between orthogonal states under the action of a *small* perturbing potential V_A, so that terms of order V_A^2 in the matrix element may be neglected. The validity of applying (1) to the O-P process in which V_A is not small must therefore be examined. We make the restriction to sufficiently high Z and low W to make the probability of both the proton and the neutron penetrating to the surface of the nucleus much smaller than that for the neutron alone, so that the short range non-Coulomb interaction $V_A'(N,p)$ of the proton with the nucleus, and the effects of direct deuteron capture may be neglected. The O-P process may then be represented by transitions between two almost orthogonal states. In the initial state the system is well described by $\chi_A(N)\varphi_W(p,n)$ where the effects of the nuclear

[1] J. R. Oppenheimer and M. Phillips, Phys. Rev. **48**, 500 (1935).
[2] P. L. Kapur, Proc. Roy. Soc. **A163**, 553 (1937).
[3] P. L. Kapur and R. Peierls, Proc. Roy. Soc. **A163**, 606 (1937).
[4] E. Lifshitz, Physik. Zeits. d. Sowjetunion **13**, 324 (1938).
[5] H. A. Bethe, Phys. Rev. **53**, 39 (1938).
[6] P. L. Kapur, Ind. J. Phys. **13**, 87 (1939).

Coulomb potential $V_c(N, p)$ and of the proton-neutron interaction $V(n, p)$ are considered in calculating $\varphi_W(p, n)$, and $V_A(N, n)$ and $V_A'(N, p)$ are neglected. The final state of the system is represented by $\chi_B(N, n)\psi_K(p)$ where $\chi_B(N, n)$ takes into account the effect of $V_A(N, n)$, but the influence of only $V_c(N, p)$ on $\psi_K(p)$ is considered, the short-range interaction of the proton with the compound nucleus being neglected. These two states are not completely orthogonal since different parts of the total interaction are left out in obtaining them, but will approach orthogonality more and more closely as the deuteron energy W is reduced making the penetration of the proton to the surface of the nucleus less and less likely. Oppenheimer's[7] method of treating transitions between almost orthogonal states may be applied to this problem. It gives (1) for the first-order term in the cross section, while the second-order correction to the matrix element is of order VV_A instead of being of order V_A^2 as given by the usual perturbation theory for transitions between orthogonal states. Although both V and V_A are large, the matrix element of their product will be small for the states under consideration, as both interactions are of short range, differing from zero only in small practically non-overlapping regions if the above restriction on the deuteron energy is made. Since no correction terms in V_A^2 appear, and the term in VV_A is small, the partial cross section to any specified final state is given by (1), subject to the condition on Z and W.

If all the variables specifying the final state except the proton energy K are integrated out of (1), the energy distribution of the outgoing protons for a given W will be obtained, and another integral over K will then give the total cross section as a function of W, i.e. the transmutation function. Carrying out these integrations in accordance with the arguments given by Bethe,[5] we obtain as a rough approximation for the proton energy distribution, leaving out numerical factors independent of K and W,

$$\sigma(K, W)dK \sim |\varphi_W(p_0, R)|^2 T(K)\xi(E_B)dK. \quad (2)$$

The principal simplifying assumptions made in obtaining (2) include the following:

(a) In order to factor out the sticking probability ξ, it is assumed that the principal contribution to (1) comes from the positions of the neutron on the surface of the nucleus. Since the neutron-nucleus interaction is of short range, regions outside the nucleus will not contribute appreciably, and since the neutron does not penetrate deeply into the nucleus undisturbed, but rapidly shares its energy with the nuclear particles, the whole neutron-nucleus interaction may be throught of as confined to a region about the surface of the nucleus much smaller than the dimensions of the polarized deuteron, which are illustrated in Fig. 3.

(b) The effects of only the spherically symmetric partial waves for both the proton and the neutron are considered in (2). The effects of the higher partial waves are important for the determination of the angular distribution of the outgoing protons, or of the absolute value of the cross section, but do not introduce appreciable changes into the estimates of the transmutation function or the proton energy distribution. If the higher partial waves were taken into account, then the first two factors in (2) would be replaced by a sum of products of similar factors taken over the various values of the angular momenta allowed by the conservation laws. The general character of the dependence of each of these factors on K and W is shown in the following sections to be the same as for the one appearing in (2), so that the summation will affect the absolute value of the cross section, but not appreciably its dependence on K and W.

(c) The integrations over the angles of the proton and the neutron are assumed to introduce no factors dependent strongly on the distance of the proton from the nucleus or on the deuteron energy W. For a given nuclear radius and a given distance of the proton from the heavy nucleus the principal contribution in integrating over the angles will come from those positions of the proton and the neutron when they are very nearly in a straight line with the heavy nucleus. As the particles deviate from the straight line position, the center of mass moves in towards the nucleus and the proton-neutron distance increases. Both these effects tend to diminish the deuteron wave function. For high values of Z and low values of W to which we are restricted the wave function will fall off quite rapidly.

The first factor in (2) gives the dependence of the penetrability of the potential barrier on the incident deuteron energy, and determines essentially the tranmsutation function. $\varphi_W(p, R)$ is the spherically symmetric partial wave of the deuteron Coulomb-field wave function normalized to unit flux. It is calculated by methods described in the next section as a function of the proton coordinate with the proton, neutron and nucleus kept in the same straight line, and the neutron held fixed at the surface of the nucleus. p_0 is the value of the proton coordinate which makes $\varphi_W(p, R)$ a maximum, and will be referred to as "the most probable proton coordinate." R is the nuclear radius.

[7] J. R. Oppenheimer, Phys. Rev. **31**, 66 (1928).

The second factor describes the transfer of kinetic energy to the proton when its coupling to the neutron is suddenly terminated by the neutron being captured, and also takes into account the gain in energy by the liberated proton under the action of the Coulomb field. We shall refer to it as the "transfer" factor. It is given by

$$T(K)=K^{\frac{1}{2}}\left|\int p^2dpS(p)f_K(p)\right|^2\equiv K^{\frac{1}{2}}|I(K)|^2, \quad (3)$$

where $S(p)\equiv\varphi_W(p, R)/\varphi_W(p_0, R)$, and $f_K(p)$ is the spherically symmetric Coulomb-field partial wave for the proton as given for instance by Mott and Massey.[8] It is in the evaluation of this factor that we differ from Bethe. His rough estimate of it yields $T(K)\sim K^{-\frac{3}{2}}$, except at very low proton energies, so that he is led to the conclusion that compared to the sticking probability the transfer factor is slowly varying, and is not important in determining the proton energy distribution. However, our re-evaluation of this factor discussed in Section III shows it to be a peaked function of K essentially determining the proton energy distribution.

The last factor $\xi(E_B)$ is called by Bethe the modified neutron sticking probability, and is defined in terms of the level spacing and the neutron width of the levels of the compound nucleus. It is introduced to give an approximate averaged out result of the effect on the proton energy distribution of the various nuclear levels into which the neutron may be captured. The function ξ is assumed to be a function of the excitation energy E_B of the nucleus formed by neutron capture measured from the ground state of the compound nucleus. Through E_B and the conservation of energy

$$W-I-I_A=K-I_B+E_B. \quad (4)$$

Therefore ξ is a function of $W-K$. Here I_A and I_B are the binding energies of the ground states of the initial and final nuclei, while I is the binding energy of the deuteron. Little direct experimental information in regard to the sticking probability is available, but a plausible assumption seems to be that for low-lying and intermediate levels of the compound nucleus it is proportional to the level density, increasing until a value of nearly unity is reached, and then remaining approximately constant for the highly excited states of the compound nucleus. Actually, of course, the effect of nuclear levels will vary irregularly from level to level depending not only on the energy, but on the angular momentum and other characteristics of the individual levels. Therefore if the neutron is captured into low-lying levels (low values of E_B, hence high values of K) whose spacing is large compared to their breadth, so that effects due to individual levels must be considered, then this way of describing the process will fail. The energy distribution of protons may show group structure at high energies, and our only reliable information about the effect of sticking probability on the high energy part of the proton energy distribution curve will be the maximum possible value of K allowed by (4) corresponding to the neutron being captured into the ground level. However, if the neutron is captured into one of the highly excited levels of the compound nucleus in the region where the spacing of the levels is comparable with their breadth, then the function ξ giving the smeared out effect of a large number of levels will be a satisfactory way of describing the influence of nuclear structure on the low energy end of the proton energy distribution curve. The transfer factor is found to have quite a sharp maximum at values of K corresponding to the neutron being captured into the higher lying levels in the region in which the sticking probability gives a good schematized description of the interaction resulting in neutron capture. Moreover, for these high levels ξ probably varies only slowly with the energy, so that the proton energy distribution is determined essentially by the peaked transfer factor.

Integrating (2) over K, and retaining only the factor most strongly dependent on W we obtain for the transmutation function

$$\sigma(W)\sim|\varphi_W(p_0, R)|^2. \quad (5)$$

The Gamow transmutation function derived on the assumption that the deuteron is not polarized, and the proton and neutron both reach the surface of the nucleus, may be obtained from (5) by replacing the most probable proton coordinate p_0 by the nuclear radius R:

$$\sigma_G(W)\sim|\varphi_W(R, R)|^2. \quad (6)$$

[8] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford, 1933), p. 39.

II. The Deuteron Wave Function and the Transmutation Function

Oppenheimer and Phillips[1] used the adiabatic and W. K. B. approximations to calculate the spherically symmetric Coulomb-field partial wave for the straight line positions of the deuteron as a function of the proton and neutron coordinates. Kapur[2] avoids the use of the adiabatic approximation, and uses a two-dimensional extension of the W. K. B. method to evaluate this wave function only at the most probable proton coordinate when the neutron is held at the surface of the nucleus, thus determining the transmutation function. His method is easily extended to determine this wave function for other straight line positions of the proton, neutron and nucleus. Lifshitz[4] obtains the same result as Kapur for the transmutation function, and the relation of his work to the two other methods is discussed at the end of Section III. To facilitate comparison all results are stated in terms of Bethe's notation. R is the nuclear radius, p and n the proton and neutron coordinates, and $r=\frac{1}{2}(p+n)$ the coordinate of the center of mass. It is convenient to introduce the reduced dimensionless quantities $\rho=IR/Ze^2$, $x=Ip/Ze^2$, $y=In/Ze^2$, $\eta=Ir/Ze^2=\frac{1}{2}(x+y)$, $\epsilon=W/I$. Most of the results can be obtained in terms of these reduced quantities, and assumptions in regard to the magnitude of the nuclear radii have to be introduced only in the final stages of the calculations. We take the nuclear radii to be given by $R=R_0A^{\frac{1}{3}}$ with $R_0=1.4\times10^{-13}$ cm which makes $R=3.5\times10^{-13}$ cm for oxygen, and $R=8.7\times10^{-13}$ cm for uranium.

The adiabatic approximation used by Oppenheimer and Phillips[1] and Bethe[5] to separate the internal motion of the deuteron from the motion of its center of mass makes use of the fact that the effective time of collision of the deuteron with the nucleus is long compared to the period of the internal motion of the deuteron, so that the change in the external forces acting on the deuteron over a period of its internal motion may be neglected. The relative motion is approximately given by the solution of the wave equation when r, the distance of the center of mass of the deuteron from the nucleus, is held fixed. The center of mass moves in an effective potential which is the energy of the relative motion expressed as a function of r. The W. K. B. method is applied separately to the two resulting equations, and gives (for $p>n$)

$$\varphi_W(p,n)=\exp\left[-\frac{2Ze^2}{\hbar}\left(\frac{M}{I}\right)^{\frac{1}{2}}F_{OP}(\epsilon,x,y)\right], \quad (7)$$

where M is the proton mass, and

$$F_{OP}(\epsilon,x,y)=\int_\eta^x du\left(1+\frac{1}{u}-\frac{1}{\eta}\right)^{\frac{1}{2}} +\int_\eta^{1/\epsilon} ds\left(\frac{1}{s}-\epsilon\right)^{\frac{1}{2}} \quad (8)$$

$$=\epsilon^{-\frac{1}{2}}f(\epsilon\eta)+\eta^{\frac{1}{2}}|1-\eta|^{-\frac{1}{2}} \times\{f(1-\eta)-f(\{1-\eta\}\{2-y\eta^{-1}\})\} \quad (9)$$

$$f(\alpha)\equiv-\sinh^{-1}(-\alpha)^{\frac{1}{2}}-[(-\alpha)(1-\alpha)]^{\frac{1}{2}} \quad \alpha\leqq0$$

$$\equiv\cos^{-1}\alpha^{\frac{1}{2}}-[\alpha(1-\alpha)]^{\frac{1}{2}} \qquad 0\leqq\alpha\leqq1. \quad (10)$$

This is imaginary for $\alpha>1$, and hence is neglected, as only the real part of F is of interest. The result is given in this form by Bethe,[5] and reduces on setting $y=0$ (zero nuclear radius) to that of Oppenheimer and Phillips.[1]

Kapur does not make the first adiabatic approximation, but directly substitutes $\varphi_W(p,n)=\exp[-1/\hbar S_K(W,p,n)]$ into the deuteron equation, and leaves out certain terms in the resulting expression. Whereas in the adiabatic approximation the variation of the relative-motion wave function with the coordinate of the center of mass is completely neglected in calculating the center-of-mass wave function, Kapur's procedure takes into account its variation with the magnitude, but not with the direction of the coordinate of the center of mass. If $S_K(W,p,n)=2Ze^2(M/I)^{\frac{1}{2}}F_K(\epsilon,x,y)$, then F_K satisfies the equation

$$F_x^2+F_y^2=\tfrac{1}{2}(1/x-\epsilon+1) \quad (11)$$

with the boundary condition (for $p>n$)

$$(F_x-F_y)_{x=y}=1. \quad (12)$$

This may be solved either by a process of minimization of a line integral as done by Kapur,[2] or by the method of characteristics. The latter method is more straightforward. The

characteristics are given by

$$dt=\frac{dx}{2F_x}=\frac{dy}{2F_y}=\frac{dF}{2(F_x^2+F_y^2)}=-\frac{dF_x}{1/2x^2}=-\frac{dF_y}{O}. \quad (13)$$

$x=y=s$ is chosen as the initial line. The values of F_x and F_y along this line as obtained from (11) and (12) are:

$$F_x(s)=\tfrac{1}{2}[1-(1/s-\epsilon)^{\frac{1}{2}}],$$
$$F_y(s)=\tfrac{1}{2}[-1-(1/s-\epsilon)^{\frac{1}{2}}] \quad (14)$$

and the value of F along the initial line is given by

$$F(s)=\int_{1/\epsilon}^{s}F'(s)ds=\int_{s}^{1/\epsilon}\left(\frac{1}{s}-\epsilon\right)^{\frac{1}{2}}ds. \quad (15)$$

From (13) we see that the characteristics are the lines of steepest descent. Integration of (13) leads to

$$F_K(\epsilon, x, y,)=2^{-\frac{1}{2}}\int_{x,\ y}^{x_1,\ x_1}dx\left(\frac{1}{x}-\epsilon+1\right)^{\frac{1}{2}}(1+y'^2)^{\frac{1}{2}}+\int_{x_1}^{1/\epsilon}\left(\frac{1}{s}-\epsilon\right)^{\frac{1}{2}}ds, \quad (16)$$

where the first integral is taken along the characteristic passing through the point (x, y) until it meets the initial line at the point (x_1, x_1). The slope of the characteristic at any point $(x\geqq y)$ is given by

$$y'=\pm\beta[2(1/x-\epsilon+1-\tfrac{1}{2}\beta^2]^{-\frac{1}{2}}, \quad (17)$$

where $+$ is taken for $\beta\geqq 2$, while $-$ is chosen for $1\leqq\beta<2$ along the initial line $x=y$, but changes to $+$ when the denominator of (17) vanishes in integrating along a characteristic. x_1 and β are determined in terms of ϵ, x and y by

$$(1/x_1-\epsilon)^{\frac{1}{2}}=\beta-1 \quad (18)$$

and

$$y-x_1=\int_{x_1}^{x}y'dx \quad (19)$$

taken along the characteristic. For $\epsilon\geqq\frac{1}{2}$, $(W\geqq\frac{1}{2}I)$ integration of (19) gives β as the root of

$$\tan^{-1}\frac{2-\beta}{(\beta^2-2+2\epsilon)^{\frac{1}{2}}}+y\frac{(\beta^2-2+2\epsilon)^{\frac{1}{2}}}{2\beta}-\frac{(\beta^2-2+2\epsilon)^{\frac{1}{2}}[\beta(\beta-1)+\epsilon-1]}{\beta[(\beta-1)^2+\epsilon]}=\pm G\left(\left[\frac{2}{x(\beta^2-2+2\epsilon)}-1\right]^{\frac{1}{2}}\right), \quad (20)$$

where $G(\alpha)\equiv\tan^{-1}\alpha+\alpha/(1+\alpha^2)$. The value of $\beta=1$ gives the characteristic which separates regions of real and complex values of F_K. In the latter region this method of obtaining F_K is inapplicable. The real values of $F_K(\beta\geqq 1)$ are given by:

$$F_K(\epsilon, x, y)=\frac{1}{\epsilon^{\frac{1}{2}}}\tan^{-1}\frac{\beta-1}{\epsilon^{\frac{1}{2}}}+\frac{\beta(\beta-1)+\epsilon-1}{\beta[(\beta-1)^2+\epsilon]}-y\frac{\beta^2+\epsilon-1}{\beta}\pm x(\beta^2+2\epsilon-2)^{\frac{1}{2}}\left[\frac{2}{x(\beta^2-2+2\epsilon)}-1\right]^{\frac{1}{2}}, \quad (21)$$

where the value of β and the sign are the same as in (20). Similar expressions are obtained for $\epsilon<\frac{1}{2}$, $(W<\frac{1}{2}I)$, but are not given here, as we are not interested in such low bombarding energies.

The difference between F_{OP} of (8) and F_K of (16) is that the paths of integration differ as shown in Fig. 1, and in the former $1/\eta\geqq\epsilon$ occurs instead of ϵ in the first integrand. We are interested primarily in $F(\epsilon, x, y)$ as a function of x for a fixed $y=\rho$. The graphs of both F_{OP} and F_K are given for a special case in Fig. 2. It is seen that the two are quite similar.

To obtain the transmutation function we must know the value of $F(\epsilon, x, \rho)$ at its minimum at $x_0=Ip_0/Ze^2$. The dependence of x_0 on ϵ for $F_{OP}(\epsilon, x, \rho)$ and a given ρ is obtained from $x_0=2\eta_0-\rho$, where η_0 is determined by $\partial F_{OP}/\partial\eta=0$ in the manner described by Bethe.[5] For $\rho<0.172$ (i.e. for those heavy nuclei whose radii satisfy the condition $R<0.172Ze^2/I=0.116Z\times10^{-13}$ cm) the

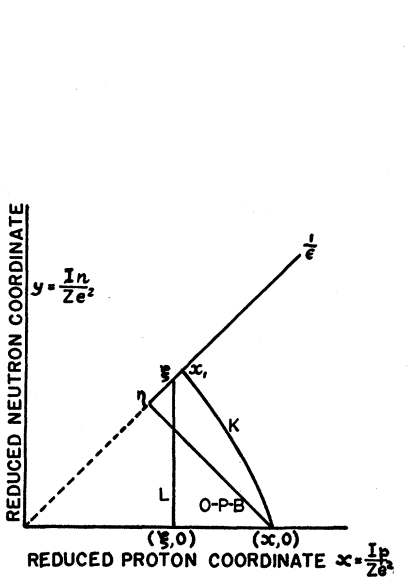

FIG. 1. Paths of integration for zero nuclear radius for the different approximations to the deuteron wave-function: O-P-B-Oppenheimer-Phillips-Bethe; K-Kapur; L-Lifshitz.

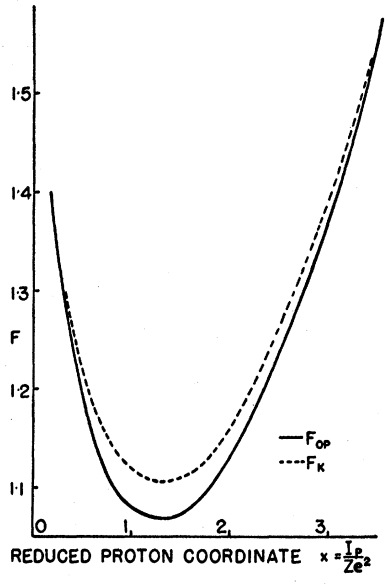

FIG. 2. Dependence of $F_{OP}(\epsilon, x, \rho)$ and $F_K(\epsilon, x, \rho)$ on the reduced proton coordinate $x=Ip/Ze^2$ for $\epsilon=0.5$, $\rho=0.17$.

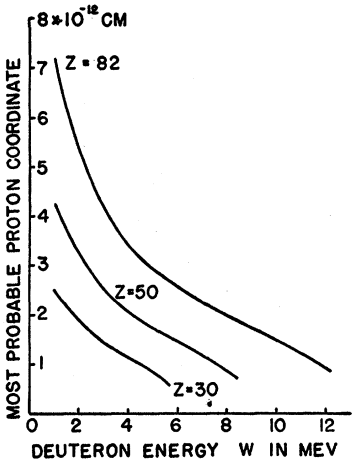

FIG. 3. Dependence on the deuteron energy W of the most probable proton coordinate $p_0=x_0\cdot Ze^2/I$ when the neutron is at the surface of the nucleus, for different values of the atomic number Z.

minimum of $F_{OP}(\epsilon, x, \rho)$ at certain energies will lie in a region of complex values of F_{OP}, but since only the real part of F is of interest, all imaginary terms are left out whenever they appear in (9), (10) or $\partial F/\partial\eta=0$. The minimum in $F_K(\epsilon, x, \rho)$ occurs at the point where a characteristic (a line of steepest descent of F_K) is perpendicular to the line $y=\rho$. From (17) we find

$$x_0=2/(\beta_0^2-2+2\epsilon), \tag{22}$$

so that β_0 is determined as a function of ϵ for a given ρ by (20) with the right side set equal to zero. The dependence of x_0 on ϵ for a given ρ is then obtained from (22), and the result is found to be practically the same as the one for the O-P-Bethe method. As in the O-P-Bethe method, for heavy nuclei ($R<0.191Ze^2/I=0.129Z\times10^{-13}$ cm, $\rho<0.191$) the minimum of F_K at certain energies lies in the region of complex values of F_K. Since this method does not give a solution in that region, neither the most probable proton coordinate nor the value of the transmutation function at those energies can be obtained by this method. The dependence on Z and W of the most probable proton coordinate $p_0=x_0Ze^2/I$ when the neutron is kept at the surface of the nucleus is shown in Fig. 3, and illustrates the extent of the polarization of the deuteron by the Coulomb field. p_0 is roughly given by Ze^2/W, ($x_0=1/\epsilon$). For a given Z the most probable proton coordinate decreases with increasing W until at $W_c=Ze^2/R-I=V_c-I$ it is found by both the O-P-Bethe and the Kapur method to be equal to the nuclear radius. W_c is thus the critical bombarding energy above which the O-P and the Gamow penetration functions coincide.

The transmutation functions of (5) and (6) are given by

$$\sigma(W)\sim|\varphi_W|^2=\exp\left[-\frac{4Ze^2}{\hbar}\left(\frac{M}{I}\right)^{\frac{1}{2}}F\right]$$
$$=\exp\left[-0.61ZF\right]. \tag{23}$$

Both $F_{OP}(\epsilon, x, \rho)$ and $F_K(\epsilon, x, \rho)$ reduce for $x=\rho$ (both proton and neutron at the surface of the nucleus) to the same value $F_G(\epsilon, \rho, \rho)=\epsilon^{-\frac{1}{2}}f(\epsilon\rho)$ which inserted in (23) gives the Gamow transmutation function. $F_G=0$ for $\epsilon=1/\rho$, ($W=Ze^2/R=V_c$), i.e., when the deuteron bombarding energy is equal to the top of the nuclear Coulomb barrier, and increases with decreasing bombarding energy. Above the critical

value $\epsilon_c=1/\rho-1$, $(W_c=Ze^2/R-I)$, F_{OP} and F_K coincide with F_G. As ϵ decreases and x_0 becomes larger than ρ, F_{OP} and F_K become smaller than F_G, making the O-P penetrability larger than the Gamow penetrability. In Fig. 4, graphs of F_G-F_{OP} and F_G-F_K are given for several ρ. It is seen that the O-P-Bethe and the Kapur approximations yield slightly different results, the deviation being most pronounced for $\rho=0$ (zero nuclear radius), and quite small for the other values of ρ considered. The dependence of the ratio of the O-P penetrability to the Gamow penetrability on the bombarding energy for different Z is calculated by means of

$$\gamma=|\varphi_W(p_0, R)/\varphi_W(R, R)|^2 =\exp[0.61Z(F_G-F_{OP})] \quad (24)$$

and is plotted on a logarithmic scale in Fig. 5. The quantity $1/\gamma$ is a measure of the probability of both the proton and the neutron reaching the surface of the nucleus compared to that of the neutron penetrating alone. Since the theory of Section I applies only when $1/\gamma$ is small, it is seen that the calculations of this paper become increasingly less reliable as the bombarding energy approaches the critical energy $W_c=Ze^2/R-I$.

The approximation for the higher partial waves of the deuteron beam would have in the second integrand in (8) in addition to the Coulomb potential $1/s(=Ze^2/rI)$ a centrifugal potential term L/s^2 with

$$L=\frac{l(l+1)}{4Z^2}\left(\frac{\hbar c}{e^2}\right)^2\frac{I}{Mc^2}=10.7\frac{l(l+1)}{Z^2}.$$

The ratio of the centrifugal to the Coulomb potential is given by $L/s=10.7l(l+1)/Z^2s$. The important values of s are $s\sim\eta_0=\frac{1}{2}(x_0+\rho)\sim 1/2\epsilon+\rho/2$. For these values $L/s\leqq 21.4l(l+1)W/Z^2I$, so that the effect of the centrifugal potential is unimportant for $l(l+1)<Z^2I/20W$. This gives $l<6$ for 2-Mev deuterons on zinc, and $l<11$ for 5-Mev deuterons on lead. Thus both the absolute value and the shape of the first few deuteron partial waves considered as functions of the proton coordinate with the neutron kept at the surface of the nucleus will not be strongly dependent on l. The fact that several partial waves contribute appreciably to the cross section leads us to expect a nonisotropic angular distribution for the outgoing protons. However, in the present paper we do not attempt to give a discussion of this question.

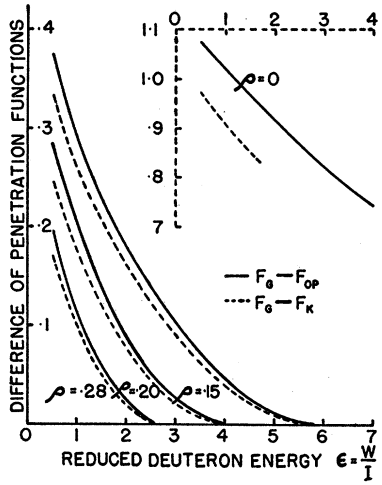

FIG. 4. Dependence on the reduced deuteron energy $\epsilon=W/I$ of $F_G(\epsilon, \rho, \rho)-F_{OP}(\epsilon, x_0, \rho)$ and $F_G(\epsilon, \rho, \rho)-F_K(\epsilon, x_0, \rho)$ for different values of the reduced nuclear radius $\rho=IR/Ze^2$.

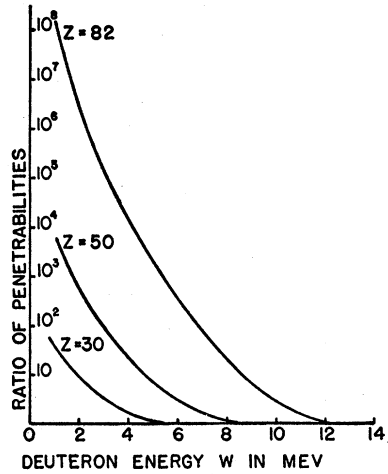

FIG. 5. Dependence on the deuteron energy W of the ratio of the Oppenheimer-Phillips to the Gamow penetrability for different values of the atomic number Z.

III. THE PROTON ENERGY DISTRIBUTION

The transfer factor $T(K)$ in the proton energy distribution as given by (3) involves the integral $I(K)=\int p^2dpS(p)f_K(p)$. The dependence of this integral on K may be estimated as follows. $S(p)$ has a maximum at $p=p_0\sim Ze^2/W$. If it were sharp like a δ-function, $I(K)\sim p_0^2f_K(p_0)$. As a

function of K this starts out flat near the origin, reaches a maximum roughly at $K\sim Ze^2/p_0$, and then oscillates with an amplitude falling off as $K^{-\frac{1}{2}}$ for large K. Since actually $S(p)$ has quite a broad maximum, it will not resolve the oscillations of $f_K(p)$ as they crowd in with increasing K, so that after a few oscillations of rapidly decreasing amplitude $I(K)$ will fall off monotonically. The position of the maximum and the general shape of $S(p)$ is about the same for the first few deuteron partial waves, and the proton partial waves differ little from one another except for the relative positions of their maxima and minima. It is therefore expected that for low proton partial waves the general dependence on K of integrals of the type $I(K)$ will be roughly the same as of the one here calculated, while for higher proton partial waves these integrals will fall off in absolute value because of the more rapid oscillation of $f_K(p)$ with p for higher l, K being chosen to fix the first maximum of $f_K(p)$ at $p\sim p_0$.

An analytic estimate of $I(K)$ is obtained as follows. For any particular Z and W, $S(p)$ can be fitted quite well by a curve of the type

$$S(p)=(p+a)^m e^{-\alpha p}=\left(-\frac{d}{d\alpha}+a\right)^m e^{-\alpha p} \tag{25}$$

with suitably chosen a, α, and an integral value of m. Using the contour integral representation for $f_K(p)$ given by Mott and Massey,[8] and interchanging in $I(K)$ the order of contour integration and integration over the proton coordinate we obtain:

$$I(K)\sim\{K^{\frac{1}{2}}[1-\exp(-2\pi\delta K^{-\frac{1}{2}})]\}^{-\frac{1}{2}}J(K) \tag{26}$$

with

$$J(K)\sim\left(-\frac{d}{d\gamma}+\zeta\right)^m\left[\frac{1+\gamma/\delta}{(K+\gamma^2)^2}\exp\{-2\delta K^{-\frac{1}{2}}\tan^{-1}(\gamma K^{-\frac{1}{2}})\}\right]$$

$$\sim\frac{g_{m+2}(\gamma,K)}{(K+\gamma^2)^{m+2}}\exp\{-2\delta K^{-\frac{1}{2}}\tan^{-1}(\gamma K^{-\frac{1}{2}})\}. \tag{27}$$

Here $g_{m+2}(\gamma, K)$ is a polynomial of the mth degree in K with a nonzero constant term. It is obtained from $g_2(\gamma, K)\equiv 1+\gamma/\delta$ by an m-fold application of the recursion relation

$$g_{s+1}(\gamma,K)=\left[\left(1+\frac{\zeta}{2\delta}K\right)+\frac{s}{\delta}\gamma+\frac{\zeta}{2\delta}\gamma^2\right]g_s-\frac{K+\gamma^2}{2\delta}\frac{dg_s}{d\gamma}. \tag{28}$$

If K is measured in Mev, then δ defined by $n=Ze^2/\hbar v=\delta/K^{\frac{1}{2}}$ is given by $\delta=0.158Z$. γ and ζ are determined in terms of α and a of (25) by

$$\gamma=1.08\alpha/\delta \qquad \zeta=\delta a/1.08, \tag{29}$$

if a and α are expressed in terms of Ze^2/I taken as the unit of length.

The dependence of the partial cross section on proton energy for a given W is then given by

$$\sigma(K)dK\sim\xi K^{\frac{1}{2}}|I(K)|^2dK=\xi T(K)dK,$$

$$T(K)=[1-\exp(-2\pi\delta K^{-\frac{1}{2}})]^{-1}\frac{\exp[-4\delta K^{-\frac{1}{2}}\tan^{-1}(\gamma K^{-\frac{1}{2}})]}{(K+\gamma^2)^{2m+4}}\cdot|g_{m+2}(K)|^2$$

$$\equiv[1-\exp(-2\pi\delta K^{-\frac{1}{2}})]^{-1}G(K)g^2(K). \tag{30}$$

$G(K)$ is the dominant factor in $T(K)$ giving a pronounced peak. The first factor is practically unity over the range of K where $G(K)$ is appreciable. $g^2(K)$ describes the oscillations mentioned at the beginning of this section. The calculation of $g_{m+2}(K)$ by means of (28) is quite laborious for high

values of m. However, numerical calculations for particular cases show that for the values of Z and W considered the first zero of $g_{m+2}(K)$ comes at a much higher value of K than the peak of $G(K)$, so that $g_{m+2}(K)$ is slowly varying over the important range of values of K, and the position and half-breadth of the maximum of the transfer factor are determined essentially by $G(K)$.

For any particular Z and W the dependence of $G(K)$ on K may be found by substituting in (30) the values of m and γ obtained by fitting a curve of the type (25) to $S(p)$ numerically calculated for that case by methods of Section II. A very rough, but helpful estimate of the dependence on Z and W of K_0, the position of the maximum of $G(K)$, and its half-breadth ΔK_0, may be obtained as follows. We first estimate the dependence on Z and W of m and γ. If we write

$$F_{\mathrm{OP}}(\epsilon, x, \rho)-F_{\mathrm{OP}}(\epsilon, x_0, \rho)=X[(x-x_0)/x_0-\log\,(x/x_0)],$$

we find that over the range of Z and W (i.e., of ρ and ϵ) of interest to us X is practically a constant ~ 0.6. x_0 was shown to be given roughly by $1/\epsilon$. So we can take

$$S(p)\equiv\exp\,[-0.3Z\{F(x)-F(x_0)\}]\sim[xe^{-\epsilon x}]^{0.18Z}$$

making $m=0.18Z$, $\gamma=1.08\times 0.18WZ/\delta I=0.57W$, $\zeta=0$. With these values $G(K)$ becomes

$$G(K)\sim\exp\,[-0.63ZK^{-\frac{1}{2}}\tan^{-1}\,(0.57WK^{-\frac{1}{2}})]/(K+0.325W^2)^{0.36Z+4}. \tag{31}$$

K_0 is determined by

$$G'(K_0)=0 \tag{32}$$

and ΔK_0 by

$$\Delta K_0=2[-G(K_0)/G''(K_0)]^{\frac{1}{2}}. \tag{33}$$

For W not too large we replace in (31) $\tan^{-1}\times(0.57WK^{-\frac{1}{2}})$ by its argument. Then (32) gives for K_0

$$K_0=W\left[\frac{1+\{1+1.3W(1+11.1/Z)\}^{\frac{1}{2}}}{2(1+11.1/Z)}\right],$$

which shows that $K_0\sim W$ for small W and large Z. Using $K_0\sim W$, and $G'(K_0)=0$, we obtain from (33)

$$\Delta K_0=2\left[\frac{0.72Z}{W^2}-\frac{0.36Z+4}{W^2(1+0.325W)^2}\right]^{-\frac{1}{2}}$$

$$=2WZ^{-\frac{1}{2}}\left[0.72-\frac{0.36+4/Z}{(1+0.325W)^2}\right]^{-\frac{1}{2}},$$

which shows that for small W and large Z, $\Delta K_0\sim 3.3WZ^{-\frac{1}{2}}$. These rough estimates give for

$Z=30$	$W=2$ Mev	$K_0\sim 2$ Mev	$\Delta K_0\sim 1.2$ Mev
$Z=82$	$W=1$ Mev	$K_0\sim 1$ Mev	$\Delta K_0\sim 0.36$ Mev.

More exact calculations using values of γ and m fitted to the above values of Z and W yield

$Z=30$	$W=2$ Mev	$K_0=1.9$ Mev	$\Delta K_0=1.4$ Mev
$Z=82$	$W=1$ Mev	$K_0=1.35$ Mev	$\Delta K_0=0.52$ Mev.

It should be emphasized again that the latter part of the above discussion of proton energies is only qualitative, and yields only the order of magnitude of K_0 and ΔK_0. However, it brings out the importance of the transfer factor in determining the proton energy distribution. Since $K_0\sim W$, the conservation of energy shows that for a proton to come off with this energy the neutron must be captured in a high lying state of the compound nucleus B with an excitation energy of $E_B\sim I_B-I_A-I\sim 6$ Mev above the ground level. For such states the "sticking probability" ξ is a valid way of describing the neutron-nucleus interaction and is expected to be roughly proportional to the level density in that region which, according to Weisskopf,[9] is given by $\omega(E_B)\sim\exp\,[2(E_B/a)^{\frac{1}{2}}]$ with a ~ 0.2 Mev. Since the half-breadth of the transfer factor is quite small, the sticking probability might vary by a factor of two or three over the breadth of the transfer factor, but the latter would still be the determining factor for the proton energy distribution.

We conclude with a reference to the work of Lifshitz[4] and Kapur.[6] Lifshitz sets out to evaluate the matrix element of the perturbing energy (really the scalar product of the initial and final wave-functions, as all non-exponential factors are left out) by using a semiclassical method (analogous to the W.K.B. approximation). In our present notation his expression for the transition probability

[9] V. Weisskopf, Phys. Rev. **52**, 295 (1937).

(using zero nuclear radius) is:

$$\sigma(\xi, W)\sim\exp\left[-\frac{4Ze^2}{\hbar}\left(\frac{M}{I}\right)^{\frac{1}{2}}F_{\mathrm{L}}(\epsilon, \xi)\right], \quad (34)$$

where

$$F_{\mathrm{L}}(\epsilon, \xi)=\frac{1}{2}\int_0^{\xi}(-2E_n)^{\frac{1}{2}}dy + \frac{1}{2}\int_{1/E_p}^{\xi}\left[2\left(\frac{1}{x'}-E_p\right)\right]^{\frac{1}{2}}dx' + \int_{\xi}^{1/\epsilon}\left(\frac{1}{s}-\epsilon\right)^{\frac{1}{2}}ds. \quad (35)$$

ξ is named by him the coordinate of the "transition point." E_n and E_p are interpreted by him as the energies of the neutron and the proton at "the instant of break-up" of the deuteron at the "transition point," the interpretation being based on a *classical* application of the energy and momentum conservation laws, and the use of the *classical* concept of the action function in a region *classically inaccessible* to the particles. E_n and E_p are given in terms of ϵ and ξ by

$$E_n=-\tfrac{1}{2}[1+(1/\xi-\epsilon)^{\frac{1}{2}}]^2$$
$$E_p=\epsilon-1+\tfrac{1}{2}[1+(1/\xi-\epsilon)^{\frac{1}{2}}]^2. \quad (36)$$

Since $1/\xi$ is connected by (36) with E_p Lifshitz interprets the variation of $\sigma(\xi, W)$ with ξ as describing the dependence of the cross section on the outgoing proton energy, and takes the transmutation function to be determined by $\sigma(\xi_0, W)$ where ξ_0 is the value of ξ which makes $\sigma(\xi, W)$ a maximum.

We compare $F_{\mathrm{L}}(\epsilon, \xi)$ of (35) with $F_{\mathrm{K}}(\epsilon, x, y)$ of (16) with $y=0$. The last integrals are of the same form. The first integral in (16) is taken along the characteristic through the point $(x, 0)$, and may be written as

$$\frac{1}{2}\int_{x,0}^{x_1,x_1}[2(1/x-E_n-E_p)]^{\frac{1}{2}}dt.$$

In (35) it is replaced by an integral along the x axis from a point $(x=1/E_p, 0)$ to $(\xi, 0)$, plus another integral along a vertical line from $(\xi, 0)$ to (ξ, ξ), as shown in Fig. 2. Thus formally F_{OP}, F_{K}, and F_{L} are very much alike. The paths of integration differ somewhat, the points at which the path of integration joins the line $x=y$ differ, but are in each case uniquely determined by the point $(x, 0)$ and the integrands along a part of the path differ, but the general dependence of any of these three functions on x is the same.

F_{K} is given explicitly by (20) and (21). The corresponding results for F_{L} are

$$F_{\mathrm{L}}(\epsilon, x, 0)=\frac{1}{\epsilon^{\frac{1}{2}}}\tan^{-1}\frac{\delta-1}{\epsilon^{\frac{1}{2}}} + \frac{1}{(\delta^2-2+2\epsilon)^{\frac{1}{2}}}\tan^{-1}\frac{2-\delta}{(\delta^2-2+2\epsilon)^{\frac{1}{2}}}, \quad (37)$$

where δ is given in terms of ϵ and x by

$$2/x=\delta^2-2+2\epsilon. \quad (38)$$

ξ is given by

$$(1/\xi-\epsilon)^{\frac{1}{2}}=\delta-1 \quad \text{(similar to (18))}. \quad (39)$$

The δ determined by (38) is in general different from the β determined by (20), so that ξ is different from x_1, and F_{L} is different from F_{K}. However, differentiation of F_{L} shows that its minimum is given by the same equation for δ_0 as is satisfied by β_0, i.e. by (20) with the right side equal to zero and $y=0$, and in that case it so happens that the positions of the minima, the values of F_{L} and F_{K} at that point, and the points ξ and x_1 fortuitously coincide, thus making Lifshitz' result for the transmutation function identical with Kapur's.

In view of the above it seems that Lifshitz' expression for the cross section (34) should be interpreted not as the square of the scalar product of the deuteron and proton wave functions given as a function of the outgoing proton energy, but rather as another approximation (analogous to the O-P-Bethe and Kapur approximations) to the square of the deuteron wave function considered as a function of the proton coordinate p with the neutron kept at the surface of the nucleus. Considered as a function of $E_p=Ze^2/p$ instead of p, (34) gives a measure of the probability of the deuteron penetrating into the classically inaccessible region and being so polarized by the Coulomb field that the proton is found at a point p where its Coulomb energy is given by $E_p=1/p$ while the neutron penetrates to the surface of the nucleus. The energy distribution of the outgoing protons, however, is determined not only by the energy gained by the proton from the Coulomb field, but also by the transfer of the kinetic energy to the proton when its connection with the neutron is severed. The two effects are taken into account in the transfer factor discussed above.

Kapur[6] interprets the process in a very classical way. Assuming the path of integration corresponding to the point (x_0, ρ) to be the *actual* path taken by the deuteron after penetrating the potential barrier he interprets the point (x_1, x_1) as the point at which the deuteron breaks up (analogous to Lifshitz' interpretation of the point ξ), and assumes the proton always to come out with an energy equal to its Coulomb energy at that point uniquely determined in terms of ρ and x_0. This interpretation is quite inconsistent, since it is based on describing the motion of the particles in terms of a well-defined classical trajectory in a region classically inaccessible to them.

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