The Meson Theory of Nuclear Forces

Part II. Theory of the Deuteron†

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With the nuclear forces as derived in the first part of this paper the deuteron problem is integrated. In §7, natural units are introduced, *viz.*, the range of the nuclear forces $1/\kappa=\hbar/\mu c$ as the unit of length and the quantity $E_0=\mu^2c^2/M$ as the unit of energy. With the adopted value of the meson mass, $\mu=177$ electron masses, we have $1/\kappa=2.18_5\times10^{-13}$ cm and $E_0=8.68$ Mev. The quantum states to be expected are discussed (§8); each state is characterized by the total angular momentum J, the total spin $S=0$ or 1, and the parity. For some states, the orbital momentum L is defined uniquely by these quantum numbers (in this case, $L=J$); for others, we have a linear combination of wave functions with two different values of L, *viz.*, $J+1$ and $J-1$. The angular coordinates are eliminated from the Schrödinger equation (§9) and the radial wave equations obtained. The order of the states is discussed qualitatively (§10) and it is made plausible that the ground state is a combination of a ^3S and a $^3D_1$ state, both in the neutral and in the symmetrical theory. The next higher triplet state is probably $^3P_1$. Then the wave equation is solved numerically for the ^1S state (§11), the position of this state being taken from experiments on the scattering of slow neutrons by protons. The method of the numerical solution is described, and a table constructed giving the interaction constant $a=(2M/\mu)f^2/\hbar c$ as a function of the cut-off distance $x_0=\kappa r_0$; a is found to depend only slightly on x_0. The next section (§12) deals with the numerical integration for the triplet (ground) state.

The following sections contain the results. In §13, the results for the neutral theory are given. A cut-off distance of 0.32 to 0.40 of the range of the forces will fit the positions of singlet and triplet state; these figures are very reasonable from general considerations. The calculations are carried out with two alternative ways of cutting off, assuming the potential inside r_0 either to be zero, or to retain the value it had at r_0. It is found that all physically significant quantities are practically independent of the method of cutting off. The value of $f^2/\hbar c$ is 0.077 or 0.080, according to the method used. The wave function contains about 6.7 percent $^3D_1$ state, the rest being ^3S. This means that the sum of the magnetic moments of proton and neutron should be about 0.04 nuclear magneton greater than the deuteron moment which is just reconcilable with the present experimental data (§14). In §15, the quadrupole moment of the deuteron is calculated. The values found are 2.70 and 2.61×10^{-27} cm² using the two methods of cutting off; the sign is positive (cigar-shape). Both sign and magnitude are in good agreement with the experiments of Kellogg, Rabi, Ramsey and Zacharias as evaluated by Nordsieck ($Q=2.7\times10^{-27}$). In §16, the $^3P_1$ state is calculated; it is shown to be unstable and to have no appreciable influence on the scattering of neutrons of moderate energy.

The symmetrical theory is discussed in §17. The calculations for this theory give unacceptable results, particularly for the cut-off distance which must be chosen considerably larger than the range of the nuclear forces ($x_0=1.3$ to 1.7). The quadrupole moment comes out about ten times too large and of the wrong sign. These results are very regrettable since only the symmetrical theory gives a natural explanation of the β-decay and of the extra magnetic moments of neutron and proton. Therefore an alternative way of cutting off is tried in §18, with even more unfavorable results. The theory of Møller and Rosenfeld is discussed (§19) but it is not considered satisfactory because of its intrinsic complication. The question remains open whether the neutral theory is correct because it gives quantitative agreement for the deuteron, or the symmetrical theory which is qualitatively preferable but in violent quantitative disagreement both with the theory of the deuteron and with the β-decay.

§7. THE WAVE EQUATION. ABBREVIATIONS

THE wave equation of a system consisting of two nuclear particles is

$$\nabla^2\psi+(M/\hbar^2)(E-V)\psi=0, \qquad (36)$$

where M is the mass of the nuclear particle, E the energy of the system and V the potential energy as given by (34). (If both particles are protons, the electrostatic repulsion should be added.) In V, the distance between the two particles occurs in the form κr; therefore it will be convenient to introduce the abbreviation

$$x=\kappa r, \quad \mathbf{x}=\kappa\mathbf{r}, \qquad (36a)$$

i.e., to measure all distances in units of $1/\kappa$, which is the range of the nuclear forces. According to (8b), $1/\kappa=2.18_5\times10^{-13}$ cm.

† Part I in Phys. Rev. **57**, 260 (1940). The sections, equations and references are consecutively numbered in Parts I and II.

If we now denote by ∇'^2 the Laplacian in the new coordinates $\mathbf{x}$, insert (34) in (36) and remember that $\kappa=\mu c/\hbar$, we obtain for the neutral theory

$$\nabla'^2\psi+\frac{ME}{\mu^2c^2}\psi-2\frac{M}{\mu}\frac{f^2}{\hbar c}\left[\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\frac{e^{-x}}{x}+\left(-\frac{3}{2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{x}\,\boldsymbol{\sigma}_2\cdot\mathbf{x}}{x^2}+\tfrac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)e^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right)\right]\psi=0. \quad (37)$$

Obviously, it will be convenient to measure the energy in units of

$$E_0=\mu^2c^2/M. \quad (38)$$

This quantity appears to be the natural unit of energy[35] in nuclear physics, just as $1/\kappa$ is the natural unit of length. E_0 is the rest energy of the meson, multiplied by the ratio of the masses of meson and nuclear particle. With a meson mass of 177 m as assumed here (Eq. (8)), we obtain

$$E_0=8.7\text{ Mev}. \quad (38a)$$

This is just of the order of nuclear binding energies per particle. It is exactly four times the binding energy $w=2.17$ Mev of the deuteron; actually, the value (8) of the meson mass has been chosen (within the limits allowed by the experimental data) to make $E_0=4w$. We shall be most concerned with the ground state of the deuteron for which E is negative; it is then convenient to put

$$E/E_0=ME/\mu^2c^2=-\epsilon^2. \quad (39)$$

According to the foregoing, we have for the ground state of the deuteron

$$\epsilon=\tfrac{1}{2}. \quad (39a)$$

This simple value makes the numerical work much easier.

Another abbreviation which is called for by the form of (37) is

$$a=2(M/\mu)(f^2/\hbar c). \quad (40)$$

This constant is a convenient expression for the strength of the forces. With our value for μ, we have

$$a=20.8f^2/\hbar c. \quad (40a)$$

Dropping again the prime at the Laplacian, (37) becomes now

$$\nabla^2\psi-\epsilon^2\psi-a\left[\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\frac{e^{-x}}{x}+\left(-\frac{3}{2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{x}\,\boldsymbol{\sigma}_2\cdot\mathbf{x}}{x^2}+\tfrac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)e^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right)\right]\psi=0. \quad (41)$$

At large distances, this equation may be separated in polar coordinates. The spherically symmetrical solution is

$$\psi=e^{-\epsilon x}/x\approx e^{-\frac{1}{2}x}/x, \quad (41a)$$

which means that the probability of finding the particles at a distance between x and $x+dx$, ψ^2x^2dx, falls off as $e^{-x}dx$.

§8. The Quantum States of the Two-Body System

The tensor interaction V_2 depends on the direction of $\mathbf{r}$ as well as its magnitude and therefore does not commute with the orbital momentum $\mathbf{L}$ of the system. The orbital momentum L will therefore in general not be a quantum number, i.e., we cannot classify the levels of the two-body system as S, P, D, etc. states. The eigenfunctions will in general be linear combinations of several terms involving spherical harmonics of different orders L.

On the other hand, the tensor interaction is, of course, invariant against simultaneous rotation of spin $\boldsymbol{\sigma}$ and spatial coordinates $\mathbf{r}$ because it depends only on the relative orientation of these two vectors. Therefore the Hamiltonian commutes with the *total angular momentum*

$$\mathbf{J}=\mathbf{L}+\mathbf{S}, \quad (42)$$

where $\mathbf{L}$ is the orbital momentum and

$$\mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) \quad (42a)$$

[35] It has always seemed artificial to me to use the so-called "nuclear units" which are based on mc^2 as unit of energy. As far as we know at present, electrons do not play an important part in nuclear forces, and even if they did, their mass would probably be irrelevant. This is already shown by the fact that all mass defects are large numbers in units of mc^2, rather than of the order unity.

the total spin. J, giving the amount of the total angular momentum, and M, giving its component in a given (z) direction, are therefore both good quantum numbers.

The Hamiltonian is also invariant against inversion, i.e., against changing the sign of all coordinates. Therefore the eigenfunctions will have a definite *parity*, even or odd, i.e., they will either remain unchanged or change sign when $\mathbf{r}$ is replaced by $-\mathbf{r}$.

Finally, the Hamiltonian commutes with the amount S^2 of the total spin although it does not commute with the Cartesian components of $\mathbf{S}$. This fact is not so obvious as the two preceding ones and depends essentially upon the fact that the spin of each nuclear particle is $\frac{1}{2}$. It can be proved by direct evaluation of the commutator $S^2V_2-V_2S^2$, using relations of the type $\sigma_{1x}\sigma_{1y}=-\sigma_{1y}\sigma_{1x}$. However, the simplest proof is by showing that V_2, applied on the wave function of any singlet state, will vanish identically. This means that the tensor interaction can have no matrix elements between a singlet and a triplet state. Therefore the tensor interaction applied to a triplet wave function must again give a triplet function. Consequently the tensor interaction does not mix singlets and triplets, in other words it commutes with the total spin S. The rest of the Hamiltonian obviously commutes with S, and therefore S is a good quantum number. It can have the values zero (singlet state) or one (triplet).

Thus the quantum numbers of our system are: the total angular momentum J, its z component M, the total spin S, and the parity. Let us now consider the states of given J and M. There are four such states, one singlet and three components of the triplet. The singlet state must have $L=J$ because $S=0$ (cf. (42)) so that in this case the orbital momentum is (more or less accidentally) quantized. We have a ^1S, ^1P, ^1D state, etc.

The triplet states can have $L=J$, $L=J+1$ and $L=J-1$. The parity of a state is determined by L, being even for even L and odd for odd L (Part I, reference 27, p. 108). Therefore the states $L=J+1$ and $J-1$ have the same parity while $L=J$ has the opposite parity. Since the parity is quantized, the state $L=J$ will be by itself, i.e., it will not be mixed with any state of a different L. Therefore here again L will be "accidentally quantized," we can speak of $^3P_1$, $^3D_2$, $^3F_3$ states, etc.

The two states $L=J+1$ and $L=J-1$ will be mixed, the eigenfunction being a linear combination of two functions involving spherical harmonics of order $J-1$ and $J+1$, respectively. The corresponding radial wave functions satisfy two simultaneous differential equations of the second order (cf. §9). There will, of course, be two such mixed states for a given J, the lower in general containing more of $L=J-1$, the higher more of $L=J+1$. For $J=1$, we get thus a mixture of a ^3S and a $^3D_1$ state: such a mixture will represent the ground state of the deuteron. For $J=2$, we have a mixture of $^3P_2$ and $^3F_2$, etc. Only $J=0$ is an exception, because $J=0$ in conjunction with $S=1$ implies necessarily $L=1$ so that we get a pure $^3P_0$ state.

Our analysis shows that at most two different values of L, and sometimes only one, occur in a given eigenfunction.[36] This makes the two-body problem manageable since it is reduced to the numerical solution of at most two simultaneous differential equations for the radial wave functions. These equations will be derived in the following section.

It need hardly be emphasized that the structure of our spectrum has no relation to that found in Russell-Saunders coupling. In particular, the spectroscopic notation is used only for convenience and does by no means imply that states in the same triplet, e.g., $^3P_0$, $^3P_1$, $^3P_2$, have any relation to each other. They will be as widely separated as states of different L or S, and states of the same J will not lie close together either.

§9. Elimination of the Angular Coordinates

The eigenfunction of a triplet state of given J and M, and given parity, may be written:

$$\psi(x, \vartheta, \varphi, M_S) = (1/x)\sum_L u_L(x)F_{JLM}(\vartheta, \varphi, M_S). \quad (43)$$

Here the sum contains at most two terms (cf. §8), the u_L are the radial wave functions ($x=\kappa r$) and the F will be called angular functions in the following. They can be expressed in terms of

[36] This was pointed out to me by R. Peierls to whom I am greatly indebted for a stimulating discussion.

ordinary spherical harmonics and of spin functions:

$$F_{JLM}=\sum_{m=M-1}^{M+1} c_{JLMm}\Phi_{LMm}, \tag{44}$$

where

$$\Phi_{LMm}=Y_{Lm}(\vartheta\varphi)\chi_{M-m}. \tag{44a}$$

Here the Y_{Lm} are normalized spherical harmonics,[37] m is the orbital momentum about the z axis (we write m instead of M_L for convenience). The χ are the usual triplet spin functions, *viz.*

$$\begin{aligned}\chi_1&=\alpha(1)\alpha(2),\\ \chi_0&=2^{-\frac{1}{2}}[\alpha(1)\beta(2)+\beta(1)\alpha(2)],\\ \chi_{-1}&=\beta(1)\beta(2).\end{aligned} \tag{44b}$$

The subscript of the spin function gives the spin in the z direction, $M_S=M-m$. Since M_S goes from -1 to $+1$, m will take the values $M-1$, M and $M+1$. The c's are constant coefficients which are given by considerations about angular momenta; they are independent of the potential; a general discussion will be found in Condon and Shortley.[38] We assume the F's normalized, then

$$\sum_{m=M-1}^{M+1}|c_{JLMm}|^2=1. \tag{45}$$

There are several other relations between the c's of different J and m expressing orthogonality and normalization of the F's.

Since the F as well as ψ are normalized, the normalization condition on the radial functions is

$$\sum_L\int u_L{}^2dx=1. \tag{46}$$

TABLE I. *Coefficients t in the tensor interaction (cf. Eq. (50))*

L	$J=L-1$		$J=L$		$J=L+1$	
	TERM	t_{JL}	TERM	t_{JL}	TERM	t_{JL}
0					$^3S_1$	0
1	$^3P_0$	2	$^3P_1$	-1	$^3P_2$	1/5
2	$^3D_1$	1	$^3D_2$	-1	$^3D_3$	2/7

$t_{J,LL'}$: $J=1$ (coupling ^3S and $^3D_1$) $t_{J,LL'}=-\sqrt{2}$
$J=2$ (coupling $^3P_2$ and $^3F_2$) $t_{J,LL'}=-(3/5)\sqrt{6}$.

The u_L obey the differential equations (neutral theory):

$$\frac{d^2u_L}{dx^2}-\left[\epsilon^2+\frac{L(L+1)}{x^2}+\frac{1}{3}a\frac{e^{-x}}{x}+t_{JL}\,ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right)\right]u_L=t_{JLL'}\,ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right)u_{L'}. \tag{47}$$

Here $-\epsilon^2$ represents the eigenvalue in our units (cf. §7), $L(L+1)/x^2$ the centrifugal force, and $\frac{1}{3}ae^{-x}/x$ the central force V_1 in which we have inserted for $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ its value (unity) for triplet states. The remaining terms arise from the tensor interaction. If $L=J$, the term on the right-hand side will be absent (§8); if $L=J\pm1$, there will be just one term on the right-hand side, with $L'=J\mp1$.

The coefficients t are constants (not depending on x) and represent the matrix elements of the characteristic term in the tensor interaction, *viz.*, of

$$T=-\frac{3}{2}\frac{\boldsymbol{\sigma}_1\cdot\mathbf{x}\,\boldsymbol{\sigma}_2\cdot\mathbf{x}}{x^2}+\tfrac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2. \tag{48}$$

The definition of the t's is

$$TF_{JLM}=t_{JL}F_{JLM}+t_{JLL'}F_{JL'M}. \tag{48a}$$

Since T commutes with J, no terms with different J and M can appear on the right-hand side, and the coefficients t can only depend on J and L but not on M. They can easily be calculated using the relations between the spherical harmonics[37] and the method of sums (cf. reference 37, p. 555). The result is

$$t_{J=L+1,\,L}=\frac{L}{2L+3}=\frac{J-1}{2J+1}, \tag{49}$$

$$t_{J=L,\,L}=-1, \tag{49a}$$

$$t_{J=L-1,\,L}=\frac{L+1}{2L-1}=\frac{J+2}{2J+1}, \tag{49b}$$

$$t_{J,\,LL'}=-3J^{\frac{1}{2}}(J+1)^{\frac{1}{2}}/(2J+1). \tag{49c}$$

Table I gives the numerical values for the smallest J.

[37] For the definition of the Y_{Lm} and relations between them see Bethe, *Handbuch der Physik*, Vol. 24, No. 1 (1933), p. 551. The definition is $Y_{Lm}=\left(\frac{2L+1}{4\pi}\right)^{\frac{1}{2}}\frac{1}{2^LL!^2}\times\left(\frac{(L-m)!}{(L+m)!}\right)^{\frac{1}{2}}\sin^m\vartheta\times\frac{d^{L+m}}{(d\cos\vartheta)^{L+m}}(\cos^2\vartheta-1)^{L+m}e^{im\varphi}$ for any (positive or negative) m.

[38] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, 1935).

§10. The Sign of the Interaction and the Order of the Quantum States

Before we proceed to the actual numerical integration of the radial wave equations (47), we shall discuss the sign and relative magnitude of the interactions in the various quantum states.

A. The neutral theory

In the *singlet states* there is an attraction

$$V_{\text{singl}} = -ae^{-x}/x. \tag{50}$$

This attraction is independent of L.

For the *triplet states with* $L=J$, the central force V_1 is repulsive, and the tensor interaction V_2 is attractive and independent of L since $t_{JL}=-1$. The total interaction is

$$V_{L=J} = V_1 + V_2 = \tfrac{1}{3}a\frac{e^{-x}}{x} - ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right)$$
$$= -ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}\right), \tag{51}$$

i.e., attractive at all distances.

The *other two triplet states*, $L=J\pm 1$, are mixed together. This mixing will be unimportant if the coupling interaction

$$V_{\text{coup}} = -3\frac{J^{\frac{1}{2}}(J+1)^{\frac{1}{2}}}{2J+1}ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right) \tag{52a}$$

is small compared with the difference between the centrifugal potentials,

$$C = \frac{(J+1)(J+2)-(J-1)J}{x^2} = \frac{2(2J+1)}{x^2}. \tag{52b}$$

This will be the case when J is large because then the particles will stay at large distances x. Then the two states with different L can be considered separately, and in each of them we shall have two repulsive interactions (central force and tensor interaction). The *repulsion* is *larger for* $L=J+1$ *than for* $L=J-1$ (cf. (49), (49b)). For very high J the two repulsions become equal, we have $t_{JL}=\frac{1}{2}$ for both values of L and

$$\lim_{L\to\infty} V_{L=J+1} = \lim V_{L=J-1}$$
$$= \tfrac{1}{2}ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{x}\right). \tag{52c}$$

On the other hand, for the ^3S state the tensor interaction does not contribute to the repulsion at all ($t_{JL}=0$) and for $^3P_2$ its contribution is rather small ($t_{JL}=\frac{1}{5}$).

Actually, the coupling between the states $L=J+1$ and $L=J-1$ will tend to lower the lower of the two, and to raise the upper. The centrifugal force is greater for $L=J+1$; and, according to the preceding paragraph, the same is true for the repulsive force arising from the "diagonal term" in the tensor interaction. Therefore the lower state will be mainly $L=J-1$, and the higher state $L=J+1$. In particular, the lower state with $J=1$ will be mainly ^3S with some $^3D_1$ mixed in (ground state of the deuteron). For $J=0$, there is only a strongly repulsive state, $^3P_0$.

Since the tensor interaction is attractive (cf. (34b)) when the vector **r** from one particle to the other is parallel to the spin **S**, the "lower" state will tend to have **r** and **S** parallel while in the "upper" state **r** and **S** will tend to be at right angles. In particular, in the ground state of the deuteron **r** will more often be parallel to **S** than this would be the case in a pure ^3S state. In such a pure state, there is no preferential direction for **r** at all; therefore *the actual ground state of the deuteron will have an oblong shape, with the major axis in the direction of the spin* **S**. This is equivalent to a *positive quadrupole moment of the deuteron which agrees in sign with the experimental result of Kellogg, Rabi, Ramsey and Zacharias.*[39]

A similar reasoning leads to an understanding of the sign of the diagonal term of the tensor interaction. At least for high orbital momenta, we can apply classical concepts. Then the orbital momentum **L** will be perpendicular to the position vector **r**. Therefore attraction is to be expected when **L** is perpendicular to **S**, and this is the case for $J=L$. When **L** is parallel or antiparallel to **S** ($J=L\pm 1$) repulsion will result.

It is of great interest to decide which of the terms of given J will actually lie lowest, the one with $L=J$ or the lower of the two other terms. The following (nonrigorous) argument decides in favor of the latter. We write the radial equations for $L=J\pm 1$ in abbreviated form:

$$\begin{aligned} d^2u_{J-1}/dr^2 - \epsilon^2 u_{J-1} &= Au_{J-1} - Du_{J+1}, \\ d^2u_{J+1}/dr^2 - \epsilon^2 u_{J+1} &= -Du_{J-1} + Cu_{J+1}. \end{aligned} \tag{53}$$

[39] J. M. B. Kellogg, I. I. Rabi, N. F. Ramsey and J. R. Zacharias, Phys. Rev. **56**, 728 (1939).

Here

$$A=\tfrac{1}{3}ae^{-x}/x+(J-1)J/x^2+(J-1)B/(2J+1),$$

$$B=ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right),$$

$$C=\tfrac{1}{3}ae^{-x}/x+(J+1)(J+2)/x^2+(J+2)B/(2J+1), \tag{53a}$$

$$D=3J^{\frac{1}{2}}(J+1)^{\frac{1}{2}}B/(2J+1).$$

Then by an argument similar to that used in the derivation of the WKB method it can be shown that the two simultaneous equations (53) are almost equivalent to a single Schrödinger equation with the potential

$$W=\tfrac{1}{2}(A+C)\pm[\tfrac{1}{4}(A-C)^2+D^2]. \tag{53b}$$

The negative sign corresponds to the lower state; inserting (53a), we obtain

$$W_-=\tfrac{1}{3}a\frac{e^{-x}}{x}+\frac{J(J+1)+1}{x^2}+\tfrac{1}{2}B-\left[\left\{\frac{2J+1}{x^2}+\frac{3B}{2(2J+1)}\right\}^2+\left(\frac{3B}{2J+1}\right)^2J(J+1)\right]^{\frac{1}{2}}. \tag{54}$$

This expression must be compared with the potential for the state $L=J$ which is

$$V_{L=J}=\tfrac{1}{3}a\frac{e^{-x}}{x}+\frac{J(J+1)}{x^2}-B. \tag{54a}$$

The difference is

$$V-W=-\frac{1}{x^2}-\tfrac{3}{2}B+\left[\left(\frac{1}{x^2}+\tfrac{3}{2}B\right)^2+\frac{4J(J+1)}{x^4}\right]^{\frac{1}{2}}, \tag{55}$$

which is always positive. Therefore *the mixed state is lower than the state* $L=J$. It may be of interest to give the value of W_- in the limiting cases: For weak coupling $((2J+1)/x^2\gg 3B/2)$ we have

$$W=\tfrac{1}{3}a\frac{e^{-x}}{x}+\frac{(J-1)J}{x^2}+\frac{J-1}{2J+1}B-\frac{9J(J+1)}{2(2J+1)^3}B^2x^2+\cdots, \tag{55a}$$

for strong coupling

$$W=\tfrac{1}{3}a\frac{e^{-x}}{x}-B+\frac{J(J+1)}{x^2}-\frac{4J(J+1)}{3Bx^4}+\cdots. \tag{55b}$$

Therefore the lowest of the triplet states with a given J is that which contains mostly the lowest possible L, *viz.*, $L=J-1$, mixed with some $L=J+1$. The next state, which is not much higher if the interaction is large, is the pure state $L=J$, while the highest state, with a strong repulsive potential, is the mixed state which contains mostly $L=J+1$ and a smaller amount of $L=J-1$.

For a given class of states, the potential is higher for higher J. Now for $J=0$, there is only one triplet state, *viz.*, ${}^3P_0$, which has a strongly repulsive potential (cf. Table I). Therefore the lowest triplet state must be found among the states with $J=1$, and according to the preceding paragraph, it must be the state which is composed of ${}^3S$ and ${}^3D_1$ with the former predominating. The next state will be ${}^3P_1$, and after this the lowest state with $J=2$ will follow, *viz.* ${}^3P_2+{}^3F_2$. That the order of the states is actually as indicated here, in particular that ${}^3P_1$ lies higher than ${}^3S+{}^3D_1$, will be shown by numerical calculation in §13 and 16.

Although L is not a true quantum number, it is interesting to investigate the states which have as their predominant part a given L. Of these, the state with $J=L$ will in general be the lowest, the state $J=L+1$ somewhat higher and $J=L-1$ by far the highest.

While the sequence of the triplet states and that of the singlet states separately can be thus established, the relative position of triplets and singlets cannot be derived on qualitative grounds. In particular, it depends on the cut-off distance x_0 whether the ^1S or the $^3S+^3D_1$ state is the lowest (§11–13).

B. Symmetrical theory

As has been mentioned repeatedly, the potential in this theory differs from that in the neutral theory by the factor $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ which is 1 for states symmetrical in the charge coordinate, -3 for antisymmetrical states. With the discussion of §6 on the symmetry of various states, we find then:

In the *singlet states* the potential is attractive for even L, and has then the same value as in the neutral theory, *viz.*,

$$V=-ae^{-x}/x \quad (S=0,\ L \text{ even}). \tag{56}$$

For odd L, the potential is repulsive and three times as large:

$$V=3ae^{-x}/x \quad (S=0,\ L \text{ odd}). \tag{56a}$$

For the *triplet states*, the potential is the same as in the neutral theory if L is *odd*, but has the opposite sign and three times the magnitude if L is even. Remembering the sign of the interaction for the various triplet states in the neutral theory, we find: *The "diagonal part" of the interaction is attractive for all triplet states of odd J, repulsive for all triplet states of even J.*

Since the centrifugal potential increases with J, it is thus immediately clear that the lowest triplet states are those with $J=1$. The competition is therefore again between the $^3P_1$ state and the mixed state $^3S+^3D_1$. However, the latter state is much more favored than in the neutral theory, because the potential in the $^3P_1$ state is the same as in the neutral theory (54a), while for the other state all terms involving the interaction constant a are multiplied by -3. Denoting by B the same expression as in (53a), we have then instead of (55)

$$V-W=\frac{4}{3}a\frac{e^{-x}}{x}+\frac{1}{2}B-\frac{1}{x^2}+\left[\left(\frac{9}{2}B-\frac{1}{x^2}\right)^2+\frac{4J(J+1)}{x^2}\right]^{\frac{1}{2}}, \tag{56b}$$

which is always positive and in general much larger than (55).

In the symmetrical theory, we can also say definitely that $^3S+^3D_1$ must be lower than ^1S, a statement which was not possible in the neutral theory. The reason is that the central force has the same value for ^1S and ^3S, *viz.*, (50a), and that the diagonal term of the tensor interaction vanishes for ^3S; the nondiagonal term which couples ^3S to $^3D_1$ will then lower ^3S below ^1S. Thus the $^3S+^3D_1$ state is the lowest of all states.

A considerably different situation obtains with respect to the relative orientation of $\mathbf{r}$ and $\mathbf{S}$, and especially to the quadrupole moment of the ground state. For odd L, as we know, the symmetrical theory agrees with the neutral theory; for even L, on the other hand, the sign of the whole interaction, and particularly the tensor interaction, is reversed; therefore in the low states of even L, $\mathbf{r}$ and $\mathbf{S}$ will tend to be perpendicular. The ground state, $^3S+^3D_1$, falls in this class. Therefore, according to the symmetrical theory, the deuteron will have a flat shape, with the short axis in the direction of the spin. It will therefore have a negative quadrupole moment, in contradiction to the experimental result.

§11. The ^1S State

We calculate the radial wave function u of the ^1S state for zero energy. u obeys the Schrödinger equation

$$d^2u/dx^2-Vu=0. \tag{57}$$

Outside the cutting-off distance x_0, we have

$$V=-ae^{-x}/x \quad (x>x_0). \tag{58}$$

For $x<x_0$, we consider the two alternative ways of cutting off discussed in §6, *viz.*

(a) The "zero cut-off"

$$V=0 \quad \text{for} \quad x<x_0. \tag{58a}$$

(b) The "straight cut-off"

$$V=V_0=-ae^{-x_0}/x_0 \quad \text{for} \quad x<x_0. \tag{58b}$$

The inner boundary condition on u is

$$u(x=0)=0, \tag{59}$$

since $u=x\psi$. The outer boundary condition may be written

$$u(x\rightarrow\infty)=c(1+\gamma x), \tag{60}$$

where c is an arbitrary normalization constant and γ is determined by the scattering of slow neutrons by protons.

The scattering cross section for slow neutrons is, neglecting the unimportant effect of the finite range of the forces (Part I, reference 27, p. 117)

$$\sigma=\frac{\pi}{\kappa^2}\left(\frac{1}{\gamma^2}+\frac{3}{\epsilon^2}\right), \tag{61}$$

where ϵ^2 is the binding energy of the deuteron in our units (cf. §7) and γ the characteristic constant of the singlet state as introduced in (60). With the value of $1/\kappa$ as given in (8b), we have

$$\pi/\kappa^2=1.50\times10^{-25}\ \mathrm{cm}^2. \tag{61a}$$

Further, according to (39a), $\epsilon=\frac{1}{2}$. The experimental value of the cross section is not very accurately known because experiments with thermal neutrons involve the effect of chemical binding which has not been calculated accurately, while experiments at higher energies are difficult and involve large experimental errors. The most recent experimental value at thermal energies is that of Carroll and Dunning,[40] *viz.*, 50.7×10^{-24} cm^2. The reduction factor for chemical binding was calculated by Arley[41] who found 2.77. This gives $\sigma=18.3\times10^{-24}$ cm^2. Direct experiments with Ag and I resonance neutrons above thermal energies) give a lower value,[42] *viz.*, 14.8×10^{-24}. Our calculations are not very sensitive to the exact value of the cross section but mainly to the fact that it is large. We have used an average of the two experimental values, *viz.*,

$$\sigma_{\mathrm{exp}}=16.8\times10^{-24}\ \mathrm{cm}^2. \tag{61b}$$

With (61) (61a) (61b) and $\epsilon=\frac{1}{2}$, we obtain

$$\gamma=0.1000. \tag{61c}$$

Thus the wave function (60) has almost horizontal tangent outside the range of the force (58). The positive sign of γ indicates that the ^1S state is virtual rather than real.[43]

[40] H. Carroll and J. R. Dunning, Phys. Rev. **54**, 541 (1938).
[41] N. Arley, Kgl. Dansk. Akad. **16**, 1 (1938).
[42] L. Simon, Phys. Rev. **55**, 792 (1939).
[43] F. G. Brickwedde, J. R. Dunning, H. J. Hoge and J. H. Manley, Phys. Rev. **54**, 266 (1938); J. Halpern, I. Estermann, O. C. Simpson and O. Stern, Phys. Rev. **52**, 142 (1938); L. W. Alvarez and K. S. Pitzer, Phys. Rev. **55**, 596 (1939).

Since γ is given experimentally, the solution of the wave equation will give the interaction constant a as a function of the cutting-off distance x_0 and of the method of cutting off ("zero" or "straight"). The integration for $x<x_0$ is elementary, we have for zero cut-off

$$u=\alpha x\quad\text{(zero)} \tag{62a}$$

(α arbitrary) and for straight cut-off

$$u=\alpha\sin\left[(ae^{-x_0}/x_0)^{\frac{1}{2}}x\right]\quad\text{(straight)}. \tag{62b}$$

For $x>x_0$, we integrate from the outside in. For large x (greater than 2 or 3), an analytical solution is used, for smaller x, numerical integration for each value of a. The point x_0 itself is determined from the condition that the outside and the inside solutions must join smoothly.

In the outside solution, we expand in powers of the potential. Obviously, such an expansion will converge rapidly for large x. We shall use it for $x>2$ when $a\leq2$, and only for $x>3$ when $a>2$. We write:

$$u=u_1+\gamma u_2, \tag{63}$$

where both u_1 and u_2 satisfy the wave equation

$$d^2u/dx^2=-a(e^{-x}/x)u, \tag{63a}$$

while their asymptotic behavior is given by

$$\lim_{x\to\infty}u_1=1, \tag{63b}$$

$$\lim_{x\to\infty}u_2=x. \tag{63c}$$

The integration gives up to terms of order a^2:

$$u_1=1-a(xEi(-x)+e^{-x})+a^2(e^{-2x}+e^{-x}Ei(-x)+(2x-1)Ei(-2x))+\cdots, \tag{64a}$$

$$u_2=x-ae^{-x}+a^2(\tfrac{1}{2}e^{-2x}+xEi(-2x))+\cdots, \tag{64b}$$

where $Ei(-x)$ is the exponential integral (cf. Jahnke-Emde, Table of Functions). The terms in a^3 can be estimated; they are only noticeable in the derivatives du/dx to which they contribute about one-thousandth of the terms linear in a.

For x between x_0 and $x_1=2$ or 3, the integration of the wave equation (63a) must be done numerically. To start the numerical integration, u and du/dx are calculated from ((63), (64)) for the joining point x_1 (see below), and then the integration carried on to smaller x. As shown by

Hartree,[44] the numerical integration is most convenient if

$$y=\log x \tag{65}$$

is used as the independent variable, and

$$F=ux^{-\frac{1}{2}}=x^{\frac{1}{2}}\psi \tag{65a}$$

as the dependent variable.

The numerical integration itself was carried out using Hartree's method, i.e., calculating the wave fundtion F at regular intervals y, $y-h$, $y-2h$, etc., and the derivative $F'=dF/dy$ at the midpoints $y-\frac{1}{2}h$, $y-\frac{3}{2}h$, etc. The interval h was chosen equal to 0.2 units of the natural logarithm; four significant figures were carried in F' and F''. As a rule, the numerical integration for a given a took about half an hour.

The starting point for the numerical integration was in general $y=0.7$, corresponding to $x=2.014$, while the first point for F' was $y=0.6$ ($x=1.822$). The values $a=1.6$, 1.7, 1.8, 2.0 were used. For larger values of a, the analytical formulae would not be accurate enough at $y=0.7$ and 0.6; therefore, the numerical integration was in this case started at $y=1.1$, $x=3.0042$ for F and $y=1.0$, $x=2.7183$ for F'. This was done for $a=2.5$, 3, 4 and 5.

The boundary condition at x_0 is a smooth join of wave function and derivative to the interior solution (62a) or (62b), respectively This means

for zero cut-off:

$$\left(\frac{1}{F}\frac{dF}{dy}\right)_{x_0}=\frac{x}{F}\frac{dF}{dx}=\frac{x}{u}\frac{du}{dx}-\frac{1}{2}=\frac{1}{2}, \tag{66a}$$

for straight cut-off:

$$\left(\frac{1}{F}\frac{dF}{dy}\right)_{x_0}=\frac{(ax_0e^{-x_0})^{\frac{1}{2}}}{\tan\,(ax_0e^{-x_0})^{\frac{1}{2}}}-\frac{1}{2}. \tag{66b}$$

In addition to these calculations with a cut-off potential, we calculated a for the case of no cutting off ($x_0=0$). In this case it was obviously not convenient to continue the numerical integration down to $x=0$. An analytical solution of (63a) was therefore obtained, *viz.*,

$$u=z^{\frac{1}{2}}J_1(2z^{\frac{1}{2}})+\tfrac{1}{3}ax^2J_2(2z^{\frac{1}{2}})-0.0417a^2x^4+(0.0083a^2+0.0229a^3)x^5+\cdots \tag{67}$$

where

$$z=ax, \tag{67a}$$

J_1 and J_2 are Bessel functions. The value of a was found by trial and error: For a given a, u was calculated by numerical integration outwards; then the solution joined to one of the type (63) and γ determined. We found $\gamma=0.1496$ for $a=1.4$ and $\gamma=0.0940$ for $a=1.5$. By interpolation we obtained $a=1.489$ for the "experimental value" $\gamma=0.1000$.

In Table II, the resulting relation between a and x_0 is given. It is seen that for moderate x_0, a changes very slowly with x_0. From the general arguments of the field theory given in §6, we expect a cutting-off distance x_0 of about $\frac{1}{3}$. According to Table II, this would mean a value of a of about 1.6 to 1.7. Thus we may say that a is almost exactly determined by the position of the singlet level of the deuteron, provided only a reasonable assumption is made about x_0. The position of the triplet state can then only be adjusted by varying x_0.

Table II has been extended to larger values of x_0 because these are needed for the symmetrical theory (§17). Also included in the table are the values of $x_0/(a-a_0)^{\frac{1}{2}}$ where $a_0=1.489$ is the value of a for $x_0=0$. This quantity is seen to vary slowly with a and is therefore most convenient for (graphical) interpolation.

That x_0 will, at least as long as it is small, be proportional to $(a-a_0)^{\frac{1}{2}}$ can be seen from a simple calculation. For small x, (67) reduces to

$$u=z-\tfrac{1}{2}z^2+\cdots. \tag{68}$$

u is the "regular" solution of the differential equation (63a). Now suppose the potential is zero for $z<z_0$. Then in this region,

$$\psi\sim z \qquad (z<z_0). \tag{68a}$$

(We write ψ for the wave function for arbitrary z_0 and reserve u for the case $z_0=0$.) When this "inside function" is joined to a solution of (63a), the latter will be a linear combination of the regular solution (69) and the irregular one, *viz.*,

$$v=1-z\log z+\cdots. \tag{69}$$

We write

$$\psi=u+\alpha v. \tag{69a}$$

Using the boundary condition of smooth join at $z=z_0$, we find (cf. 68a)

$$\alpha=-\tfrac{1}{2}z_0^2. \tag{69b}$$

[44] D. R. Hartree, Phys. Rev. **46**, 738 (1934); Proc. Camb. Phil. Soc. **24**, 89, 111 (1928).

TABLE II. *Relation between cutting-off distance x_0 and interaction constant a for the singlet state.*

a	1.489	1.6	1.7	1.8	2	2.5	3	4	5
x_0 Zero Cut-off	0	0.2447	0.3554	0.4458	0.5959	0.8933	1.121	1.486	1.732
x_0 Straight Cut-off	0	0.4056	0.5687	0.7012	0.9196	1.3295	1.635	2.101	2.428
$x_0/(a-a_0)^{\frac{1}{2}}$(Zero)	0.689	0.735	0.773	0.799	0.834	0.888	0.912	0.926	0.924
$x_0/(a-a_0)^{\frac{1}{2}}$(Straight)	1.194	1.217	1.238	1.257	1.287	1.322	1.330	1.326	1.296

The regular function (68) satisfies the boundary condition (60) at large x for $a=a_0$. The new function (69a) will satisfy the same boundary condition if a is changed by an amount proportional to α, so that

$$a-a_0=c'z_0^2=cx_0^2. \tag{70}$$

The constants c' and c could only be determined by actually calculating v. It is simpler to obtain c from an extrapolation of the data in Table II.

For straight cut-off, we have instead of (68a)

$$\psi\sim\sin(z/z_0^{\frac{1}{2}}) \tag{71}$$

and therefore for the condition of smooth join

$$\left(\frac{z}{\psi}\frac{d\psi}{dz}\right)_{z_0}=\frac{z_0^{\frac{1}{2}}}{\tan z_0^{\frac{1}{2}}}=1-\tfrac{1}{3}z_0+\cdots. \tag{71a}$$

Solving for α, we obtain

$$\alpha=-\tfrac{1}{6}z_0^2. \tag{71b}$$

Therefore for straight cut-off (cf. (70))

$$a-a_0=\tfrac{1}{3}cx_0^2, \tag{71c}$$

which shows that the limit, for $x_0\rightarrow 0$, of $x_0/(a-a_0)^{\frac{1}{2}}$ is $3^{\frac{1}{2}}$ times greater for straight cut-off than for zero cut-off.—The same results could also be obtained by treating the cutting off as a perturbation of the potential (63a) and applying the Schrödinger-Born theory.

§12. The Triplet State (Ground State), Calculation

As has been shown in §8–10, the ground state of the deuteron is a mixture of a ^3S and a $^3D_1$ state. For convenience, we shall denote the respective radial wave functions of the S and D component by χ and φ. Then the complete wave function is

$$\psi=x^{-1}[\chi(x)F_{10M}+\varphi(x)F_{12M}], \tag{72}$$

where the F_{JLM} are the angular functions defined in (44). The radial wave functions satisfy the two simultaneous differential equations

$$\begin{aligned} d^2\chi/dx^2&=A\chi-B\sqrt{2}\varphi,\\ d^2\varphi/dx^2&=(A+B+6/x^2)\varphi-B\sqrt{2}\chi, \end{aligned} \tag{73}$$

where

$$A=ae^{-x}/3x+\epsilon^2, \tag{73a}$$

$$B=ae^{-x}\left(\frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{3x}\right). \tag{73b}$$

The potential will be cut off at small distances, and we consider again the two alternatives

(a) Zero cut-off

$$A=B=0 \quad \text{for} \quad x<x_0. \tag{74a}$$

(b) Straight cut-off

$$\left.\begin{aligned} A&=A_0=e^{-x_0}/3x_0+\epsilon^2,\\ B&=B_0=e^{-x_0}\left(\frac{1}{x_0^3}+\frac{1}{x_0^2}+\frac{1}{3x_0}\right), \end{aligned}\right\}\quad (x<x_0). \tag{74b}$$

Because of the strong divergence of B (73b), the result will be very sensitive to the value of x_0. The "experimental" value of ϵ is $\frac{1}{2}$ (cf. (39a)).

The conditions on the wave functions $\chi\varphi$ are that they vanish for $x\rightarrow\infty$ and for $x=0$. Disregarding these conditions, there would be four linearly independent solutions of the two simultaneous equations (73). Two of these solutions may be chosen so as to be finite at ∞; they will then behave as $e^{-\epsilon x}$ as can easily be seen from (73) (73a). The two others will become infinite as $e^{+\epsilon x}$. Alternatively, the four solutions may be combined in such a way that two of them vanish at $x=0$, the two others will then not vanish. But for a given value of x_0 and ϵ, there will in general not be any solution[45] which vanishes both at $x=0$ and $x=\infty$. For a given x_0, there will only be certain eigenvalues of ϵ for which such a solution exists. Actually, our procedure is the reverse: Since ϵ is known from experiment, we seek the value of x_0 which permits a regular solution for the given ϵ (and a).

The integration is similar to that of the preceding section but complicated by the fact that we have now two pairs of solutions where we had only a single solution before.

Inside solution

The inside solution is trivial in the case of zero cut-off; we have then the two independent regular solutions

$$\chi_3=c_3x \qquad \varphi_3=0 \tag{75a}$$

and

$$\chi_4=0 \qquad \varphi_4=c_4x^3 \tag{75b}$$

with arbitrary constants c_3, c_4.

For the straight cut-off, the solutions are

[45] From the two regular solutions at small x, we can always form a linear combination such that its continuation to large x does not contain *one* of the functions which become infinite for $x=\infty$. However, the selected linear combination *will* in general contain the other of these functions. The condition for an eigenvalue is then that the coefficient of this other irregular function also vanishes. This is clearly one condition and can be satisfied by the variation of one parameter, e.g., ϵ.

much more complicated. For practical purposes it is convenient to make use of the fact that B (cf. (74b)) is much larger than A if x_0 is of the expected magnitude (about 0.4). If A is neglected entirely, the solution of Eq. (73) will be a function of

$$z = B_0^{\frac{1}{3}} x \tag{76}$$

only. The influence of A can then be considered by expanding in powers of

$$b = \frac{A_0}{B_0} = \frac{\frac{1}{3}a + x_0\epsilon^2 e^{x_0}}{a(x_0^{-2}+x_0^{-1}+\frac{1}{3})}; \tag{76a}$$

in practice, only terms up to b^2 need to be considered. In these variables, the wave equation (73) becomes

$$d^2\chi/dz^2 = b\chi - \sqrt{2}\varphi,$$
$$d^2\varphi/dz^2 = -\sqrt{2}\chi + (1+b+6/z^2)\varphi. \tag{76b}$$

Both the regular solutions of (76b) contain only odd powers of z, one of them contains also a logarithmic term. Similarly as in (75), there is one solution in which χ is the "leading" wave function and one in which it is φ. The first pair of solutions is

$$\chi_3 = z - 0.00900z^5 - \cdots + \log z(0.020z^5 + \cdots) + b(0.1667z^3 - \cdots) + b\log z(\cdots) + \cdots, \tag{77a}$$

$$\varphi_3 = 0.01298z^5 + 0.01202z^7 + \cdots - \log z(0.2828z^5 + \cdots) + b(0.0113z^5 + \cdots) + \cdots. \tag{77b}$$

The second pair in which φ is the leading function does not contain the $\log z$:

$$-\chi_4 = 0.07071z^5 + 0.002405z^7 + \cdots + b(0.0041z^7 + \cdots), \tag{77c}$$

$$\varphi_4 = z^3 + 0.0714z^5 + \cdots + b(0.0714z^5 + \cdots) + \cdots. \tag{77d}$$

The values of χ and φ and their derivatives at x_0 were computed from these formulae to 4 significant figures. The value of z corresponding to x_0 is (cf. (74b), (76))

$$z_0 = [ae^{-x_0}(1/x_0 + 1 + x_0/3)]^{\frac{1}{3}}. \tag{78}$$

Outside solution

For very large x, a solution of (73) was obtained in terms of a power series. Unlike the case of the singlet state, only the terms without a and those linear in a can be expressed in terms of tabulated functions, this difference being due to the energy term ϵ^2 in the wave equation. There is again one pair of functions in which χ is "leading" while φ vanishes for $a=0$, and another pair in which the reverse is true.

The pair in which χ is the leading function is, up to terms linear in a:

$$\chi_1 = e^{-\epsilon x} + (a/6\epsilon)G(x), \tag{79a}$$

$$\varphi_1 = -\frac{\sqrt{2}a}{8\epsilon^4}\left\{\left[\frac{2\epsilon^3+3\epsilon-3}{x^2} + \frac{(2\epsilon-1)\epsilon^2}{x}\right]e^{-(1+\epsilon)x} + \frac{3-4\epsilon^2}{2\epsilon}F(x)\right\}, \tag{79b}$$

where

$$G(x) = -e^{-\epsilon x}Ei(-x) + e^{\epsilon x}Ei[-x(1+2\epsilon)], \tag{79c}$$

$$F(x) = -e^{-\epsilon x}Ei(-x)\left(\frac{1}{x^2}+\frac{\epsilon}{x}+\frac{\epsilon^2}{3}\right) + e^{\epsilon x}Ei(-x(1+2\epsilon))\left(\frac{1}{x^2}-\frac{\epsilon}{x}+\frac{\epsilon^2}{3}\right). \tag{79d}$$

The second pair of functions in which φ is the leading one is:

$$\chi_2 = -\frac{\sqrt{2}a}{12}\left\{\left[\frac{1}{x^3}+\frac{1+\epsilon}{x^2}+\frac{2\epsilon-1}{x}\right]e^{-(1+\epsilon)x} + \frac{3-4\epsilon^2}{12\epsilon}G(x)\right\}, \tag{80a}$$

$$\varphi_2 = e^{-\epsilon x}\left(\frac{1}{x^2}+\frac{\epsilon}{x}+\frac{\epsilon^2}{3}\right) + a\left\{\left[\frac{1}{6x^3} + \left(-\frac{3}{16\epsilon^4}+\frac{3}{16\epsilon^3}+\frac{3}{8\epsilon^2}-\frac{1}{4\epsilon}+\frac{1}{6}+\frac{1}{12}\epsilon\right)\frac{1}{x^2} + \frac{(2\epsilon^2+3)(2\epsilon-1)}{48\epsilon^2 x}\right]e^{-(1+\epsilon)x} + \left[\frac{3}{32\epsilon^5}-\frac{5}{16\epsilon^3}+\frac{1}{12\epsilon}\right]F(x)\right\}. \tag{80b}$$

TABLE III. *Cut-off distances x_0 for the ground state.*

	Zero Cut-off	Straight Cut-off
Neutral Theory, $a=1.6$	0.3092	0.4051
Neutral Theory, $a=2$	0.3628	0.4989
Symmetrical Theory, $a=2$	1.0560	1.374
Symmetrical Theory, $a=3.2$	1.2975	1.733

These functions simplify considerably when the value $\frac{1}{2}$ is inserted for ϵ.

The terms in a^2 were calculated using an approximate half-analytical procedure. For this purpose, the terms linear in a in the wave functions (79), (80) were approximated as closely as possible by exponentials, and the same was done for the terms proportional to a in the wave equation ((73), (73a,b)). The term $6/x^2$ which is not important for large x was replaced by a suitable average. The resulting wave equation could then be integrated in an elementary way in exponentials. The terms in a^2 never contributed more than 1 part in 1000 to the main wave functions χ_1 and φ_2 and 1 part in 100 to their derivatives, at the point where numerical integration was started.

Intermediate region

In the region between x_0 and $x_1=2$ or 3, the wave equations (73) were integrated numerically. The method used was the same as in §11, i.e., $y=\log x$ was introduced as independent variable, and

$$g=x^{-\frac{1}{2}}\chi \qquad f=x^{-\frac{1}{2}}\varphi \tag{81}$$

as dependent variables. Because of the rapid change of the potentials A and B and of the wave functions, it was necessary to use a smaller interval, *viz.*, $h=0.1$. The numerical integration was started at $y_1=0.7$ for the neutral theory, at $y_1=1.1$ for the symmetrical theory.

The functions g_1, f_1, g_2, f_2 increase exponentially towards smaller x. The actual wave function, on the other hand, will reach a maximum and then decrease; it must therefore be a small difference between the large functions g_1, f_1 and g_2, f_2. To improve the accuracy and save labor we calculated, instead of the second pair of functions, a linear combination of the two pairs which was somewhat closer to the actual wave function (but still "on the side of" g_2, f_2). We chose for the neutral theory

$$g_{(2)}=0.6g_2+g_1, \qquad f_{(2)}=0.6f_2+f_1. \tag{82a}$$

For the symmetrical theory and $a=2$, we took

$$g_{(2)}=2g_2-g_1, \qquad f_{(2)}=2f_2-f_1, \tag{82b}$$

while for greater a we used g_2, f_2 themselves. Even so, it was necessary to carry 5 significant figures in g_1, f_1, g_2, f_2 for small x in order to get the fourth figure in the result reasonably correct.

The increased number of wave functions (four instead of one), the smaller interval (0.1 instead of 0.2), the greater accuracy required, and the more complicated differential equation, all together make the integration about 20 to 30 times more laborious than in the case of the singlet state. The integration was therefore carried out only for very few values of a. Table II suggests that the correct value of a will be in the neighborhood of 1.6; this value was therefore chosen. Even before the singlet integration was performed, the triplet state had been calculated with $a=2$. These two values, $a=1.6$ and 2, were sufficient to determine the correct a and x_0 and the correct wave function by interpolation.

All the foregoing formulae and discussions hold for the neutral theory. In the symmetrical theory, the potential has to be multiplied by -3 for the ground state (§4). This can be done formally by replacing a by $-3a$. This makes obviously the convergence of the expansion in powers of a worse; therefore y_1 was chosen equal to 1.1 instead of 0.7. The first integration was made with $a=2$ but this was found considerably too small. $a=3.2$ was estimated to be about correct for the straight cut-off, and this estimate was confirmed by actual integration. For the zero cut-off, a was found to be about 3.8 by extrapolation.

Determination of cut-off distance

x_0 is determined by the condition of continuity of the functions and their derivatives, *viz.*,

$$\begin{aligned} c_1g_1+c_2g_2&=c_3g_3+c_4g_4\\ c_1f_1+c_2f_2&=c_3f_3+c_4f_4\\ c_1dg_1/dy+c_2dg_2/dy&=c_3dg_3/dy+c_4dg_4/dy\\ c_1df_1/dy+c_2df_2/dy&=c_3df_3/dy+c_4df_4/dy. \end{aligned} \tag{83}$$

The functions 1, 2 are the outer solutions, obtained by numerical integration, the functions 3, 4 are the inside solutions. Since the functions are all given (for given a), the equations (83) represent four linear homogeneous equations for the four coefficients $c_1c_2c_3c_4$. Therefore the determinant must vanish

$$\Delta\equiv\begin{Vmatrix} g_1 & g_2 & g_3 & g_4\\ f_1 & f_2 & f_3 & f_4\\ dg_1/dy & dg_2/dy & dg_3/dy & dg_4/dy\\ df_1/dy & df_2/dy & df_3/dy & df_4/dy \end{Vmatrix}. \tag{84}$$

(84) is the equation for the determination of x_0. (Strictly speaking, x_0 is the largest root of (84).)

For zero cut-off, the inside functions are very simple (cf. (75a,b)), and (83), (84) reduce to

$$-\frac{c_1}{c_2}=\frac{dg_2/dy-\frac{1}{2}g_2}{dg_1/dy-\frac{1}{2}g_1}=\frac{df_2/dy-(5/2)f_2}{df_1/dy-(5/2)f_1}. \tag{84a}$$

The two fractions in the middle and on the right-hand side were computed from the numerical solution at various points; the point where they are equal was then found by interpolation. For the straight cut-off, the inside functions are more complicated. In this case, the determinant (84) itself was calculated at various points and then interpolated.

The results obtained are given in Table III. For the neutral theory and straight cut-off, $a=1.6$ gives (accidentally) almost exactly the same value of x_0 as was obtained for the same theory and the same a for the ^1S state (cf. Table II, 0.4051 and 0.4056). This shows that 1.6 is the correct value of a for this theory. Extrapolation using the result for $a=2$ gives $a=1.5997$, $x_0=0.4050$. For the final wave function, we took the one computed for $a=1.6$.

In the case of *zero cut-off*, *neutral theory*, $a=1.6$ gives a larger x_0 for the triplet state (0.3092) than for the singlet state (0.2447) while for $a=2$ the relation is reversed (0.3628 and 0.5959). That x_0 will depend less strongly on a for the ground state is to be expected from the strong divergence of the potential in this state. Since x_0/a (in the triplet state) does not change much from $a=1.6$ to 2 this quantity was used for (linear) interpolation. Then the values of x_0 for triplet and singlet state agree for $a=1.664$. Direct linear interpolation of x_0 for the triplet state would give $a=1.663_3$. The corresponding value of x_0 is 0.318, by both methods of interpolation.

For the symmetrical theory, straight cut-off, $a=3.2$ gives approximate agreement between triplet and singlet state (1.733 and 1.741). If x_0^2/a for the triplet state is linearly interpolated between $a=2$ and 3.2, triplet and singlet will intersect at $a=3.16_5$, $x_0=1.725$. The triplet wave function for $a=3.2$ was taken as final.

The correct value of a for the symmetrical theory, zero cut-off, is obviously greater than 3.2. In view of the generally unsatisfactory results for this theory (too large x_0) it was considered sufficient to extrapolate the results for $a=2$ and 3.2. We assumed x_0^2 for the triplet state to be a linear function of a and found that $a=3.76$ will make x_0 for singlet and triplet agree and will give $x_0=1.396$. The value of a may easily be in error by ± 0.05 unit and x_0 by ± 0.02 unit, these estimates being obtained by using different methods of extrapolation. The wave function was also obtained by extrapolation from the results for $a=2$ and 3.2, assuming linearity in a. This assumption was checked by the correct behavior of the wave function near x_0, *viz.*, $\chi \sim x^{\frac{1}{2}}$ and $\varphi \sim x^{2.5}$.

The wave functions obtained as described were normalized by the condition (46), *viz.*,

$$\int_0^\infty (\chi^2+\varphi^2)dx=1. \tag{85}$$

TABLE IV. *Results.*

	Neutral Theory		Symmetrical Theory	
	Zero Cut-off	Straight Cut-off	Zero Cut-off	Straight Cut-off
a	1.664	1.600	3.76	3.16_5
$f^2/\hbar c$	0.0800	0.0770	0.181	0.152
x_0	0.318	0.405	1.396	1.725
Max. of $B\sqrt{2}$ (Mev)	625	288	38.4	15.1
c_1	0.323	0.322	0.286	0.267_5
c_2	0.122_5	0.119	-0.446	-0.390
Percent D	6.80	6.63	18.03	18.52
Q in 10^{-27} cm^2	2.70_5	2.61_5	-20.0	-17.8_5
r'^2_{Av} in 10^{-26} cm^2	3.31	3.28	8.31	7.51
$\epsilon=Q/r'^2_{\text{Av}}$	0.081_5	0.080	0.241	0.238

§13. Results for the Neutral Theory

The calculations in the preceding section gave cut-off distances $x_0=0.405$ for the straight cut-off and 0.318 for the zero cut-off. These values are very reasonable indeed, and are, in fact, just of the order of magnitude which we expected from general considerations of the field theory in §6. It was mentioned in that section that Møller and Rosenfeld found the ratio of second-order to first-order interaction to be about

$$f^2/\hbar c x^2. \tag{86}$$

Our calculations give about 0.08 for $f^2/\hbar c$ (cf. Eq. 87) and therefore 0.5 or 0.8 for the above ratio for straight and zero cut-off, respectively. These figures are quite close to unity.

That x_0 must be smaller for the zero cut-off is evident: When the potential is zero inside, instead of having a finite value corresponding to attraction, this must be compensated by a larger potential outside; and the outside potential is increased most effectively by making x_0 smaller.

A cut-off at $x_0=0.3$ or 0.4 represents a relatively slight modification of the Yukawa potential ae^{-x}/x. This can be seen from the small change of a necessary to compensate the effect of the cutting off for the singlet state: Our a is 1.6 or 1.66 instead of the value 1.489 found without cutting off (§11). In particular, the general behavior of the singlet potential is

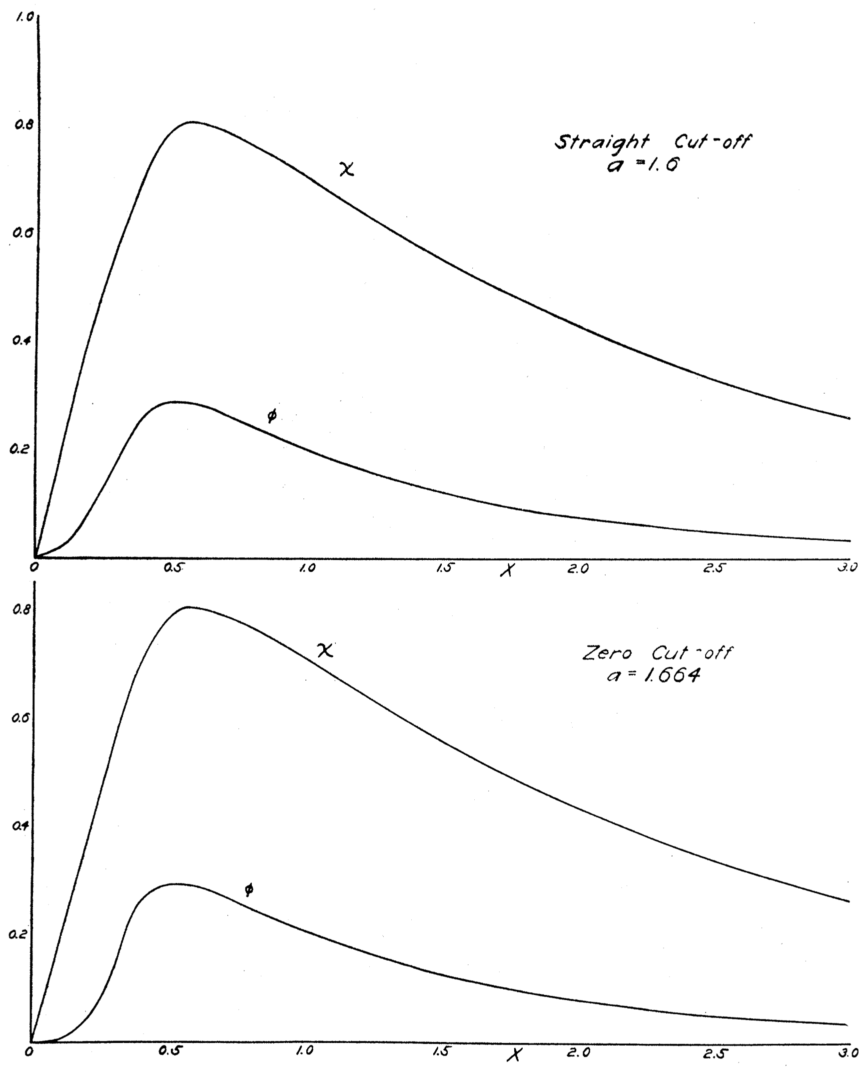

FIG. 1. Radial wave functions of the S and D state of the deuteron. Neutral theory, two ways of cutting off.

preserved, *viz.*, the exponential decrease at large distances and the stronger than exponential increase at small x. Especially the latter may be helpful for removing the discrepancies in the older theory of the three- and four-body problems.[46]

The interaction constant a does not depend appreciably on the method of cutting off. This is satisfactory and makes it probable that the actual theory which may be developed in the future will give a similar value for a. The most important reason for this insensitivity is that a is principally determined by the singlet state, and that this state is not sensitive to the cutting off (§11). The same fact makes a also insensitive to the mass of the meson. A change of μ would only change the natural unit of length $1/\kappa$, and therefore also the constant γ, in the inverse ratio of μ. Because of the small value of γ, even a change of the meson mass by a factor of two either way would cause only a small absolute change of γ, and therefore a small change of a. We estimate that a doubling of μ would increase a by about 0.08, i.e., five percent.

[46] W. Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).

The actual value of a is considerably less than that reported in previous papers. Sachs and Goeppert-Mayer[47] find $a=2.83$ for $\mu \doteq 177m$. This difference is due to the fact that these authors assumed that the *triplet state* is determined by a potential of the Yukawa type ae^{-x}/x; the lower energy of the triplet state requires of course a larger a than the one found in this paper from the singlet state.

The absolute value of the interaction is most easily appreciated when instead of a we use the quantity $f^2/\hbar c$. From (40), we have

$$f^2/\hbar c=(\mu/2M)a. \tag{87}$$

Therefore, with our value $\mu=177m=0.096_2M$, we find

$$f^2/\hbar c=0.0800 \quad \text{for zero cut-off,} \tag{87a}$$

$$f^2/\hbar c=0.0770 \quad \text{for straight cut-off.} \tag{87b}$$

This means that f^2 is about 11 times the square of the electronic charge, e^2, or about the geometrical mean between e^2 and $\hbar c$.

In Table IV, we have listed the more important results. After the values of a and x_0, we have given the maximum value (obtained at x_0) of the "coupling potential" between S- and D-state, $B\sqrt{2}$ (cf. (73), (73b)). It is seen that this quantity does not reach excessive values but remains considerably smaller than the rest energy of the deuteron (1860 Mev). This makes it improbable that the cutting off is due to simple relativistic effects (cf. §6).

The next two rows in Table IV give the coefficients c_1 and c_2 in the normalized wave function

$$\begin{aligned}\chi&=c_1\chi_1+c_2\chi_2\\ \varphi&=c_1\varphi_1+c_2\varphi_2.\end{aligned} \tag{88}$$

These coefficients are seen to be almost the same for straight and zero cut-off, and what difference there is, is due to the difference in a. For the same a, the coefficients for the two methods of cutting off agree within the accuracy of the calculation (about 1 in 500). The reason is that a small change in these coefficients would mean a considerable change in the shape of the wave function, owing to the rapid increase of $\chi_1\varphi_1\chi_2\varphi_2$ towards smaller x.

[47] R. G. Sachs and M. Goeppert-Mayer, Phys. Rev. **53**, 991 (1938).

The wave functions χ and φ themselves are shown in Fig. 1 for the two ways of cutting off. The two graphs are seen to be very similar. Of the two wave functions, χ behaves very similar to the usual wave function of the deuteron and has a flat maximum at $x\approx 0.6$, i.e., outside the cut-off distance but within the range of the nuclear forces. At large distances, χ will always behave as $e^{-\epsilon x}$ independent of the special form of the nuclear forces as long as the forces have short range.[48] The radial wave function of the D state, φ, has a steeper maximum at a slightly smaller distance ($x\approx 0.5$).

In Table IV, we have also given the percentage of D function contained in the ground state, *viz.*, 100 times the quantity

$$p_D=\int \varphi^2 dx. \tag{88a}$$

This percentage is again almost independent of the method of cutting off as is to be expected from the similarity of the wave function. The value of p_D is 6.6 to 6.8 percent, i.e., rather small, in spite of the large effect which the mixing of S and D state has on the energy. The reason for this is the large value of the coupling potential $B\sqrt{2}$.

§14. The Magnetic Moment

As Schwinger[49] has pointed out, the percentage of D function in the ground state is important for the magnetic moment of the deuteron. For a pure S state, the magnetic moment may be assumed to be equal to the sum of the intrinsic moments of proton and neutron,

$$g_S=\mu_P+\mu_N. \tag{89}$$

In the past the neutron moment has always been calculated from this formula, g_S being identified with the measured magnetic moment of the deuteron. For a pure $^3D_1$ state the moment would be given by the Landé formula

$$g_D=\tfrac{1}{2}(g_S+g_L)+\frac{S(S+1)-L(L+1)}{2J(J+1)}(g_S-g_L) \tag{90}$$

$$=\tfrac{3}{2}g_L-\tfrac{1}{2}g_S, \tag{90a}$$

[48] H. A. Bethe and R. Peierls, Proc. Roy. Soc. **149**, 176 (1934).

[49] J. Schwinger, Phys. Rev. **55**, 235 (1939), Abs. 13; and further unpublished results.

since $L=2$, $S=J=1$. Here g_L is the g value of the orbital momentum, and we have

$$g_L=\tfrac{1}{2}, \tag{90b}$$

since the proton carries one-half of the total orbital momentum of the deuteron. All moments are of course given in nuclear magnetons. If we have a mixture of a $^3S_1$ and a $^3D_1$ state, the moment will be

$$\begin{aligned} g&=(1-p_D)g_S+p_Dg_D \\ &=g_S-\tfrac{3}{2}p_D(g_S-\tfrac{1}{2}). \end{aligned} \tag{91}$$

The cross terms between the S and D wave function do not contribute to the magnetic moment.

The observations give almost exactly $g=g_S$, i.e., the observed moment of the deuteron is within the experimental error equal to the sum of the moments of neutron and proton. The measurements of Kellogg, Rabi, Ramsey and Zacharias[39] give for the deuteron

$$g=0.855\pm 0.006,$$

for the proton

$$\mu_P=2.785\pm 0.02,$$

whereas Bloch and Alvarez[50] find for the neutron

$$\mu_N=-1.93_5\pm 0.02.$$

Therefore the difference between the deuteron moment and the moment expected for a pure S state is

$$g-g_S=0.005\pm 0.03 \tag{91a}$$

nuclear magnetons.

From the theoretical Eq. (107), we should have

$$g-g_S=-0.53p_D. \tag{91b}$$

Therefore a value of p_D of 0.067, as it follows from our theory, would be just outside the limits of the experimental error. However, if it were not known that the deuteron had a quadrupole moment, the result of the experiments would doubtless be taken to indicate exact additivity of the magnetic moments of neutron and proton in the deuteron. Whether a p_D of 6.7 percent is actually permissible, can only be decided after a further increase in the precision of the experimental values of the moments.

Schwinger's theory[49] gives a greater percentage of D function in the deuteron ground state, *viz.*, about 10 percent or more. He introduces a central force and a tensor interaction which are constant inside a certain radius $r_1\approx e^2/mc^2$ and zero outside. The magnitude of the central force is assumed different in triplet and singlet state and is adjusted to give the correct binding energies, and the magnitude of the tensor interaction is adjusted so as to give the observed value of the electrical quadrupole moment. Since our theory also gives the quadrupole moment correctly, the difference between the two theories apparently arises from the different shape of the radial functions. This point requires further consideration. At the moment, Schwinger's result seems somewhat worse than ours.

§15. The Quadrupole Moment

As has been pointed out repeatedly, the deuteron will, according to the present theory, have a quadrupole moment. We define the quadrupole moment in the same way as Kellogg, Rabi, Ramsey and Zacharias,[39] *viz.*,

$$Q=(3z'^2-r'^2)_{\text{Av}}=r'^2(3\cos^2\vartheta-1)_{\text{Av}}, \tag{92}$$

where z' and r' are the coordinates of the proton with respect to the center of gravity of the deuteron, i.e., one-half of the coordinates used in this paper which give the position of the proton with respect to the neutron. The average is to be taken for the magnetic state $M=1$; a positive quadrupole moment thus indicates that the deuteron is elongated along the direction of its spin, and vice versa. Putting $r'=\frac{1}{2}r=x/2\kappa$ we have

$$Q=(1/4\kappa^2)\int x^2(3\cos^2\vartheta-1)\psi^2 d\tau. \tag{92a}$$

The wave function ψ of the state $M=1$ is (cf. (72), (44))

$$\begin{aligned}\psi=&x^{-1}\chi(\kappa)Y_{00}(\vartheta,\varphi)\alpha(1)\alpha(2)+x^{-1}\varphi(x)10^{-\frac{1}{2}}\\ &\times[Y_{20}(\vartheta\varphi)\alpha(1)\alpha(2)+3^{\frac{1}{2}}Y_{21}(\vartheta\varphi)\{\alpha(1)\beta(2)\\ &+\beta(1)\alpha(2)\}/\sqrt{2}+6^{\frac{1}{2}}Y_{22}(\vartheta\varphi)\beta(1)\beta(2)].\end{aligned} \tag{93}$$

[50] L. W. Alvarez and F. Bloch, Phys. Rev. **57**, 111 (1940).

Inserting (93) in (92a) we obtain

$$Q=(1/10\kappa^2)\int x^2dx(\chi\varphi\sqrt{2}-\tfrac{1}{2}\varphi^2). \quad (94)$$

In other words, the $^3D_1$ state *alone* would give a negative contribution to the quadrupole moment $(-\frac{1}{2}\varphi^2)$. The interference term between S and D function, $\chi\varphi\sqrt{2}$, is positive provided χ and φ have the same sign which is the case for our neutral theory. As long as φ is not too large which again is fulfilled in our theory the positive term will predominate. Thus *the quadrupole moment will be positive, in agreement with the experiments of Kellogg, Rabi, Ramsey and Zacharias.*[39] This was already deduced from qualitative considerations in §10.

The integral in (94) was calculated from the wave functions for the two types of cut-off. The result is again almost independent of the cut-off and the small existing difference of about three percent is mostly due to the difference in a. A correct theory would therefore give a similar value provided the principle of cutting off singlet and triplet potential in the same way is preserved. The final result (cf. Table IV) is

$$\begin{aligned} Q&=2.70_5\times10^{-27}\ \text{cm}^2 \quad \text{(zero cut-off)}\\ Q&=2.61_5\times10^{-27}\ \text{cm}^2 \quad \text{(straight cut-off)}. \end{aligned} \quad (95)$$

This value should be compared with the experimental value

$$Q=2.74\times10^{-27}\ \text{cm}^2 \quad \pm2\ \text{percent (experimental)}, \quad (95a)$$

which has been derived by Nordsieck[51] from the experiments of Rabi and collaborators.[39] The agreement is excellent.

This is practically[52] the first time that a calculation in the theory of simple nuclear systems gives a quantitative agreement. In all previous instances, any new phenomenon either could not be treated with the theoretical concepts deduced from the phenomena known previously or disagreed with them. E.g., the cross section for the scattering of slow neutrons did not agree with the value deduced previously from the binding energy of the deuteron but required for its explanation the postulation of the singlet state of the deuteron—a postulate which was well confirmed later. The binding energy of the triton cannot be deduced from deuteron data but can be adjusted at will by changing the range of the forces. The binding energy of the α-particle disagrees with the forces deduced from the two- and three-body systems. In our theory, there are for the first time more experimental data than there are arbitrary parameters: The data are the positions of triplet and singlet state of the deuteron, the mass of the meson and the quadrupole moment. The parameters are the nuclear unit of length, $1/\kappa$, the interaction constant a, and the cut-off distance x_0. This makes it possible to calculate one datum from the three others. This has been achieved by using the principle of simplicity in the fundamental theory, particularly by the assumption that the spin-independent interaction is absent (Single Force Hypothesis, §5). The good agreement obtained justifies this hypothesis.

To judge the significance of Q for the shape of the deuteron we compare it with the average value of r'^2_{Av} where r' is the distance of the proton from the center of gravity. We have

$$\begin{aligned} r_{\text{Av}}'^2&=(1/4\kappa^2)\int(\chi^2+\varphi^2)x^2dx\\ &=3.31\times10^{-26}\ \text{cm}^2 \quad \text{for zero cut-off}\\ &=3.28\times10^{-26}\ \text{cm}^2 \quad \text{for straight cut-off}. \end{aligned} \quad (96)$$

For comparison we note that for infinitely short range of the nuclear forces $r'^2_{\text{Av}}=2.39\times10^{-26}$ cm^2, i.e., the finite range causes a moderate increase in the size of the deuteron as is to be expected. We define the excentricity of the deuteron

$$\epsilon=Q/r'^2_{\text{Av}}. \quad (96a)$$

ϵ would have the value 2 if the proton and neutron were constrained to move on a straight line in the direction of the spin. In our case (cf. (95), (96)) we find $\epsilon=0.081_5$ for the zero

[51] A. Nordsieck, Bull. Am. Phys. Soc., New York Meeting, February, 1940, Abstract No. 9. I am grateful to Dr. Nordsieck for communicating his results to me before publication.

[52] As other instances, the capture cross section of protons for slow neutrons, and the cross section of the photoelectric effect of the deuteron for 2.62-Mev γ-rays may be quoted; also to some extent the energy dependence of the scattering of neutrons by protons. However, the theory of all these phenomena is almost independent of the range and nature of the nuclear forces so that the agreement proves only the short range of the forces. Our case is quite different in this respect.

and 0.080 for the straight cut-off showing that the deuteron does not differ too much from spherical shape.

§16. The $^3P_1$ State

As has been shown in §9, the potential in the $^3P_1$ state is strongly attractive. It would therefore be conceivable that this state might be lower than the $^3S+^3D_1$ state. We have made it plausible in §10 by a semi-quantitative argument that this is not the case. However, because of the importance of this question we have made a direct numerical calculation of the $^3P_1$ state. This calculation was also meant to decide whether the $^3P_1$ state is higher than the ^1S state which question could not be investigated by the method of §10.

We have calculated the $^3P_1$ wave function for zero energy and zero cut-off. The radial wave function obeys the wave equation (cf. (51))

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= \left(\frac{2}{x^2} - a\frac{e^{-x}}{x^3}(1+x)\right)\psi \quad (x>x_0) \\ \frac{d^2\psi}{dx^2} &= \frac{2\psi}{x^2} \quad (x<x_0) \end{aligned} \tag{97}$$

$$x_0=0.318, \quad a=1.664.$$

The inside solution is simply $\psi\sim x^2$. The asymptotic solution for very large x is

$$\psi=\psi_1+\gamma\psi_2 \tag{98}$$

with

$$\psi_1\sim x^2, \quad \psi_2\sim 1/x. \tag{98a}$$

For values of x which are still large but not infinite we expand in powers of a and find:

$$\psi_1=x^2-ae^{-x}(1+1/x)+\tfrac{1}{4}a^2e^{-2x}/x^2+\cdots, \tag{99}$$

$$\begin{aligned} \psi_2=\frac{1}{x}-\frac{a}{24}\left[e^{-x}\left(\frac{6}{x^2}-\frac{2}{x}+1-x\right)-x^2Ei(-x)\right] \\ +\frac{a^2}{360}\left[e^{-2x}\left(\frac{9}{x^3}-\frac{12}{x^2}-\frac{7}{x}-8+16x\right)\right. \\ -15e^{-x}Ei(-x)(1+1/x) \\ \left.+32x^2Ei(-2x)\right]+\cdots. \end{aligned} \tag{99a}$$

The values of ψ_1 and ψ_2 were calculated at $y=\log x=0.7$ and the values of their derivatives at $y=0.65$. Then both solutions were integrated numerically towards smaller x, using the same method as in §11 and 12 and interval $h=0.1$. Finally, a linear combination of the two solutions was joined at $x_0=0.318$ to the inside solution $\psi\sim x^2$. The result is (cf. (98))

$$\gamma=2.39. \tag{99b}$$

This value of γ is positive showing that there is no bound $^3P_1$ state. This confirms the qualitative result of §9 that the $^3S+^3D_1$ state is the ground state. Moreover, the value of γ is not excessively large as it would be if a virtual state existed in the neighborhood of zero energy. This shows that the $^3P_1$ state is also "higher" than the ^1S.

To estimate the scattering of $^3P_1$ neutrons we write down the wave function at a finite energy:

$$\psi=-\cos(kx+\delta)+\sin(kx+\delta)/kx. \tag{100}$$

Here k^2 is the relative energy of neutron and proton in nuclear units of 8.7 Mev, x the distance of the two particles and δ the phase shift. If $kx\ll 1$ but still $x\gg 1$ so that the nuclear forces are negligible, (100) reduces to

$$\psi=\tfrac{1}{3}(kx)^2\cos\delta+\sin\delta/kx. \tag{100a}$$

Comparison with (98) gives

$$\tan\delta\approx\delta=\tfrac{1}{3}k^3\gamma=0.80k^3. \tag{100b}$$

As example let us consider the scattering of D−D neutrons. The kinetic energy is about 3 Mev, therefore the relative kinetic energy $1\frac{1}{2}$ Mev and $k^2=1.5/8.7=0.17$. Eq. (100b) then gives for the phase shift of $^3P_1$ the value $\delta=0.056=3.2°$ which is small but not negligible. In spite of the strong attraction, the $^3P_1$ state is therefore only of slight importance for the scattering of neutrons of moderate energy. The angular distribution of the scattered neutrons will be almost spherically symmetrical as in the older theory and in experiment.[53]

§17. Results of the Symmetrical Theory

The main results of the calculations have already been given in Table IV, §13. The most

[53] The angular distribution of the scattered neutrons as a function of the phase shifts of the ^1S, ^1P, ^3S, $^3P_{0,1,2}$ waves has been given by Hoisington, Share and Breit, Phys. Rev. **56**, 884 (1939).

conspicuous feature is the enormous value of the cut-off distance, $x_0=1.396$ and 1.725 for zero and straight cut-off. It can be said that the potential must be cut off before it has really begun to act. This is also shown by the smallness of the maximum value of the coupling potential $B\sqrt{2}$ which is 38.4 and 15 Mev, respectively.

It is evident that these results are entirely unacceptable from the standpoint of general nuclear theory. A potential which exists only at distances *greater* than 3×10^{-13} cm (zero cut-off) is obviously in violent contradiction with the known short range character of nuclear forces. Such a potential could never explain that the binding energies of the triton and the α-particle are large compared with that of the deuteron; it would lead to nuclear radii at least three times as large as the observed ones, and it would mean a very considerable influence of the P waves in the scattering of 2-Mev protons by protons for which no such influence has been observed.[54] The potential obtained with straight cut-off which is constant up to 4×10^{-13} cm and then falls off rather slowly is equally out of the question.

Moreover, it seems inconceivable that the theory of the meson field should break down at such enormous distances. The relativistic corrections are minute, the potentials being only 38 and 15 Mev. Likewise, the second-order interaction (cf. (86)) should give only 0.09 or 0.05 times the first order. That the field theory would be worthless in the case of such an early breakdown goes without saying.

In view of this situation, it seems hardly necessary to give additional evidence against the symmetrical theory. However, it should be mentioned that the *quadrupole moment has the wrong sign*, negative instead of positive (cf. Table IV, and reference 50). This was already realized by Heitler[55] and it seems very difficult to devise a mechanism to change this result. The enormous size of the quadrupole moment, 7 to 8 times the observed value, is only in keeping with the long range of the nuclear forces.

The reason for these results can be appreciated most easily by a comparison with the neutral theory. The central force V_1 (cf. (34)) has the same value for singlet and triplet state in the symmetrical theory so that the tensor interaction is only responsible for the difference between these two states. In the neutral theory, the central force is repulsive in the triplet state, therefore the tensor interaction must overcome this repulsion, must provide an attraction equal to that in the singlet state, and in addition make the triplet state lower than the singlet. This means that the tensor interaction must have a much smaller effect in the symmetrical than in the neutral theory. But the analytical expression for the tensor interaction, in terms of a and x, is three times larger in the symmetrical theory. Therefore its effect can only be reduced by cutting the interaction off at a very much larger distance x_0.

This effect is further enhanced by the fact that an increase of x_0 means a reduction of the effective potential in the singlet state which must be compensated by an increase of a (cf. Table II). In fact, the final value of a is twice as large or more in the symmetrical than in the neutral theory. An increase of a increases the tensor interaction still further and thus necessitates an even larger cut-off radius x_0, etc.

Our result that the symmetrical theory disagrees violently with experiment is very regrettable indeed. For all *qualitative* arguments are in favor of the symmetrical and speak against the neutral theory. In this connection we do not think primarily of the fact that charged mesons have been observed in cosmic rays while there is up to the present no experimental evidence for neutral ones: The existence of neutral mesons is indispensable for *any* theory of nuclear forces, symmetrical as well as neutral. But the symmetrical theory is capable of explaining at least in principle the *β-disintegration* and the *magnetic moments* of neutron and proton.

There is as yet no actual theory of the extra magnetic moments, and it is already clear that such moments could not be expected in the approximation of the field theory which has been considered in this paper. However, it seems at least possible qualitatively to interpret the moments as due to charged mesons which are present part of the time. With neutral mesons no such possibility seems to exist; moreover, we

[54] G. Breit, H. M. Thaxton and L. Eisenbud, Phys. Rev. **55**, 1018 (1939).

[55] W. Heitler, Report to the Solvay Conference 1939.

have assumed in our theory that neutrons and protons have exactly the same meson fields around them so that it would be impossible for them to have different signs of the extra magnetic moment if that moment is due to the mesons.

Another strong qualitative argument for the symmetrical theory is the β-disintegration. In the symmetrical theory this process will proceed according to Yukawa's scheme, *viz.*,

$$\begin{aligned} N&\rightarrow P+M^-, \quad M^-\rightarrow\epsilon^-+n \\ P&\rightarrow N+M^+, \quad M^+\rightarrow\epsilon^++n \end{aligned} \tag{101}$$

(N=neutron, P=proton, M^+ and M^-=positive and negative meson, ϵ=electron, n=neutrino). Obviously no such scheme for the β-disintegration is available when only neutral mesons interact with the nuclear particle. In the neutral theory it would therefore be necessary to fall back on the original Fermi theory in which a direct interaction was postulated between nuclear particle and electron-neutrino field. There are many points in favor of an indirect interaction via the meson field as in (101): The direct splitting of a nuclear particle into three particles seems to lead to much more serious divergences at high energies than the splitting into two particles postulated in (101). No other instance is known in nature where the fundamental process is a simultaneous emission of two particles whereas the emission of light or of gravitational waves is quite analogous to (101) in that a Bose particle is emitted or absorbed by a Fermi particle. Moreover, the assumption that mesons are an intermediate stage in β-emission obviously provides a much more unified picture of nuclear phenomena; and finally, it seems rather unlikely that the charged meson which, after all, is known to exist should play no role at all in nuclear phenomena.

On the other hand, it is well known that there is a serious quantitative discrepancy in the meson theory of β-decay. As Nordheim[56] has pointed out the observed lifetimes of β-active nuclei require an extremely short life for the meson itself. With the latest data on β-decay, and the most favorable theoretical assumptions, the half-life of the meson comes out[57] to be 9×10^{-9} sec. which should be regarded as an upper limit. With such a short life, the meson could not be observed in cosmic rays whereas actual cosmic-ray data set a *lower* limit of about 2×10^{-6} sec. to the lifetime. The discrepancy of a factor 200 cannot be removed by a reasonable change of the mass μ of the meson; since the theoretical life is inversely proportional to μ^4, the mass would have to be reduced by a factor of 4 to about 45 electron masses which is entirely irreconcilable with direct experimental determinations and also would lead to an impossible range of the nuclear forces, of about 10^{-12} cm.

This quantitative discrepancy in the β theory must be regarded as a serious argument against the symmetrical theory in its present form.

§18. An Alternative Method of Cutting Off

Since the qualitative arguments for the symmetrical theory are certainly very strong we attempted to improve the quantitative agreement by using different ways of cutting off. In a recent discussion in which he emphasized the arguments for the symmetrical theory, Heisenberg suggested to cut off the potential *before* the differentiations implied in Eq. (18). Such a procedure would correspond to a cut-off in momentum space which seems to be the only reasonable way to achieve a relativistically invariant cut-off.[58] Moreover, it would seem promising for a solution of our difficulties because it might be expected that the tensor interaction being a second derivative would be more strongly reduced than the central force, which would indeed bring about better agreement with the experimental positions of triplet and singlet states (cf. §17).

Unfortunately, this is not the case but the results are even less favorable than in our earlier calculations, §12 and 17. Let $\frac{1}{2}v$ be the potential before the differentiation so that the spin-dependent interaction is

$$V=-\tfrac{1}{2}\boldsymbol{\sigma}_1\cdot\text{curl curl }(\boldsymbol{\sigma}_2 v)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2. \tag{102}$$

In the Yukawa theory, $v=(2f^2/\kappa^2)(e^{-\kappa r}/r)$. For

[56] L. W. Nordheim and G. Nordheim, Phys. Rev. **54**, 254 (1938); L. W. Nordheim, *ibid.* **55**, 506 (1939); H. Yukawa, V. and S. Sakata, Nature **143**, 761 (1939).

[57] H. A. Bethe and L. W. Nordheim, to be published shortly in *The Physical Review.*

[58] G. Wataghin, Zeits. f. Physik **88**, 92 (1934); W. Heisenberg, *ibid.* **110**, 251 (1938).

our present investigation, we shall assume that v is a certain unknown function of r which is already "cut off," i.e., which does not diverge strongly at small r. The central force is then

$$V_1=\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\ \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\ \nabla^2 v \qquad (102a)$$

and the tensor force

$$V_2=\tfrac{1}{2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left(-\frac{\boldsymbol{\sigma}_1\cdot\mathbf{r}\ \boldsymbol{\sigma}_2\cdot\mathbf{r}}{r^2}+\tfrac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)r\frac{d}{dr}\left(\frac{1}{r}\frac{dv}{dr}\right). \qquad (102b)$$

We see therefore that both V_1 and V_2 are second derivatives of v so that there is no reason to suppose that V_2 is more strongly reduced by the cutting off than V_1.

In order to investigate the general potential v further, we note that the radial factor in V_1 plays the role of the charge density associated with v. It is therefore easy to calculate, for a given V_1, the corresponding v and V_2. Let us introduce the abbreviations

$$\rho=\nabla^2 v=\frac{1}{r}\frac{d^2}{dr^2}(rv), \qquad (103)$$

$$w=r\frac{d}{dr}\left(\frac{1}{r}\frac{dv}{dr}\right), \qquad (103a)$$

so that ρ is proportional to the central force V_1, w to the tensor interaction V_2. Then we have

$$v(r)=-\int_0^r(1/r)\rho(r')r'^2dr'-\int_r^\infty(1/r')\rho(r')r'^2dr'+A/r+B. \qquad (104)$$

It seems reasonable to require that the potential v at large distances should approach zero and should do so faster than $1/r$. This means

$$B=0, \qquad A=\int_0^\infty\rho(r')r'^2dr', \qquad (104a)$$

i.e., A is the "total charge." Then

$$v=\int_r^\infty(1/r-1/r')\rho(r')r'^2dr'. \qquad (104b)$$

In order to give attraction in the singlet state, ρ must be in the main positive, and therefore v also positive, just as in the original Yukawa theory. For small r, v will behave as

$$v=A/r+c_0+c_1r+c_2r^2, \qquad (104c)$$

where A is the total charge (104a) and $c_1=0$ if ρ is finite at $r=0$. Inserting into (103a), we obtain

$$w=3A/r^3-c_1/r+3c_3r+\cdots. \qquad (105)$$

Thus we see that w diverges as $1/r^3$ unless A is zero. Since any such divergence would lead to infinite binding energy, we have to require

$$A=\int_0^\infty\rho(r)r^2dr=0. \qquad (105a)$$

This means that ρ, and therefore V_1, must change sign somewhere in order to avoid divergences of w. In other words, V_1 is more effectively cut off than V_2, in contrast to the expectation mentioned above. The singlet potential, if attractive at large r, must be repulsive at small r, whereas the tensor interaction may have the same sign for all r.

As a simple example we assume for ρ the following "rectangular" function:

$$\begin{aligned}\rho&=-a &&\text{for}\quad r<r_1\\ \rho&=b &&\text{for}\quad r<r<r_2\\ \rho&=0 &&\text{for}\quad r_2<r.\end{aligned} \qquad (106)$$

The neutrality condition (105a) requires then

$$a=b(r_2^3/r_1^3-1) \qquad (106a)$$

and we have

$$\begin{aligned}w&=br_2^3/r^3 && (r_1<r<r_2)\\ w&=0 && (r<r_1\ \text{and}\ r>r_2).\end{aligned} \qquad (106b)$$

This shows that w is relatively smallest if r_1 is very close to r_2. Since smallness of w is desirable we assume

$$r_2-r_1=\delta\ll r_2. \qquad (107)$$

Then (129a) gives

$$a\approx 3b\delta/r_1. \qquad (107a)$$

The potential for the singlet state is (cf. (102a)) simply $-\rho$. After a simple calculation, and with the abbreviations

$$x=(3b\delta r_1)^{\frac{1}{2}},\quad \eta=\gamma r_1,\quad \gamma=0.100\kappa=4.58\times10^{11}\ \text{cm}^{-1}, \qquad (107b)$$

we obtain

$$\frac{x}{\tanh x}-\tfrac{1}{3}x^2=\frac{\eta}{1+\eta}. \qquad (108)$$

Because of the smallness of γ (cf. (107b)), η will be quite small provided r_1 is of the order of magnitude of the range of the nuclear forces, i.e., a few times 10^{-13} cm. For $\eta=0$, the solution of (108) is

$$x(\eta=0)=3.015. \qquad (108a)$$

For the triplet state in the symmetrical theory we find

$$\left(\frac{z+y}{x^2}-\frac{1}{3}\right)\left(\frac{f(z)+g(y)}{x^2}-\frac{4}{3}\right)=2, \qquad (109)$$

where $$y=\epsilon r_1, \quad \epsilon=0.500\kappa=5\gamma, \tag{109a}$$

$$z=(x^2+y^2)^{\frac{1}{2}}, \tag{109b}$$

$$f(z)=z+\frac{z-2}{z^2/3-z+1}, \tag{109c}$$

$$g(y)=y+(y+2)/(y^2/3+y+1). \tag{109d}$$

Terms of order e^{-2z} have been neglected since z is about 6.5. If we assume a moderate range of the nuclear forces, x will be about 3 (cf. (108a)), y will be small and $z\approx 3$ (cf. (109b)). This will make the first factor of (109) vanish, while the second factor has the value $-\frac{2}{3}$. This shows that there can be no solution with a moderate range of the forces. The reason is obviously that the right-hand side, as well as the subtracted term 4/3, is too large. It can be shown easily that the right-hand side of (109) arises from the coupling of S and D state, and the 4/3 from the attractive potential in the D state itself. These quantities are therefore too large to allow a moderate range of the forces, as we know already from the calculations in §12, 17.

The actual solution of the Eqs. (108), (109), with $y=5\eta$, was found by trial and error. It is

$$\begin{array}{c} x=2.3595, \quad y=5.963, \quad z=6.414 \\ r_1=y/\epsilon=2.61\times 10^{-12}\text{ cm.} \end{array} \tag{110}$$

This result is obviously out of the question, the range of the forces being more than ten times too large. The result is very much worse than the previous result, in §17, where a cutting-off distance of 3.7×10^{-13} cm was required. This shows that the cutting off before differentiation makes matters considerably more unfavorable for the symmetrical theory.

For the neutral theory, we get again a perfectly reasonable result. We have to replace x^2 by $-\frac{1}{3}x^2$ in all formulae referring to the triplet state, i.e., Eq. (109) to (109d). The result is, instead of (110)

$$\begin{array}{c} x=2.8840, \quad y=0.744, \quad z=1.490 \\ r_1=3.25\times 10^{-13}\text{ cm.} \end{array} \tag{110a}$$

This range of the forces is only slightly too large and, correspondingly, the quadrupole moment will also come out somewhat too large, *viz.*, about 5×10^{-27} cm^2. It is evident that a minor modification will make this theory agree with experiment—e.g., going back to the original theory discussed in §§1 to 17.

For the sake of completeness, we finally investigated a potential which is repulsive in the narrow region from r_1 to r_2 and attractive inside r_1. Such a potential is not very plausible and quite contrary to all usual assumptions about nuclear forces. It gave a slightly better result, *viz.* $y\approx 1.8$, $r_1\approx 8\times 10^{-13}$ cm for the symmetrical theory. Its main advantage is to give the correct sign of the quadrupole moment, but the range of the forces is still much too large.

The results of this section can therefore be described as wholly negative.

§19. Other Possibilities. Conclusion

Møller and Rosenfeld[18, 59] have proposed to introduce two entirely separated meson fields, one of which is described by a vector wave function just as our field, while the other has a pseudoscalar wave function. The interaction between two nuclear particles consists then of two parts: The part due to the vector field which we have considered in this paper may be written

$$V=(f^2/\kappa^2)[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\nabla^2(e^{-\kappa r}/r) \\ -\boldsymbol{\sigma}_1\cdot\text{grad}\,(\boldsymbol{\sigma}_2\cdot\text{grad}\,(e^{-\kappa r}/r))]. \tag{111}$$

The interaction due to the pseudoscalar field has the form

$$V'=(f'^2/\kappa^2)\boldsymbol{\sigma}_1\cdot\text{grad}\,(\boldsymbol{\sigma}_2\cdot\text{grad}\,(e^{-\kappa r}/r)).$$

Choosing, e.g., $f'=f$, the grad grad terms will cancel and thus the tensor interaction will vanish. There remains then only the central force $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\nabla^2$. If we choose f' slightly larger than f, we shall retain a small tensor interaction whose sign is opposite to that in the vector theory.

It is evident that this will solve our difficulties, at least qualitatively: The smallness of the tensor interaction will enable us to get the correct position of singlet and triplet state with a small, and therefore reasonable, cutting-off radius. The change of sign will give the correct (positive) sign of the quadrupole moment.

Møller and Rosenfeld have pointed out[59] that their assumption will also solve the discrepancies

[59] C. Møller and L. Rosenfeld, Nature **144**, 476 (1939). I am indebted to Drs. Møller and Rosenfeld for sending me the manuscript before publication.

in the theory of the β-decay. They assume that one of the two kinds (vectorial and pseudoscalar) of mesons disintegrates rapidly into electron plus neutrino while the other kind disintegrates only slowly. The second kind would then be the one found in cosmic rays while the first kind is responsible for the β-decay of nuclei.

While it must be admitted that the Møller-Rosenfeld theory solves the difficulties of the symmetrical meson theory, I do not think that a solution should be sought on these lines. I believe that the solution of the problem of nuclear forces ought to be fundamentally simple, and this cannot be said of the Møller-Rosenfeld proposition. It seems that one type of mesons, either represented by a vector or by a pseudoscalar field, is entirely sufficient to account qualitatively for all properties of nuclei, i.e., the forces between the nuclear particles as well as the β-transformation, and it seems rather superfluous to have a second type. Moreover, the theory would bring us back to the previous stage of nuclear physics in that it contains as many (or more) adjustable constants as there are experimental data to fit. It can even be said that the situation is worse than before because there has been added to nuclear theory a very complicated field theory, without achieving any greater definiteness in the results.

If the symmetrical meson theory is fundamentally correct, I believe it is much more likely that its quantitative aspects will be corrected by a better understanding of the cut-off process. It is true that our alternative method of cutting off in §18 only made matters worse, but we do not know the correct method yet.

A possibility of preserving at least one essential feature of the symmetrical theory, plus the single force hypothesis (§5), is suggested by Schwinger's calculations.[49] The feature in question is the equality of the central forces for triplet and singlet state. We may try to keep this feature but disregard all the information of the meson theory on the shape and relative magnitude of the potentials. Then we would simply assume "old-fashioned" square wells or Gaussian potentials, etc., for both the central and tensor force and adjust the constants so as to give the correct binding energies for the singlet and triplet state of the deuteron. (Since the range of the forces is then also unknown, the binding energy of the triton, or the analysis of the proton-proton scattering,[15] is required for its determination.) Schwinger's calculations show that with such assumptions a value for the quadrupole moment of the deuteron can be obtained which agrees well with the observed value (§15). This agreement is made possible by assuming the tensor interaction much smaller than the symmetrical field theory would predict.

The single force hypothesis itself seems to be well substantiated by our results. The introduction of a spin-independent force U (cf. (19a)) into the symmetrical theory would lower the triplet and raise the singlet state (cf. §4D). Actually, we have found that the tensor interaction already lowers the triplet state too much (§17), therefore the force U would make the disagreement even worse.

In the neutral theory, U would be felt mostly in the singlet state in which it is subtracted from V_1. To make up for this, the interaction constant a would have to be raised; to compensate for the effect of the raise in a on the triplet state, x_0 must be increased as well, and therefore the quadrupole moment would come out somewhat larger. Since the neutral theory in its present form seems to agree well with experiment, there is no reason to introduce the force U.

In the present situation, there seem to be two alternative assumptions on nuclear forces which are distinguished by inherent simplicity and seem to us to merit further investigation. One is the symmetrical theory with the single force hypothesis but arbitrary choice of shape, range and depth of the force (see the discussion of Schwinger's theory above). It is superior to older nuclear theory in that the dependence of the forces on charge and spin follows from general principles being simply given by $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\ \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$, but it shares with the older theory the arbitrariness of the constants in the force. Obviously, in such a theory we could not say anything about the interaction of mesons with nuclear particles because we would consider the nuclear forces following from the meson theory as incorrect. This would invalidate all calculations about the production of mesons in nuclear collisions, about the meson theory of β-disintegrations, etc.

The second alternative is to consider, for the time being, the neutral meson theory as correct. Although it may not be the final solution it may be a promising working hypothesis for the problem of nuclear forces as such, i.e., disregarding the problems of β-disintegration and the magnetic moment. In view of the good result for the quadrupole moment it seems worth while to investigate the consequences of the neutral theory for other problems, particularly the three- and four-body problem. The steep increase of the potential at small distances seems rather promising of a solution of the well-known discrepancies[46] in that it may increase considerably the binding energy of the triton. It should be realized that a treatment of systems containing more than two particles is quite difficult with a force as complicated as that following from the meson theory but it may be possible to find some simplification which preserves the most important features—such as the potential discussed in §18.

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Electromagnetic Self-Energy of Mesotrons

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We have calculated the electromagnetic self-energy of a mesotron of unit spin according to the Proca-Yukawa-Kemmer theory by an approximate method equivalent to the second order of perturbation theory. The method enables the result to be separated into various parts whose physical significance can be seen more clearly than that of the total result, but is of course open to the same objection as perturbation theory for such calculations. The various parts of the self-energy are due to the longitudinal electric field (Coulomb part), the transverse electric field (arising from the spin), the magnetic field (also from spin) and a part due to the influence of the zero point fluctuations of the radiation field on the motion and therefore on the proper field of the particle. In terms of a "cut-off radius" a, introduced to make the various contributions finite, but which will later be set equal to zero, the first and last parts diverge as $1/a^2$. The second and third parts diverge as $1/a^4$, but are of opposite sign and cancel to this order. The whole self-energy diverges as $1/a^2$ in our approximation. These results are compared with similar ones for spins 0 and $\frac{1}{2}$ recently obtained by Weisskopf.

I. Introduction—Some Equations of Mesotron Theory

With a notation similar to those of Yukawa,[1] Proca,[2] and Kemmer,[3] the system mesotrons+electromagnetic field is characterized by the four vectors $A_\mu U_\mu U_\mu^*$, whose space-components are treated as "generalized coordinates" depending parametrically on x, y, z, as well as μ, and by the canonical conjugates of these "coordinates," $A_k^\dagger U_k^\dagger U_k^{*\dagger}$ ($k=1,2,3$).

The Hamiltonian function is (it is understood that we sum our repeated suffixes, including those occurring in the square of a quantity having a suffix—Latin suffixes go from 1 to 3 and Greek ones from 1 to 4)

$$H=\int\left[(1/8\pi)|G_{kj}|^2+4\pi\kappa^2c^2|U_j^\dagger|^2+(1/4\pi)|U_j|^2+(1/8\pi)\{(F_{kj})^2+(4\pi cA_j^\dagger)^2\}+4\pi c^2|\{\partial/\partial x^j+(ie/\hbar c)A_j\}U_j^\dagger|^2\right]dV, \quad (1)$$

where $\kappa=(c/h)\times$mass of mesotron and we have used the abbreviations

$$G_{\mu\nu}=\frac{1}{\kappa}\left[\{\partial/\partial x^\mu-(ie/\hbar c)A_\mu\}U_\nu-\{\partial/\partial x^\nu-(ie/\hbar c)A_\nu\}U_\mu\right], \quad (2)$$

$$F_{\mu\nu}=\partial A_\nu/\partial x^\mu-\partial A_\mu/\partial x^\nu=(\boldsymbol{B},\boldsymbol{E})\text{ (electromagnetic field).} \quad (3)$$

[1] H. Yukawa, Proc. Phys. Math. Soc. Japan **20**, 1 (1938).
[2] A. Proca, J. de phys. et rad. **7**, 347, 532 (1936).
[3] N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938).