On After-Effects in Solid Dielectrics

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The discharge- and residual charge-curves observed in anomalous dielectrics are discussed and two general relations between them are established.

It is generally known that the capacity of a condenser which shows the phenomenon of residual charges, depends on the charging time. For instance a Leyden jar, which can give successive sparks, discharges slowly if a potential difference has been applied for some minutes, before but will discharge very rapidly if the potential difference has been applied only for a few seconds. This behavior seems often very irregular. We should like to point out some very simple, but general relations existing there, which result from the principle of superposition, always used successfully for the explanation of phenomena of this kind.

Let us consider the three following experiments characteristic of the behavior of anomalous dielectrics: (Fig. 1).

(I) A charging potential U_0 is applied to the condenser during an interval T, which is so long that at its end the after-effects produced by the application of the potential U_0 have dispersed completely. Then U_0 is withdrawn and one of the plates of the condenser remains insulated while the other one is connected to earth. The discharge is then very slow and does not depend on T.

(II) The potential U_0 is applied during an interval t_0 much smaller than T. The discharge now depends on t_0 and is much faster than in the foregoing case.

(III) Finally U_0 is applied during the interval T, then it is removed and the condenser is short-circuited during t_0, and then one of the plates is again insulated. A potential difference will appear again between the plates and we have a residual-charge effect.

These three effects are related to the fundamental property of the anomalous dielectrics, i.e., the anomalous- or after-current and can be described mathematically, if we assume the validity of the principle of superposition introduced into the theory of after-effects by Hopkinson,[1] and subsequently used by many authors, such as v. Schweidler,[2] Whitehead,[3] Neufeld,[4] and Yager.[5]

Let $\Delta U\varphi(t-\tau)$ be the after-current produced by the discontinuous variation ΔU of the applied potential difference at an instant τ. Then the principle of superposition enables us to express the after-current i flowing in the instant t and produced by any continuous variation of the potential, as follows,

$$i=\int_{-\infty}^{t}\frac{dU(\tau)}{d\tau}\varphi(t-\tau)d\tau. \qquad (1)$$

If we assume that a continuous variation of the potential difference between the plates begins at $t=0$ and take into account the leakage current U/R and the displacement current CdU/dt, the

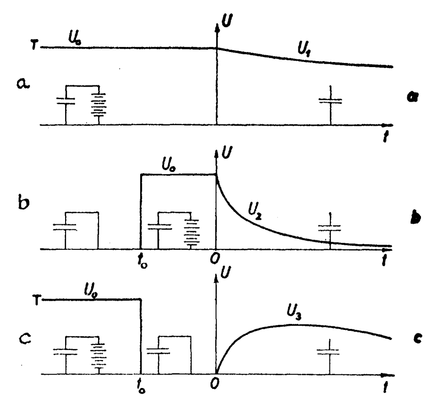

FIG. 1. *a*, Discharge after complete charge. *b*, Discharge after instantaneous charge. *c*, The residual charge.

[1] L. Hopkinson, Phil. Trans. **166**, 489 (1876); **167**, 599 (1878).
[2] E. v. Schweidler, Ann. d. Physik **24**, 711 (1907).
[3] J. B. Whitehead, *Lectures on Dielectric Theory and Insulation* (McGraw-Hill, New York, 1927).
[4] J. Neufeld, J. Frank. Inst. **222**, 327 (1936).
[5] W. A. Yager, Physics **7**, 434 (1936).

total current-flow across the dielectric is given by

$$J=C\frac{dU}{dt}+\frac{U}{R}+\int_0^t \frac{dU(\tau)}{d\tau}\varphi(t-\tau)d\tau+i(t), \quad (2)$$

where $i(t)$ is the after-current produced by all the potential difference variations which eventually had occurred before $t=0$.

If the plates are insulated the total current J must become 0.

$$J=0. \quad (3)$$

We thus obtain $U(t)$ as the solution of an integro-differential equation

$$C\frac{dU}{dt}+\frac{U}{R}+\int_0^t \frac{dU(\tau)}{d\tau}\varphi(t-\tau)d\tau+i(t)=0. \quad (4)$$

With regard to the mathematical description the three cases referred to above differ only in the form of the right member $i(t)$. In (I) it has become 0; in (II) it is given by $U_0\varphi(t+t_0)$, and in (III) it has the same absolute value, but the opposite sign. Hence we may write more generally

$$i(t)=\begin{cases}0 & \text{for case (I)}\\ +i_0(t) & \text{for case (II)}\\ -i_0(t) & \text{for case (III).}\end{cases} \quad (5)$$

The rigorous evaluation of the functions $U(t)$ is rather complicated,[6] but we can deduce some simple relations among the curves $U(t)$ in all three cases, reducing them to a unique one.

For this purpose we write at first Eq. (4) in the form

$$F[U(t)]=-i(t), \quad (6)$$

observing the linearity of the operator F:

$$F[U_1(t)]+F[U_2(t)]=F[U_1(t)+U_2(t)]. \quad (7)$$

If $U_1(t)$, $U_2(t)$ and $U_3(t)$ are, respectively, the potential difference curves in the three cases concerned here, they satisfy the equations

$$F[U_1(t)]=0 \quad \text{with } U_1(0)=U_0, \quad (8)$$

$$F[U_2(t)]=-i_0(t) \quad \text{with } U_2(0)=U_0, \quad (9)$$

$$F[U_3(t)]=i_0(t) \quad \text{with } U_3(0)=0. \quad (10)$$

Summing up (9) and (10) and using relation (7) we find

$$F[U_2(t)+U_3(t)]=0 \quad \text{with } U_2(0)+U_3(0)=U_0. \quad (11)$$

Comparing this with (8) we have

$$U_2(t)+U_3(t)=U_1(t). \quad (12)$$

The three curves are therefore correlated in a very simple manner and it is always possible to deduce one of them from the two other ones. But even between these we find a direct relation.

By the substitution

$$dU/dt=\psi(t), \quad (13)$$

introduced by F. M. de Oliveira Castro[7] into the theory of Eq. (4) and using the new kernel

$$\rho(t)=\varphi(t)/C+1/RC, \quad (14)$$

we write (4) in the form

$$\psi_1(t)+\int_0^t \psi_1(\tau)\rho(t-\tau)d\tau=-U_0/RC \quad (15)$$

for case (I),

$$\psi_2(t)+\int_0^t \psi_2(\tau)\rho(t-\tau)d\tau=-\frac{U_0}{RC}-\frac{1}{C}\int_0^t\frac{di_0(\tau)}{d\tau}d\tau-\frac{i_0(0)}{C} \quad (16)$$

for case (II).

Eq. (15) is an integral equation of the Volterra-type with constant right member and (16) the same equation with variable right member. The equations being linear it seems logical to apply a principle of superposition, for we may consider $i(t)$ as an external "force" acting on a system of which we know the "movement" produced by the constant "force." So we try the solution in the form

$$\psi_2(t)=f(t)+\int_0^t\frac{di_0(\tau)}{d\tau}g(t-\tau)d\tau, \quad (17)$$

where $g(t)$ and $f(t)$ are two arbitrary functions.

Introducing (17) into (16) and noting the identity

$$\int_0^t \rho(t-\tau)d\tau\int_0^\tau\frac{di_0(\sigma)}{d\sigma}g(t-\sigma)d\sigma =\int_0^t\frac{di_0(\tau)}{d\tau}d\tau\int_0^{t-\tau}g(\sigma)\rho(t-\tau-\sigma)d\sigma, \quad (18)$$

[6] For an approximative solution see B. Gross, Zeits. f. Physik **107**, 217 (1937); a more rigorous treatment is given by F. M. de Oliveira Castro, Zeits. f. Physik **114**, 116 (1939).

[7] F. M. de Oliveira Castro, Ann. Acad. Bras. Sc. **11**, 150 (1939).

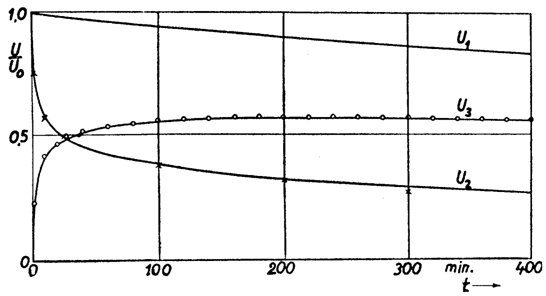

FIG. 2. Measurements with carnauba-wax. U_1, Discharge after 24-hour charge. U_2, Discharge after 10-sec. charge. U_3, Residual charge after 24-hour charge and subsequent 10-sec. short circuit. The circles of curve U_3 are calculated by aid of relation (12). The crosses of curve U_2 are calculated by aid of relation (23).

which follows from the well-known Dirichlet-formula, we obtain

$$\int_0^t \frac{di_0(\tau)}{d\tau}\left[g(t-\tau)+\int_0^{t-\tau} g(\sigma)\rho(t-\tau-\sigma)d\sigma+\frac{1}{C}\right]d\tau$$
$$+\left[f(t)+\int_0^t f(\tau)\rho(t-\tau)d\tau+\frac{U_0}{RC}+\frac{i_0(0)}{C}\right]=0. \quad (19)$$

This equation will be satisfied, if $f(t)$ and $g(t)$ satisfy equations of the type (15), because then each of the two members of (19) will disappear separately. Thus it follows that

$$f(t)=[1+(R/U_0)i_0(0)]\psi_1(t) \quad (20)$$

and $$g(t)=(R/U_0)\psi_1(t). \quad (21)$$

By introducing (20) and (21) into (17) we obtain, after a further integration, $U_2(t)$ in function of $U_1(t)$. Noting the identity

$$\int_0^t \frac{di_0(\tau)}{d\tau}g(t-\tau)d\tau=\frac{d}{dt}\int_0^t i_0(t-\tau)g(\tau)d\tau - i_0(0)g(t), \quad (22)$$

there results finally

$$U_2(t)=U_1(t)+\frac{R}{U_0}\int_0^t \frac{dU_1(\tau)}{d\tau}i_0(t-\tau)d\tau. \quad (23)$$

We are now able to calculate the discharge-curve for every possible charging-time t_0, if we know the discharge-curve for complete charge and the expression of the after-current.

We have tested these results experimentally. Some measurements were made with carnauba-wax, a substance which was found to show dielectric absorption in a high degree. The full lines of Fig. 2 give the functions U_1, U_2, U_3 obtained by measurement; the circles give the values calculated for U_3 by subtracting U_2 from U_1; the crosses give the values calculated for U_2 by using relation (23). The constant RC was determined in such a way as to obtain a good agreement for $t=100$ minutes; for $\varphi(t)$ we put $\beta t^{-0.75}$. We see that the calculated values fit well the measured ones.

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