theory as shown in Fig. 2. Fig. 3 gives the shape of the diffraction maxima. The increasing diffuseness with $|\Delta|$ is well illustrated; but the numerical values for the half-widths are—as Fig. 4 shows—smaller than the theory demands. This discrepancy may be real, but more likely is due to the limited accuracy of our photographic measurements.

More accurate and more extensive measurements, including observations at different temperatures and with different crystals, are of course desirable and are already under way. It may be mentioned that we have observed the new diffraction maxima with calcite crystals and that the maxima show the same general behavior as Δ is varied. Indeed, the theory of diffuse scattering should apply also to perfect crystals since the coupling interaction between incident and scattered radiation is weak when no Bragg scattering takes place.

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The Equations of Motion in Electrodynamics

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A formulation of the equations of motion of singularities in classical electromagnetic theory is obtained. The general method introduced by Einstein, Infeld and Hoffmann leads in a simple way, without any difficulty with "infinities," to the equations of motion obtained before by Dirac. It is shown further that Dirac's introduction of the inertial term into the equations of motion correspond in this method to the assumption of an energy-momentum tensor of matter. The attempt to remove this arbitrary assumption leads in a simple and natural way to the general theory of relativity, in which the equations of motion are obtained from the Einstein-Maxwell field equations.

1. INTRODUCTION

IN Maxwell's theory the motion of charged particles represented as singularities of the field is not determined by the field equations. Dirac[1] has shown that the equations of motion are suggested by the conservation law for the electromagnetic energy-momentum tensor. By virtue of this law, it is demonstrated that the integral representing the flux of energy and momentum over a thin tube surrounding the world-line of a singularity (electron) depends only on the conditions at the ends of the tube. Then the equations of motion are obtained by assuming a simple expression for this flux. However, the integral takes an infinite value as the tube shrinks to the world-line, and Dirac is compelled to remove this difficulty artificially by equating the flux to an expression containing an infinite term and a finite term representing the product of mass and acceleration. This, and the formal complications of the paper, constitute its weak points.

The above procedure leads to familiar equations for the motion of electrons, but whereas these equations were formerly considered to be approximate, Dirac concludes that there is good reason to assume them exact.

The general method of obtaining equations of motion in the theory of relativity introduced by Einstein, Infeld and Hoffmann[2] appears at first sight fundamentally different from that of Dirac. It is shown here, however, that the former method can be adapted to the problem of motion in an electromagnetic field. The method involves only two-dimensional and not three-dimensional surface integrals, avoids the difficulties of "infinities," and leads in a simple manner, without the use of δ-functions, to the results obtained by Dirac. But we believe that the advantage of the

[1] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).

[2] Einstein, Infeld and Hoffmann, Ann. Math. **39**, 65 (1938).

method used in this paper lies not only in the simplification of the derivation of the equations of motion. It gives also a natural transition to the problem of motion of charged particles in the general theory of relativity.

We have tried to formulate the argument in such a way that it may be understood without a knowledge of the quoted papers.

2. The Surface Integrals and the Equations of Motion

Throughout this section we shall use an essentially 3-dimensional notation, Latin letters running from 1 to 3, and repetition of an index implying summation over this range. The notation ", n" and ", o" will be used to denote ordinary derivatives with respect to the coordinates x^n and time, respectively. The velocity of light is taken as unity throughout.

We write down the conservation laws for the electromagnetic energy-momentum tensor:

$$T_{mn,\,n}=T_{mo,\,o}, \tag{2.1}$$

$$T_{on,\,n}=T_{oo,\,o}, \tag{2.2}$$

where

$$T_{mn}=\text{Maxwell stress tensor}=F_{ms}F_{ns}-F_{mo}F_{no}-\tfrac{1}{4}\delta_{mn}F_{rs}F_{rs}+\tfrac{1}{2}\delta_{mn}F_{so}F_{so}, \tag{2.3}$$

$$T_{on}=\text{Poynting vector}=F_{so}F_{sn}, \tag{2.4}$$

$$T_{oo}=\text{Energy density}=\tfrac{1}{2}F_{so}F_{so}+\tfrac{1}{2}F_{rs}F_{rs}, \tag{2.5}$$

$$\left(\delta_{mn}=\begin{cases}1 \text{ if } m=n\\ 0 \text{ if } m\neq n\end{cases}\right).$$

$F_{so}=E_s$ and $F_{mn}=\epsilon_{mns}H_s$ where E_s and H_s are the electric and magnetic field, respectively, and ϵ_{mns} is the "permutation symbol" defined in the following manner: $\epsilon_{mns}=0$ when two of m, n, s are the same, $=1$ when m, n, s form an even permutation of 1, 2, 3, $=-1$ when m, n, s form an odd permutation of 1, 2, 3.

Equations (2.1) and (2.2) break down only at the point occupied by the singularity, which we assume to be a simple electric pole. Let us surround the singularity with a small sphere, and accept the validity of the equations everywhere outside. Furthermore, let us choose a Lorentz frame of reference in which the singularity is instantaneously at rest at the origin at some moment t. All our calculations will refer to this moment t and to this coordinate system. It is because of this special choice of coordinate system that it is convenient to use throughout the three-dimensional notation.

We introduce now a vector ψ_m and a scalar ψ_o satisfying the equations

$$\Delta\psi_m=-T_{mo,\,o}, \tag{2.6}$$

$$\Delta\psi_o=-T_{oo,\,o}, \tag{2.7}$$

where Δ is the Laplacian operator. Both ψ_m and ψ_o will always exist outside the small sphere if we assume the electromagnetic field to be known, and the T's to be calculated from it. We can therefore write (2.1) and (2.2) in the form

$$(T_{mn}+\psi_{m,\,n})_{,\,n}=0, \tag{2.8}$$

$$(T_{on}+\psi_{o,\,n})_{,\,n}=0. \tag{2.9}$$

Assuming that the energy-momentum tensor vanishes rapidly enough at infinity, we obtain from Green's theorem, by virtue of Eqs. (2.8) and (2.9), that the four surface integrals

$$\int(T_{mn}+\psi_{m,\,n})\lambda^n dS, \tag{2.10}$$

$$\int(T_{on}+\psi_{o,\,n})\lambda^n dS, \tag{2.11}$$

(where λ^n are the direction-cosines of the normal to the surface of integration), are independent of the shape and size of the surface chosen, and hence can depend only on quantities characterizing the singularity, in particular the coordinates of the singularity and their time-derivatives. We see from (2.6) and (2.7) that ψ_m and ψ_o are both arbitrary to within a harmonic function. The only solid harmonic function which can give a contribution to the surface integrals is one of the form $f(t)/r$ where $r^2=x^sx^s$. If we take as the surface of integration a sphere with center at the singularity (the origin) the contribution to the surface integral of a term of this type is $-4\pi f(t)$, i.e., an arbitrary function of t. For

$$(f(t)/r)_{,\,n}=-(x^n/r^3)f(t),\ \lambda^n=x^n/r, \text{ and } dS=r^2d\omega,$$

where $d\omega$ is the element of solid angle subtended at the origin. Hence

$$\int(f(t)/r)_{,\,n}\lambda^n dS=-\int f(t)d\omega=-4\pi f(t).$$

It follows, then, that we can give the integrals (2.10) and (2.11) arbitrary values by choosing appropriately the functions $f(t)$.

To assign consistent values to the four integrals means to determine the motion of the singularity, because the integrals depend only on the coordinates of the singularity and their derivatives. We proceed in the following manner: we assume that the integrals are equal to zero—

$$\int (T_{mn}+\psi_{m,n})\lambda^n dS=0, \tag{2.12}$$

$$\int (T_{on}+\psi_{o,n})\lambda^n dS=0, \tag{2.13}$$

and shift the task of determining the motion to the choice of the arbitrary harmonic functions. We could equally well assume any value for the integral, making a corresponding change in the arbitrary harmonic function. Therefore our assumptions (2.12) and (2.13) do not imply any loss of generality.

The apparent anomaly of having four equations where three would appear to determine the motion completely may be met by choosing the arbitrary function in the last equation in such a way that no new restriction is placed on the motion.

We will write

$$\psi_m=\Psi_m-\bar{\psi}_m, \tag{2.14}$$

$$\psi_o=\Psi_o-\bar{\psi}_o, \tag{2.15}$$

where $\bar{\psi}_m$, $\bar{\psi}_o$ are the arbitrary harmonic functions, and Ψ_m, Ψ_o are the solutions of the Poisson equation which do not contain in their development with respect to r an harmonic function of the type $1/r$.

We let the coordinates of the singularity be $\eta^r(t)$, where $\eta^r(t)=0$, $\dot{\eta}^r(t)=0$ in our special coordinate system, at the instant to which our calculations refer. (Dots indicate derivatives with respect to time.) We assume that the field of the singularity is represented by the advanced or retarded potential, and that the external field is superimposed upon this.

We are now prepared to determine the equations of motion by choosing the arbitrary function $\bar{\psi}_m$. Since $\eta^r(t)=0$ and $\dot{\eta}^r(t)=0$ the simplest possible assumption aside from the trivial one which omits this term altogether is

$$\bar{\psi}_m=m_o(\ddot{\eta}^m/r);$$

with this choice the equations of motion (2.12) become

$$m_o\ddot{\eta}^m=-\frac{1}{4\pi}\int (T_{mn}+\Psi_{m,n})\lambda^n dS, \tag{2.16}$$

where the integral on the right-hand side is taken over an arbitrary surface surrounding the singularity.

The form taken by the equations of motion will depend upon the actual field from which T_{mn} is calculated. The usual procedure is to divide the field into two parts, the "external" field (which Dirac calls the "incoming" field) and the field of the singularity, taken as a retarded potential. This procedure is, however, arbitrary. From the formal point of view the solution corresponding to $\frac{1}{2}$ (retarded+advanced) potential, which is known as the "standing wave" solution and which does not specify a privileged direction for the flow of time, is the simplest which can be gained from the use of the "new approximation method."[3] It was this solution which was discussed at the time of Bohr's theory, since it represented an orbital motion without radiation. The addition of radiation seems from this point of view arbitrary, being obtained by superimposing upon the "standing field" a field equal to $\frac{1}{2}$ (retarded-advanced) potential.

The complete calculation of the equations of motion is carried out in Appendix I. If we use the "standing wave" solution for the field of the electron we obtain from (2.16) the equations

$$m_o\ddot{\eta}^m=e_{\text{ext}}\bar{E}_m, \tag{2.17}$$

where ${}_{\text{ext}}\bar{E}_m$ is evaluated at $x^s=0$ and at the moment t.

With regard to the fourth equation (2.13), the contribution from the field is identically zero (Appendix I), and hence the added harmonic function $\bar{\psi}_o$ must be taken equal to zero, since we have laid down the condition that this equation should place no new restriction upon the motion.

Hence the equations of motion formulated by (2.12) and (2.13) and by the choice of the harmonic functions $\bar{\psi}_m=m_o(\ddot{\eta}^m/r)$ and $\bar{\psi}_o=0$

[3] L. Infeld, Phys. Rev. 53, 836 (1938).

leads through the "standing wave" solution to (2.17) for the special coordinate system in which the equations are formulated.

In the case where we take the retarded potential and leave the harmonic functions as before (2.16) leads to the equations

$$m_o\dot{u}^m = e_{\text{ext}}\bar{E}_m + \tfrac{2}{3}e^2\ddot{u}^m, \tag{2.18}$$

where $u^m = \dot{\eta}^m$; and (2.13) again imposes no new restriction upon the motion.

We wish to remove the restriction imposed by the choice of a particular coordinate-system. This will involve a change to four-dimensional notation. We will consider a four-space with coordinates x^μ. Greek letters will be assumed to take the values 0, 1, 2, 3 where x^o is the time-coordinate and x^1, x^2, x^3 are the space-coordinates, as before. We will assume that the four-space has signature $+ - - -$. The vector v^μ will be defined as the four-dimensional velocity-vector of the electron. Accents will indicate differentiation with respect to arc-length in space-time. Then it may be shown that in the "standing wave" case the equations of motion may be written

$$m_o v'^\mu = e_{\text{ext}}\widetilde{F}_\nu{}^\mu v^\nu \tag{2.19}$$

and in the case in which the retarded-potential solution is chosen, they become

$$m_o v'^\mu = e_{\text{ext}}\widetilde{F}_\nu{}^\mu v^\nu + \tfrac{2}{3}e^2 v''^\mu + \tfrac{2}{3}e^2 v'^2 v^\mu, \tag{2.20}$$

where ${}_{\text{ext}}\widetilde{F}_\nu{}^\mu$ is evaluated at the world-point of the singularity. We can prove this by verifying that these equations reduce to (2.17) and (2.18), respectively, in the particular coordinate system chosen.

In this special coordinate-system in which our calculations have been carried out, since $\dot{\eta}^r = 0$, we have the following values for the components of v^μ and their derivatives:

$$\begin{matrix} v^o = 1, & v'^o = 0, & v''^o = \dot{u}^r\dot{u}^r, \\ v^m = 0, & v'^m = \dot{u}^m, & v''^m = \ddot{u}^m. \end{matrix} \tag{2.21}$$

Hence

$$v'^2 = v'^o v'^o - v'^m v'^m = -\dot{u}^m\dot{u}^m. \tag{2.22}$$

Now Eqs. (2.19) can be written

$$m_o v'^o = e_{\text{ext}}\widetilde{F}_\nu{}^o v^\nu, \quad m_o v'^m = e_{\text{ext}}\widetilde{F}_\nu{}^m v^\nu.$$

In our special coordinate-system the first of these is an identity. The second reduces to

$$m_o\dot{u}^m = e_{\text{ext}}\widetilde{F}_o{}^m = e_{\text{ext}}\bar{E}_m,$$

which is precisely (2.17).

Consider the additional terms $(\frac{2}{3}e^2v''^\mu + \frac{2}{3}e^2v'^2v^\mu)$ which occur in (2.20). From (2.21) and (2.22) we have:

$$\tfrac{2}{3}e^2v''^o + \tfrac{2}{3}e^2v'^2v^o = \tfrac{2}{3}e^2\dot{u}^r\dot{u}^r - \tfrac{2}{3}e^2\dot{u}^r\dot{u}^r = 0$$

and

$$\tfrac{2}{3}e^2v''^m + \tfrac{2}{3}e^2v'^2v^m = \tfrac{2}{3}e^2\ddot{u}^m;$$

therefore (2.20) is likewise an identity when $\mu = 0$ and it reduces to (2.18) when $\mu = m$.

Hence Eqs. (2.19) and (2.20) are valid in one coordinate-system. Since they are vector equations, it follows that they are valid in all Lorentz coordinate-systems, and therefore give the general form of the equations of motion.

Dirac assumes that (2.20) and not (2.19) represent the rigorous equations of motion. The present discussion tends to show that any choice between them is from the formal point of view arbitrary.

3. The Energy-Tensor of Matter

In order to obtain the equations of motion we have made three fundamental assumptions: (1) A special choice for the arbitrary additive harmonic functions in (2.12) and (2.13). (2) A division of the field into "external" or "incoming" field, and the field of the electron. (3) A special choice for the field of the electron ("standing" or "retarded").

The second and third assumptions cannot be removed, since they are characteristic of the whole structure of electrodynamics. This is not true, however, of the first assumption. We ask how we may generalize our scheme of equations to rid ourselves of this assumption, and obtain a logically simpler though formally more complicated scheme.

We may show first that the assumption (1) corresponds to the choice of an energy-momentum tensor for matter. From Eq. (2.12) we had

$$\int T_{mn}\lambda^n dS + \int \psi_{m,\,n}\lambda^n dS = 0,$$

where $\psi_m = \Psi_m - \bar{\psi}_m$, $\bar{\psi}_m$ being a harmonic function.

If we define a "mechanical" energy-momentum tensor

$$M_{mn} = -\bar{\psi}_{m,n} - \bar{\psi}_{n,m} + \delta_{mn}\bar{\psi}_{s,s}, \qquad (3.1)$$

where $\bar{\psi}_m = m_0\ddot{\eta}^m/r$, then M_{mn} is symmetric in m and n and satisfies the condition

$$M_{mn,n} = 0. \qquad (3.2)$$

Then $\int M_{mn}\lambda^n dS$ taken over any surface enclosing the singularity is independent of the size or shape of the surface. If a sphere with center at the singularity is again used, it follows immediately that the contribution to the integral of the last two terms of M_{mn} vanishes identically. The first term yields $\int \bar{\psi}_{m,n}\lambda^n dS$, which has the value $-4\pi m_0\ddot{\eta}^m$. Hence (2.12) can now be written

$$\int (E_{mn} + M_{mn})\lambda^n dS = 0, \qquad (3.3)$$

E_{mn} being $T_{mn} + \Psi_{m,n} + \Psi_{n,m} - \delta_{mn}\Psi_{s,s}$. Furthermore (2.13) may be put into the form

$$\int (E_{on} + M_{on})\lambda^n dS = 0 \qquad (3.4)$$

if we take $M_{on} = 0$, $T_{on} + \Psi_{o,n} = E_{on}$ in the particular coordinate-system employed.

We now prove a lemma:

If we have a set of functions $B_{ab\cdots kl}$, skew-symmetric in the indices k and l, the surface integral

$$\int_{(S)} B_{ab\cdots kl,l}\lambda^k dS$$

taken over an arbitrary closed surface S which may enclose but may not pass through a singularity of the field, vanishes identically.

For let us set

$$B_{ab\cdots 23} = C_1, \quad B_{ab\cdots 31} = C_2, \quad B_{ab\cdots 12} = C_3.$$

Then we can write the integral

$$\int_{(S)} \mathrm{curl}_n\, \mathbf{C} dS,$$

which may be transformed by Stokes' theorem into a line integral around the rim of the surface; but since the surface is closed this rim must have zero length, so that the integral vanishes identically.

By virtue of the lemma, we deduce that if the T's satisfy relations of the form

$$E_{mn} + M_{mn} = K_{mnl,l}, \qquad (3.5)$$

$$E_{on} + M_{on} = K_{onl,l}, \qquad (3.6)$$

where K_{mnl}, K_{onl} are skew-symmetric in the indices n and l, then the equations of motion have the form (3.3) and (3.4). Such relations are therefore sufficient to ensure that the equations of motion have this form.[4]

4. The Equations of Motion of a Charged Particle in General Relativity

The difficulty which still remains is the arbitrariness of the choice of the tensor M_{mn}. We shall show now how this may be removed by an appeal to the general relativity theory.

Let us attempt to derive equations of motion from field equations by a method similar to that used by Einstein and Infeld.[5] We start from the Einstein-Maxwell field equations

$$G_{\mu\nu} + kT_{\mu\nu} = 0, \qquad (4.1)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, $R_{\mu\nu}$ being the Ricci tensor and $T_{\mu\nu}$ the electromagnetic energy-momentum tensor. We shall take $k=1$, as may obviously be done if we choose appropriately the units of mass and charge.

If Eq. (4.1) above be contracted, since $T=0$ it follows that $R=0$, and hence the field equations may be written

$$R_{\mu\nu} + T_{\mu\nu} = 0. \qquad (4.2)$$

We shall write the fundamental metric tensor of general relativity in the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad (4.3)$$

where

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We shall make the following assumptions: (i) We neglect nonlinear gravitational terms in $R_{\mu\nu}$. (ii) We neglect gravitational-electromagnetic interactions. The second assumption is equivalent to treating the term $T_{\mu\nu}$ as though the metric tensor of space-time were $\eta_{\mu\nu}$.

[4] Cf. M. H. L. Pryce, Proc. Roy. Soc. **A168**, 302 (1938).

[5] A. Einstein and L. Infeld, Ann. Math. (In print.)

Now Eq. (4.2) are six independent equations in the ten unknowns $h_{\mu\nu}$. Hence we may add to these equations four nontensorial "coordinate conditions" on the $h_{\mu\nu}$. We introduce the quantities

$$\gamma_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}. \tag{4.4}$$

The coordinate conditions which will be imposed are

$$\eta^{\nu\rho}\gamma_{\mu\nu,\rho}=0. \tag{4.5}$$

With the aid of the assumption (i) and the coordinate conditions, we derive the relation[6]

$$R_{\mu\nu}=\tfrac{1}{2}\Box\gamma_{\mu\nu}, \tag{4.6}$$

where $\Box$ is the d'Alembertian operator

$$\eta^{\rho\sigma}(\partial^2/\partial x^\rho\partial x^\sigma).$$

Hence the field equations become

$$-\Box\gamma_{\mu\nu}=2T_{\mu\nu}. \tag{4.7}$$

We may write these as follows:

$$\begin{aligned}&(a) && \gamma_{mn,ss}-\gamma_{mn,oo}=2T_{mn},\\ &(b) && \gamma_{on,ss}-\gamma_{on,oo}=2T_{on},\\ &(c) && \gamma_{oo,ss}-\gamma_{oo,oo}=2T_{oo},\end{aligned} \tag{4.8}$$

and the coordinate conditions (4.5) may be written:

$$\begin{aligned}&(a) && \gamma_{mn,n}-\gamma_{mo,o}=0,\\ &(b) && \gamma_{on,n}-\gamma_{oo,o}=0.\end{aligned} \tag{4.9}$$

Now

$$(\gamma_{mn,s}-\gamma_{ms,n})_{,s}=\gamma_{mn,ss}-\gamma_{ms,ns}=\gamma_{mn,ss}-\gamma_{mo,no}$$

and

$$(\gamma_{on,s}-\gamma_{os,n})_{,s}=\gamma_{on,ss}-\gamma_{os,ns}=\gamma_{on,ss}-\gamma_{oo,no},$$

by virtue of the coordinate-conditions (4.9) (a), (4.9) (b), respectively. Hence (4.8) (a) and (4.8) (b) may be written

$$\begin{aligned}&(a) && K_{mns,s}=(\gamma_{mn,s}-\gamma_{ms,n})_{,s}\\ &&&\quad=2T_{mn}+\gamma_{mn,oo}-\gamma_{mo,no}.\\ &(b) && K_{ons,s}=(\gamma_{on,s}-\gamma_{os,n})_{,s}\\ &&&\quad=2T_{on}+\gamma_{on,oo}-\gamma_{oo,no}.\end{aligned} \tag{4.10}$$

[6] A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge University Press, second edition, 1930), pp. 128–129, (57.4) and (57.5).

As a consequence of the lemma of §2 we have, since K_{mns}, K_{ons} are skew-symmetric in the indices n and s,

$$\begin{aligned}&(a) && \int K_{mns,s}\lambda^n dS=0,\\ &(b) && \int K_{ons,s}\lambda^n dS=0,\end{aligned} \tag{4.11}$$

the integrals being over surfaces surrounding the singularity of the field at $\eta^r(x^o)$. Because of the field-equations (4.10) the equations (4.11) lead to

$$\begin{aligned}&(a) && \int(\gamma_{mn,oo}-\gamma_{mo,no}+2T_{mn})\lambda^n dS=0,\\ &(b) && \int(\gamma_{on,oo}-\gamma_{oo,no}+2T_{on})\lambda^n dS=0,\end{aligned} \tag{4.12}$$

and we call Eqs. (4.12) the "equations of motion" of the electron. For, as in §3, the integrals are independent of the shape and size of the surface of integration, and so can depend only on quantities characterizing the singularity, in particular the coordinates of the singularity and their time derivatives.

These equations of motion are obtained by virtue of *both* the field equations and the coordinate conditions. In fact, the equations of motion (4.12) are conditions which ensure the consistency of Eqs. (4.8) and (4.9). Or, to formulate it differently, (4.8) and (4.9) can be satisfied only by virtue of the equations of motion. This becomes evident if we attempt to solve the equations completely by means of the "new approximation method."[7]

Thus we see that we are led to "equations of motion" in general relativity theory without any such assumptions as the first one in §3. This is because the general theory of relativity has provided us with an energy tensor of matter, derived from the Ricci tensor $R_{\mu\nu}$.

We may write the solutions of the field equations (4.7) in the form

$$\gamma_{\mu\nu}=\Gamma_{\mu\nu}+\bar{\gamma}_{\mu\nu}, \tag{4.13}$$

where $\Gamma_{\mu\nu}$ are particular solutions of Eqs. (4.8) obtained by a process of direct calculation and

[7] This has been done by Mr. P. R. Wallace in a thesis at present in preparation.

$\bar{\gamma}_{\mu\nu}$ are solutions of the corresponding homogeneous set:

$$\square\bar{\gamma}_{\mu\nu}=0. \quad (4.14)$$

The quantities $\Gamma_{\mu\nu}$ are purely electromagnetic in nature. On the other hand, the $\bar{\gamma}_{\mu\nu}$ may be solved by retarded (or "standing") potentials of the same form as those given for the electromagnetic potentials. In choosing the solution we are guided by the consideration that when the particle is not charged, i.e., when $e=0$, the field must reduce, in the first approximation, to the Newtonian gravitational field. Therefore the $\bar{\gamma}_{\mu\nu}$ represent the gravitational field of a particle of mass m and zero charge.

Neither $\Gamma_{\mu\nu}$ nor $\bar{\gamma}_{\mu\nu}$ will be chosen to satisfy the coordinate conditions separately, but solutions may be obtained such that their sum satisfies them by virtue of the equations of motion.

Because of our simplifying assumptions, involving neglect of nonlinear gravitational terms and electromagnetic-gravitational interactions, we have been able to obtain a "splitting" of the field into gravitational and electromagnetic parts. With these assumptions, the problem demonstrates a formal similarity to that of §3. The field equations (4.10) may be written, as in (3.5) and (3.6), in the form

$$E_{mn}+M_{mn}=K_{mns,\,s}, \quad E_{on}+M_{on}=K_{ons,\,s},$$

where K_{mns}, K_{ons} are skew-symmetric in n and s. The $E_{\mu\nu}$ may be considered as arising from the $T_{\mu\nu}$ and the $\Gamma_{\mu\nu}$, and the $M_{\mu\nu}$ as coming from the $\bar{\gamma}_{\mu\nu}$. Then the equations of motion (4.12) take the form of (3.3) and (3.4). But if the problem had been attempted in its most general form, the energy-momentum tensor would have appeared in a form involving electromagnetic-gravitational interactions, and could not have been split in the above manner.

Thus the theory of relativity provides a scheme which includes both the gravitational and electromagnetic fields, whereas before the gravitational field was introduced merely through an arbitrary harmonic function and defined only in a special coordinate system.

The coordinate conditions (4.5) are invariant under a Lorentz transformation, from which it follows that we may, as before, choose a coordinate-system in which the electron is instantaneously at rest at some time t, and carry out the calculation of the surface integrals with reference to this particular coordinate-system and this particular time. The removal of this restriction, once the integrals have been calculated, will be carried out precisely as in §2.

The calculation of the equations of motion from Eqs. (4.12) involves the computation of three types of integrals: (i) those arising from $\Gamma_{\mu\nu}$, (ii) those arising from the $\bar{\gamma}_{\mu\nu}$, (iii) the integrals of T_{mn}, T_{on}. The latter have already been calculated in Appendix I. In Appendix II it is shown that the integrals of type (i) are all zero, while from those of type (ii) there is a contribution of $m_o\ddot{\eta}^m$ to the equation of motion (4.12) (a), and no contribution to (4.12) (b). It follows that the equations of motion of the electron, derived from (4.12) (a), are exactly the same as those derived in §2, and that, as required, (4.12) (b) is an identity, and places no new restriction upon the motion.

We have therefore shown that the equations of motion of an electron in an external field, as deduced under stated assumptions from the general theory of relativity, may have the form either of (2.19) or of (2.20).

The whole problem has concerned the development of the equations of a single electron. The method is easily generalized to deal with the problem of n electrons in an external field. In determining the equations of motion of one of the electrons, one has merely to add to the external field the fields of the other electrons.

Appendix I

All the integrals in these Appendices are evaluated over a sphere with center at the singularity. Only expressions of the order $1/r^2$ in the integrands can give contributions to the integrals, since the integrals of other expressions would necessarily depend on the size of the sphere of integration and must therefore vanish. All integrations then correspond to the spherical surface of integration and to the terms of order $1/r^2$ in the integrands. The final result refers, of course, to an arbitrary surface.

We wish (i) to evaluate $(1/4\pi)\int T_{mn}\lambda^n dS$ and $(1/4\pi)\int T_{on}\lambda^n dS$, and (ii) to show that $\int \Psi_{m,n}\lambda^n dS$ and $\int \Psi_{o,n}\lambda^n dS$ are both zero. For the former, it is necessary to calculate the terms of order $1/r^2$ in T_{mn} and T_{mo}, omitting those which contain u^r which is zero. For the latter, we have to calculate the terms of order $1/r^3$ in T_{mo} and T_{oo}, omitting those which contain u^r more than once. (This may be seen from an examination of (2.6) and (2.7).) Now

$$F_{mn}=\gamma_{m,n}-\gamma_{n,m}+{}_{\text{ext}}F_{mn}, \tag{I.1}$$

$$F_{mo}=\gamma_{m,o}-\gamma_{o,m}+{}_{\text{ext}}F_{mo}, \tag{I.2}$$

where γ_o is the electromagnetic scalar potential and γ_m the vector potential of the field of the electron and ${}_{\text{ext}}F_{\mu\nu}$ is the external field. Since none of F_{mn}, F_{mo} approach infinity faster than $1/r^2$ as r approaches zero, and they appear quadratically in the T's, F_{mn} and F_{mo} influence the first calculation only through terms of order -2, -1 and zero in r.

The solutions for the potentials satisfying Maxwell's equations are:[3]

$$\gamma_o=e\sum_{l=1}^{\infty}\frac{1}{(2l-2)!}\frac{d^{2l-2}}{dt^{2l-2}}(r^{2l-3})-e\sum_{l=1}^{\infty}\frac{1}{(2l-1)!}\frac{d^{2l-1}}{dt^{2l-1}}(r^{2l-2}), \tag{I.3}$$

$$\gamma_m=-e\sum_{l=1}^{\infty}\frac{1}{(2l-2)!}\frac{d^{2l-2}}{dt^{2l-2}}(\gamma^{2l-3}u^m)+e\sum_{l=1}^{\infty}\frac{1}{(2l-1)!}\frac{d^{2l-1}}{dt^{2l-1}}(r^{2l-2}u^m), \tag{I.4}$$

where the first sum in each case represents the solution: ($\frac{1}{2}$ retarded$+\frac{1}{2}$ advanced) potential, i.e., the "standing wave" solution, and the second represents the solution: ($\frac{1}{2}$ retarded$-\frac{1}{2}$ advanced) potential. Therefore the complete expressions represent the retarded potential solutions.

The symbol "$\sim$" will be used to designate the relevant terms of an expression. Also, when an expression is divided into two parts by a comma, the part preceding the comma will be understood to have arisen from the "standing wave" solution alone, while the part following it will represent the terms added when the retarded potential solution is used.

From the above expressions for the potentials we derive, keeping only terms which can influence the calculations (i) or (ii):

$$\gamma_o\sim\frac{e}{r}-\frac{e}{2}\frac{(x^s-\eta^s)}{r}\dot{u}^s+\frac{3e}{8}\frac{(x^l-\eta^l)(x^s-\eta^s)}{r}\dot{u}^l\dot{u}^s+\frac{3e}{8}r\dot{u}^s\dot{u}^s,\ -eu^s\dot{u}^s+\frac{e}{3}(x^s-\eta^s)\ddot{u}^s, \tag{I.5}$$

$$\gamma_m\sim-\frac{e}{r}u^m+\frac{e}{2}\frac{(x^s-\eta^s)}{r}\dot{u}^s u^m+\frac{e}{2}\frac{(x^s-\eta^s)}{r}u^s\dot{u}^m-\frac{e}{2}r\ddot{u}^m,\ +e\ddot{u}^m-e(x^s-\eta^s)\dot{u}^s\dot{u}^m. \tag{I.6}$$

We deduce the relevant terms of

$$F_{mo}\sim{}_{\text{ext}}F_{mo}+e\frac{x^m}{r^3}-\frac{e}{2}\frac{1}{r}\dot{u}^m-\frac{e}{2}\frac{x^s x^m}{r^3}\dot{u}^s+\frac{e}{4}\frac{x^s}{r}\dot{u}^s\dot{u}^m-\frac{3e}{8}\frac{x^m}{r}\dot{u}^s\dot{u}^s+\frac{3e}{8}\frac{x^l x^s x^m}{r^3}\dot{u}^l\dot{u}^s,\ +\frac{2}{3}e\ddot{u}^m, \tag{I.7}$$

and

$$F_{mn}\sim{}_{\text{ext}}F_{mn}+e\frac{x^n}{r^3}u^m-e\frac{x^m}{r^3}u^n-\frac{e}{2}\frac{x^s x^n}{r^3}(\dot{u}^s u^m+u^s\dot{u}^m)+\frac{e}{2}\frac{x^s x^m}{r^3}(\dot{u}^s u^n+u^s\dot{u}^n)-\frac{e}{2}\frac{x^n}{r}\ddot{u}^m+\frac{e}{2}\frac{x^m}{r}\ddot{u}^n,\ +0. \tag{I.8}$$

In these latter expressions we have put $\eta^r=0$ throughout.

Of the expressions occurring in the various integrands, the only ones whose integrals are not zero are those of the form

$$x^{S_1}x^{S_2}\cdots x^{S_p}/r^{p+2},\quad (S_1, S_2, \cdots S_p=1, 2, 3),$$

where p is an odd integer. Hence, so far as the calculation (i) is concerned we may write

$$-T_{mn}\sim e\frac{x^m}{r^3}{}_{\text{ext}}F_{no}+e\frac{x^n}{r^3}{}_{\text{ext}}F_{mo}-\delta_{mn}e\frac{x^s}{r^3}{}_{\text{ext}}F_{so},+\frac{2}{3}e^2\frac{x^m}{r^3}\ddot{u}^n+\frac{2}{3}e^2\frac{x^n}{r^3}\ddot{u}^m-\frac{2}{3}\delta_{mn}e^2\frac{x^s}{r^3}\ddot{u}^s, \tag{I.9}$$

$$T_{on}\sim e\frac{x^s}{r^3}{}_{\text{ext}}F_{sn}. \tag{I.10}$$

Consequently the integrals arising from (I.9) and (I.10) are $e_{\text{ext}}F_{mo}$, $+\frac{2}{3}e^2\ddot{u}^m$, and

$$\frac{1}{4\pi}\int T_{on}\lambda^n dS=e\delta_{mn\ \text{ext}}\tilde{F}_{mn}=0,$$

since ${}_{\text{ext}}\tilde{F}_{mn}$ is skew-symmetric.

Let us now consider (ii). The equations satisfied by Ψ_m, Ψ_o are

$$\Delta\Psi_m=-T_{mo,\,o}, \tag{2.6}$$

$$\Delta\Psi_o=-T_{oo,\,o}. \tag{2.7}$$

Using the expressions (I.7) and (I.8) to form $T_{mo,\,o}$ and $T_{oo,\,o}$ and remembering that we are interested only in terms of order $1/r^3$, we have

$$T_{mo,\,o}=(F_{ms}F_{os})_{,o}\sim 2e^2\frac{x^s}{r^4}\dot{u}^s\dot{u}^m-\frac{1}{2}e^2\frac{x^m}{r^4}\dot{u}^s\dot{u}^s-\frac{3}{2}e^2\frac{x^mx^lx^s}{r^6}\dot{u}^l\dot{u}^s,$$

and

$$T_{oo,\,o}=\tfrac{1}{2}(F_{so}F_{so}+F_{rs}F_{rs})_{,o}\sim -e^2\frac{x^s}{r^4}\ddot{u}^s$$

if we omit terms which contain u^r. It follows that all the terms of $\Psi_{\mu,\,n}$ will contain an even number of factors x^s, and hence that their integrals will vanish μ.

Appendix II.

By inspection of the integrals (4.12) we see that only the following types of terms in the $\gamma_{\mu\nu}$ could give nonzero contributions to the integrals: (i) terms of γ_{mn} of order $1/r$ and not containing $\dot{\eta}^s$; or terms of order $1/r^2$, which may contain $\dot{\eta}^s$ at most twice. (ii) terms of γ_{on} of order $1/r$ which contain $\dot{\eta}^s$ once only or not at all; or terms of order $1/r^2$ which may contain $\dot{\eta}^s$ at most twice. (iii) terms of γ_{oo} of order $1/r$, only if they contain a single $\dot{\eta}^s$ or higher derivative of η^s.

Consider first $\Gamma_{\mu\nu}$. The following solutions of (4.8) may be verified, where we include only terms of the above types, and omit also terms whose divergence vanishes identically:

$$\begin{aligned}
&\text{(a)}\quad \Gamma_{oo}=\left[\frac{e^2}{r^2}\right]+e^2\frac{(x^s-\eta^s)}{r^2}\ddot{\eta}^s+\cdots,\\
&(b)\quad \Gamma_{on}=-\frac{e^2}{r^2}\dot{\eta}^n+e^2\frac{(x^n-\eta^n)}{r^2}\dot{\eta}^s\ddot{\eta}^s+2e^2\frac{(x^r-\eta^r)(x^s-\eta^s)(x^n-\eta^n)}{r^4}\dot{\eta}^m\ddot{\eta}^s+\cdots,\\
&\text{(c)}\quad \Gamma_{mn}=\frac{e^2}{r^2}\dot{\eta}^m\dot{\eta}^n+e^2\frac{(x^m-\eta^m)(x^s-\eta^s)}{r^4}\dot{\eta}^n\dot{\eta}^s+e^2\frac{(x^n-\eta^n)(x^s-\eta^s)}{r^4}\dot{\eta}^m\dot{\eta}^s-\frac{3e^2}{2}\frac{(x^m-\eta^m)}{r^2}\ddot{\eta}^n\\
&\qquad\qquad -\frac{3e^2}{2}\frac{(x^n-\eta^n)}{r^2}\ddot{\eta}^m+2\delta_{mn}e^2\frac{(x^s-\eta^s)}{r^2}\ddot{\eta}^s+\cdots.
\end{aligned}\tag{II.1}$$

If now we calculate $(\Gamma_{mn,oo}-\Gamma_{mo,no})$ and $(\Gamma_{on,oo}-\Gamma_{oo,no})$ omitting from them all terms which involve $\dot{\eta}^s$, it will be found that the remaining terms are all of the types whose surface integrals are zero.

The omission of terms whose divergence vanishes identically is justified, since the surface integrals of such terms must also vanish identically.

The solutions (II.1) were obtained by the "new approximation method,"[3] adopting the same procedure by which the potentials (I.3) and (I.4) were determined.

Thus we have shown that the $\Gamma_{\mu\nu}$ give no contribution to the integrals occurring in the equations of motion (4.12). Let us now turn to the $\bar{\gamma}_{\mu\nu}$.

The retarded potentials which we choose to satisfy (4.14) are:[3]

$$\text{(a)}\qquad \bar{\gamma}_{oo}=-2m_0\sum_{l=1}^{\infty}\frac{1}{(2l-2)!}\frac{d^{2l-2}}{dx^{o\,2l-2}}(r^{2l-3})+2m_0\sum_{l=1}^{\infty}\frac{1}{(2l-1)!}\frac{d^{2l-1}}{dx^{o\,2l-1}}(r^{2l-2}),$$

$$\text{(b)}\qquad \bar{\gamma}_{on}=2m_0\sum_{l=1}^{\infty}\frac{1}{(2l-2)!}\frac{d^{2l-2}}{dx^{o\,2l-2}}(r^{2l-3}\dot{\eta}^n)-2m_0\sum_{l=1}^{\infty}\frac{1}{(2l-1)!}\frac{d^{2l-1}}{dx^{o\,2l-1}}(r^{2l-2}\dot{\eta}^n),\qquad \text{(II.2)}$$

$$\text{(c)}\qquad \bar{\gamma}_{mn}=-2m_0\sum_{l=1}^{\infty}\frac{1}{(2l-2)!}\frac{d^{2l-2}}{dx^{o\,2l-2}}(r^{2l-3}\dot{\eta}^m\dot{\eta}^n)+2m_0\sum_{l=1}^{\infty}\frac{1}{(2l-1)!}\frac{d^{2l-1}}{dx^{o\,2l-1}}(r^{2l-2}\dot{\eta}^m\dot{\eta}^n).$$

If we inspect (II.2) (a), (b), and (c) for terms of the types specified above, we find that there are none of type (i) or type (iii). Only one term of type (ii) is to be found, and this is the first term of the first sum of (II.2) (b). This will not contribute to the integral (4.12) (b), since its contribution to $\bar{\gamma}_{on,oo}$ will contain $\dot{\eta}^r$ which is zero. It will, however, give a contribution to (4.12) (a), for

$$-\frac{1}{4\pi}\int 2m_0\left(\frac{1}{r}\right)_{,n}\ddot{\eta}^m\lambda^n dS=2m_0\ddot{\eta}^m.$$

This is the source of the inertial term in the equation of motion.

The same obviously holds when we take in (II.2) the "standing" rather than the "retarded" gravitational potentials.

If the solutions given in (II.1) are pushed one step further, so as to involve terms which do not become infinite near the singularity, and the $\gamma_{\mu\nu}$ are formed from these and (II.2), it may be verified that in the neighborhood of the singularity and in our special coordinate system we have

$$\gamma_{on,n}-\gamma_{oo,o}=0$$

and

$$\gamma_{mn,n}-\gamma_{mo,o}=-\frac{2}{r}(m_0\ddot{\eta}^m-e_{\text{ext}}\widetilde{F}_{mo})=0^7$$

by virtue of the equations of motion.

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