Stationary States of Scalar and Vector Fields

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In the presence of an electrostatic field (but not for a static world scalar or magnetostatic field) the stationary solutions of the quadratic relativistic wave equation for scalar particles do not form an orthonormal set. In spite of this they may in general be used to introduce normal coordinates for the quantum theory of this field. If the wave field and its canonical conjugate are expanded in terms of these stationary solutions, then the commutation laws for the amplitudes follow from the wave field commutators and the assumption of the integrability of the classical wave equation for arbitrary initial function and time derivative. Parallel considerations are applied to the vector field. An alternative method is described that involves the introduction of orthonormal functions and the construction of a "particle Hamiltonian."

I.

THE scalar and vector wave fields may be quantized according to Bose statistics by taking canonical commutation laws for the wave fields and their canonical conjugates. However, it has not been clear that the Hamiltonian can be brought to diagonal form by means of a unitary transformation when there is a static electric field present. When the electric field vanishes, and there are only static magnetic and gravitational fields present, the Hamiltonian is readily brought to diagonal form. One can easily understand why this is true. If we make a time Fourier analysis of the field we find: (*a*) all of the frequencies are real; (*b*) to every positive frequency there corresponds a negative frequency with equal magnitude, and the space dependence of the Fourier coefficients of the positive frequencies is the same as that of the corresponding negative frequencies; (*c*) the Fourier coefficients form a complete set of orthogonal functions in the space variables.

The situation is quite different when there is a static electric field present: (a') the frequencies are not necessarily real, although there exist potentials for which they are; (b') there is no direct correspondence between the positive and negative frequencies or their Fourier coefficients; (c') the Fourier coefficients do not form a set of orthogonal functions. If the frequencies are real, Pauli[1] has shown, in spite of (b') and (c'), that the total charge and energy take their canonical form in terms of the Fourier amplitudes of an unquantized field. Our principal task is to show that Bose quantization of these amplitudes is equivalent to the canonical commutation laws for the wave fields.

Furthermore, in the cases where (a,b,c) are true, it is possible to show that the frequencies and Fourier coefficients are the eigenvalue and eigenfunction, respectively, of a Hermitian operator, which may be interpreted as a "particle-Hamiltonian" for the individuals of an Einstein-Bose ensemble. In this paper we show how such a "particle-model" may be constructed in the case of the breakdown of (b,c), although not if any of the frequencies are complex.

II.

We will be concerned in this section only with electrostatic fields $\mathcal{E}=-\nabla V$. The Lagrangian for the scalar field then becomes:[2]

$$\mathfrak{L}=\left(\frac{\partial}{\partial t}-ieV\right)\psi^*\left(\frac{\partial}{\partial t}+ieV\right)\psi-\nabla\psi^*\cdot\nabla\psi-\psi^*\psi. \quad (1)$$

The momenta canonically conjugate to ψ and ψ^* are:

$$\begin{aligned}\pi&=\partial\mathfrak{L}/\partial(\partial\psi/\partial t)=\dot{\psi}^*-ieV\psi^*,\\ \pi^*&=\partial\mathfrak{L}/\partial(\partial\psi^*/\partial t)=\dot{\psi}+ieV\psi.\end{aligned} \quad (2)$$

The Hamiltonian for the system, obtained in the usual way, is:

$$H=\int Udv, \quad (3)$$

[1] W. Pauli, Princeton mimeographed notes (1935).

[2] W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 709 (1934).

with

$$U=\pi^*\pi+\nabla\psi^*\cdot\nabla\psi+\psi^*\psi+\rho V, \qquad (4)$$

where the charge density ρ is given by

$$\rho=ie(\pi^*\psi^*-\pi\psi). \qquad (5)$$

These expressions will be considered to be symmetrized whenever we use them quantum-mechanically.

From the Hamiltonian (3) we can immediately find the Hamiltonian equations of motion, which are, of course, equivalent to the Lagrangian equations.

$$\begin{aligned}\dot\psi&=\delta H/\delta\pi=\pi^*-ieV\psi,\\ \dot\psi^*&=\delta H/\delta\pi^*=\pi+ieV\psi^*,\end{aligned} \qquad (6)$$

$$\begin{aligned}\dot\pi&=-\delta H/\delta\psi=\Delta\psi^*-\psi^*+ieV\pi,\\ \dot\pi^*&=-\delta H/\delta\psi^*=\Delta\psi-\psi-ieV\pi^*.\end{aligned} \qquad (7)$$

The usual method of quantizing this Hamiltonian is to suppose that the ψ, ψ^*, π, and π^* satisfy the following commutation relations:

$$\begin{aligned}[\psi(x),\pi(x')]&=i\delta(x-x'),\\ [\psi^*(x),\pi^*(x')]&=i\delta(x-x'),\end{aligned} \qquad (8)$$

and all other commutators zero. With these commutators it is easily shown that the time rate of change of any function $f(\psi,\psi^*,\pi,\pi^*)$ is given by:

$$i\dot f=[f,H] \qquad (9)$$

In particular, the equations of motion for ψ, ψ^*, π, and π^* are (6) and (7).

One can also see that:

$$i\dot q=[q,H]=0, \qquad (10)$$

where

$$q=\int\rho dv \qquad (11)$$

is the total charge of the system.

Because of (10), both q and H should be simultaneously diagonalizable. It is, however, quite difficult to carry out the diagonalization explicitly using the commutation relations (8), because of the occurrence of V in the right-hand side of Eqs. (6). For this reason, we have carried out the quantization in a different manner, but we will show later that this quantization is equivalent to (8).

III.

If we substitute π^* from (6) into (7), treating ψ as a classical field, we find that ψ must satisfy a second-order partial differential equation, namely, the Klein-Gordon[3] equation

$$(\partial/\partial t+ieV)^2\psi=\Delta\psi-\psi, \qquad (12)$$

with a similar equation for ψ^*.

"Any"[4] arbitrary function $\psi(x,t)$ may be written

$$\psi(x,t)=\int_{-\infty}^{\infty}dE_k e^{-iE_k t}\psi_k(x). \qquad (13)$$

If $\psi(x,t)$ is to satisfy the K-G equation, then the $\psi_k(x)$ must satisfy:

$$(E_k-eV)^2\psi_k=\psi_k-\Delta\psi_k. \qquad (14)$$

As a consequence of this, the general solution of (12) may be written:

$$\psi=\Sigma_k a_k\psi_k, \qquad (15)$$

where the $\psi_k(x)$ are all of the solutions of (14) satisfying suitable boundary conditions, and where

$$\dot a_k=-iE_k a_k. \qquad (16)$$

(For the present, E_k is assumed to be real.)

It should be noted that, in general, the functions ψ_k are not orthogonal to one another, but satisfy an orthogonality relation of the form:

$$\int\psi_k^*(E_k+E_l-2eV)\psi_l dv=0 \quad (k\neq l), \qquad (17)$$

which has the weighting factor (E_k+E_l-2eV).

For all k, l,

$$\int\psi_k^*(E_k+E_l-2eV)\psi_l dv=\epsilon_k\delta_{kl}, \qquad (18)$$

where ϵ_k is a number, which depending upon the particular ψ_k, may be either positive or negative. The case where some of the ϵ_k vanish will be treated separately; it is connected with the Klein paradox. This division of the ψ_k into states with positive and negative ϵ_k corresponds, as we shall see, to the division of the system into states of

[3] K-G will be used as an abbreviation for Klein-Gordon.

[4] The quotation marks mean that this is true for any function which is sufficiently regular and which remains bounded at positive and negative infinite times.

positive and negative charge. For the total charge is given by

$$q=ie\int(\pi^*\psi^*-\pi\psi)dv. \qquad (19)$$

Because $\pi^*=\dot{\psi}+ieV\psi$, we find

$$\pi^*=-i\Sigma_k a_k(E_k-eV)\psi_k, \text{ etc.} \qquad (20)$$

The total charge then becomes:

$$q=\Sigma_k a_k^* a_k \epsilon_k, \qquad (21)$$

in which the division into states of positive and negative charge is evident.

A similar simple calculation gives the Hamiltonian its canonical form:

$$H=\Sigma_k a_k^* a_k E_k \epsilon_k. \qquad (22)$$

Transition to q Numbers

Our procedure for quantizing this Hamiltonian, which up to this point has been treated classically, is to find commutation laws for the a_k and a_l^* so that for any function $f(a_k,a_l^*)$ we have $i\dot{f}=[f,H]$, or, in particular,

$$i\dot{a}_k=[a_k,H]=E_k a_k. \qquad (23)$$

We suppose that $[a_k,a_l]=[a_k^*,a_l^*]=0$, and then

$$\begin{aligned} E_k a_k &= \Sigma_l \epsilon_l E_l[a_k a_l^* a_l - a_l^* a_l a_k], \\ E_k a_k &= \Sigma_l \epsilon_l E_l[a_k,a_l^*]a_l, \end{aligned} \qquad (24)$$

from which we see that we must take:

$$[a_k,a_l^*]=\delta_{kl}/\epsilon_k. \qquad (25)$$

The characteristic values of the symmetrized form of $a_k^* a_k$ are:

$$\tfrac{1}{2}[a_k^* a_k + a_k a_k^*]=(N_k+\tfrac{1}{2})/|\epsilon_k|; \quad N_k=0,1,2,\cdots. \qquad (26)$$

The total charge and energy then become:

$$q=e\Sigma_k(N_k+\tfrac{1}{2})\epsilon_k/|\epsilon_k|, \qquad (27)$$

$$H=\Sigma_k E_k(N_k+\tfrac{1}{2})\epsilon_k/|\epsilon_k|, \qquad (28)$$

where

$$\epsilon_k/|\epsilon_k|=\pm 1. \qquad (29)$$

In order that the total charge vanish when there are no particles present, we must set the conditionally convergent series $\Sigma_k \epsilon_k/|\epsilon_k|$ equal to zero. This can be done even when electrostatic fields are present. Then the charge becomes:

$$q=e\Sigma_k N_k \epsilon_k/|\epsilon_k|. \qquad (30)$$

IV.

We have given a quantum theory of this field by quantizing the normal coordinates a_k. We must show that this method of quantization is equivalent to taking the Einstein-Bose commutation laws (8) for ψ, ψ^*, π, and π^* and all other commutators zero.

We discover, using the commutation laws for the a_k and a_k^*, that:

$$[\psi(x),\psi^*(x')]=\Sigma_k \epsilon_k^{-1}\psi_k(x)\psi_k^*(x'), \qquad (31)$$

$$[\psi(x),\pi(x')]=i\Sigma_k \epsilon_k^{-1}\psi_k(x)\psi_k^*(x') \times[E_k-eV(x')], \qquad (32)$$

$$[\pi^*(x),\pi(x')]=\Sigma_k \epsilon_k^{-1}\psi_k(x)\psi_k^*(x') \times[E_k-eV(x)][E_k-eV(x')]. \qquad (33)$$

All other commutators vanish, except the commutator of $\psi^*(x)$ and $\pi^*(x')$ which is the complex conjugate of (32).

If the commutation laws (31), (32), and (33) are to be the same as the usual ones, then the functions ψ_k must satisfy the following identities:

$$\Sigma_k \epsilon_k^{-1}\psi_k(x)\psi_k^*(x')=0, \qquad (34)$$

$$\Sigma_k \epsilon_k^{-1}\psi_k(x)\psi_k^*(x')E_k=\delta(x-x'), \qquad (35)$$

$$\Sigma_k \epsilon_k^{-1}\psi_k(x)\psi_k^*(x')E_k^2=2eV(x)\delta(x-x'). \qquad (36)$$

The validity of these identities will be proven by constructing the Green's functions for the K-G equation. We observe that every solution of the K-G equation which remains bounded in time may be written:

$$\psi=\Sigma_k a_k^0 e^{-iE_k t}\psi_k(x), \qquad (15a)$$

in which the a_k^0 are arbitrary numerical constants, and the time variation of ψ is written explicitly. Since ψ satisfies a second-order differential equation in time, we must specify not only the initial value of ψ but also its time derivative at the initial time, say $t=t'$, in order to calculate ψ at other times. This means that the values of a_k^0 depend not only upon the initial values of ψ, but also upon those of $\dot{\psi}$. Furthermore, because "every" solution ψ may be written in the form (15a), there must exist a_k^0 so that ψ

and $\dot{\psi}$ will take on arbitrary initial values at $t=t'$.

From the fact that the differential equation is linear in ψ and of second order in the time derivatives, there will exist two Green's functions $G(x,t;x',t')$ and $L(x,t;x',t')$ by means of which we can calculate the value of ψ at t from the values of ψ and $\dot{\psi}$ at t', namely:

$$\psi(x,t)=\int G(x,t;x',t')\psi(x',t')dv' + \int L(x,t;x',t')\dot{\psi}(x',t')dv'. \quad (37)$$

We will construct the functions G and L as bilinear forms in $\psi_k(x)$ and $\psi_k^*(x')$. We write for the initial values of ψ and $\dot{\psi}$ at $t=t'$:

$$\psi(x',t')=\Sigma_k a_k{}^0 e^{-iE_k t'}\psi_k(x'), \quad (38)$$

$$\dot{\psi}(x',t')=-i\Sigma_k a_k{}^0 E_k e^{-iE_k t'}\psi_k(x'), \quad (39)$$

knowing that $a_k{}^0$ exist which will give them any pre-assigned values.

If we multiply $\psi(x',t')$ by $e^{iE_l t'}\psi_l^*(x')\times[E_l-2eV(x')]$, and $\dot{\psi}(x',t')$ by $ie^{iE_l t'}\psi_l^*(x')$, add the two results and integrate over all space, we find:

$$\epsilon_l a_l{}^0=e^{iE_l t'}\int \psi_l^*(x') \times\{[E_l-2eV(x')]\psi(x',t')+i\dot{\psi}(x',t')\}dv', \quad (40)$$

or, since we are supposing all $\epsilon_k\neq 0$,

$$a_l{}^0=\epsilon_l{}^{-1}e^{iE_l t'}\int \psi_l^*(x') \times\{[E_l-2eV(x')]\psi(x',t')+i\dot{\psi}(x',t')\}dv'. \quad (41)$$

If we place this value of $a_l{}^0$ in Eq. (15a), we find the following expressions for G and L:

$$L(x,t;x't')=i\Sigma_k\epsilon_k{}^{-1}e^{iE_k(t'-t)}\psi_k(x)\psi_k^*(x'), \quad (42)$$

$$G(x,t;x't')=\Sigma_k\epsilon_k{}^{-1}e^{iE_k(t'-t)} \times\psi_k(x)\psi_k^*(x')[E_k-2eV(x')]. \quad (43)$$

These expressions (42) and (43) are the Green's functions for the K-G equation and give ψ and $\dot{\psi}$ their initial values (38) and (39) at $t=t'$. Since ψ and $\dot{\psi}$ are arbitrary at $t=t'$, we see from Eq. (37) that:

$$L(x,t';x',t')=i\Sigma_k\epsilon_k{}^{-1}\psi_k(x)\psi_k^*(x')=0, \quad (44)$$

$$G(x,t';x',t')=\Sigma_k\epsilon_k{}^{-1}\psi_k(x)\psi_k^*(x') \times[E_k-2eV(x')]=\delta(x-x'). \quad (45)$$

Equation (44) is the identity (34), and Eq. (45) is equivalent to identity (35). By differentiating Eq. (37) with respect to the time, one finds, for the same reasons, that:

$$dG(x,t;x't')/dt\,|_{t=t'}=-i\Sigma_k\epsilon_k{}^{-1}\psi_k(x) \times\psi_k^*(x')E_k[E_k-2eV(x')]=0, \quad (46)$$

which is equivalent to identity (36), since identity (35) is true. The differentiation of L in (35) with respect to time gives us a repetition of the proof of identity (35).

In this way we have proven that our method of quantizing the scalar field is equivalent to the usual method of quantization under the condition:

that for arbitrary initial ψ and $\dot{\psi}$, the K-G *equation may be integrated to give a ψ which remains bounded in time.* (47)

There exist potentials for which this condition will not be satisfied: if this happens it can be shown that the Hamiltonian and charge cannot be simultaneously diagonalized when the commutation relationships (8) are satisfied.

V.

The methods of Sections II to IV may be employed to extend the results there obtained for the Pauli-Weisskopf scalar field, to the case of the mesotron vector field. The mesotron Hamiltonian has been given by Yukawa and Sakata[5] in terms of the vector field-coordinates ψ, ψ^* and their respective canonically conjugate momenta π, π^*[6] as:

$$H=\int\{\pi\cdot\pi^*+(D^*\cdot\pi)(D\cdot\pi^*) +(D^*\times\psi^*)\cdot(D\times\psi)+\rho V+\psi^*\cdot\psi\}dv, \quad (48)$$

[5] H. Yukawa and S. Sakata, "The Interaction of Elementary Particles III," Proc. Physico-Math. Soc. Japan, Ser. 3, 20, 319 (1938).

[6] Vector signs will be omitted, for convenience, on the symbols like ψ, D, α, G. $(\cdot)$ and $[\times]$ have their usual meanings.

where A, V are the vector and scalar potentials of the electromagnetic field (assumed time-independent) and where we define

$$D=\nabla-ieA, \qquad D^*=\nabla+ieA, \tag{49}$$

and where the charge-density ρ is given by

$$\rho=ie(\pi^*\cdot\psi^*-\pi\cdot\psi). \tag{50}$$

Using Green's theorem, one may write (48) in the form

$$H=\int\{\pi\cdot[\pi^*-D(D\cdot\pi^*)-ieV\psi] \\ +\psi^*\cdot[\psi+D\times(D\times\psi)+ieV\pi^*]\}dv. \tag{51}$$

By varying π and ψ^* and their complex conjugates, we may obtain the classical Hamiltonian equations of motion:

$$\dot{\pi}^*=-\delta H/\delta\psi^*=-D\times(D\times\psi)-\psi-ieV\pi^*, \tag{52}$$

$$\dot{\psi}=\delta H/\delta\pi=\pi^*-D(D\cdot\pi^*)-ieV\psi, \tag{53}$$

and conjugates. Under restrictions similar to those in (13), one may make a Fourier analysis of ψ and π^* with respect to the time:

$$\psi=\Sigma_k e^{-iE_kt}\psi_k, \quad \pi^*=\Sigma_k e^{-iE_kt}\pi_k^*. \tag{54}$$

In order, then, to satisfy (52) and (53), we must have:

$$i(E_k-eV)\pi_k^*=\psi_k+D\times(D\times\psi_k), \tag{55}$$

$$-i(E_k-eV)\psi_k=\pi_k^*-D(D\cdot\pi_k^*); \tag{56}$$

and, for the present, we shall assume E_k real. From (55) and (56) one easily obtains:

$$\int\{(E_k-eV)\psi_l^*\cdot\pi_k^*+(E_l-eV)\psi_k\cdot\pi_l\}dv=0, \tag{57}$$

$$\int\{(E_k-eV)\pi_l\cdot\psi_k+(E_l-eV)\pi_k^*\cdot\psi_l^*\}dv=0, \tag{58}$$

which may be combined to give the analog of (17):

$$(E_k-E_l)i\int\{\psi_l^*\cdot\pi_k^*-\psi_k\cdot\pi_l\}dv=0, \tag{59}$$

or

$$i\int\{\psi_l^*\cdot\pi_k^*-\psi_k\cdot\pi_l\}dv=\epsilon_k\delta_{kl}. \tag{60}$$

As before, the sign of ϵ_k will determine the sign of the charge of particles in the kth state, so that, for the moment, we shall exclude the possibility $\epsilon_k=0$, treated separately in the succeeding paper.

We may now proceed to the quantum theory by symmetrizing all bilinear expressions in π, ψ, π^*, ψ^*, and by redefining ψ, π^* in terms of a sequence of operators a_k independent of the space-variables but dependent on the time. We shall take:

$$\psi=\Sigma_k a_k\psi_k, \qquad \pi^*=\Sigma_k a_k\pi_k^*, \tag{61}$$

corresponding to the general classical solutions of (54). Applying (60) to (50) and (51), one finds

$$q=\int\rho dv=\Sigma_k(a_k^*a_k+a_ka_k^*)\epsilon_k/2, \tag{62}$$

$$H=\Sigma_k E_k(a_k^*a_k+a_ka_k^*)\epsilon_k/2, \tag{63}$$

for the total charge and energy, respectively. (This is comparable to (21) and (22) in the Pauli-Weisskopf case.) To obtain the quantum analog of the canonical equations, we must take

$$i\dot{a}_k=E_ka_k=[a_k,H], \tag{64}$$

which, in general, will require

$$[a_k,a_l^*]=\delta_{kl}/\epsilon_k, \quad [a_k,a_l]=[a_k^*,a_l^*]=0. \tag{65}$$

This, of course, results in Bose quantization of the states numbered by k, as shown by the results (26) to (30), which now follow exactly as before.

It is not clear, however, that these equations represent the quantization of the mesotron field, until one can demonstrate that ψ and π defined by (61) and (65) satisfy canonical commutation laws as in (8). For the vector-functions ψ, π, one must then be able to prove:

$$\begin{aligned}[]
[\alpha\cdot\psi(x),\beta\cdot\pi(x')]&=i(\alpha\cdot\beta)\delta(x-x'),\\
[\alpha\cdot\psi^*(x),\beta\cdot\pi^*(x')]&=i(\alpha\cdot\beta)\delta(x-x'),
\end{aligned} \tag{66}$$

all other commutators zero, for arbitrary constant vectors, α and β. By (65), we find, however:

$$\begin{aligned}[]
[\alpha\cdot\psi(x),\beta\cdot\psi^*(x')]&=\Sigma_k\epsilon_k^{-1}\alpha\cdot\psi_k(x)\beta\cdot\psi_k^*(x'),\\
[\alpha\cdot\psi(x),\beta\cdot\pi(x')]&=\Sigma_k\epsilon_k^{-1}\alpha\cdot\psi_k(x)\beta\cdot\pi_k(x'),\\
[\alpha\cdot\pi^*(x),\beta\cdot\pi(x')]&=\Sigma_k\epsilon_k^{-1}\alpha\cdot\pi^*(x)\beta\cdot\pi(x'),\\
[\alpha\cdot\pi^*(x),\beta\cdot\psi^*(x')]&=\Sigma_k\epsilon_k^{-1}\alpha\cdot\pi^*(x)\beta\cdot\psi^*(x'),
\end{aligned} \tag{67}$$

all other commutators zero. If these are to be equivalent to (66), the following identities must then be shown to be valid:

$$\Sigma_k \epsilon_k^{-1}\alpha\cdot\psi_k(x)\beta\cdot\psi_k^*(x') = \Sigma_k\epsilon_k^{-1}\alpha\cdot\pi_k^*(x)\beta\cdot\pi_k(x') = 0, \quad (68)$$

$$\Sigma_k\epsilon_k^{-1}\alpha\cdot\psi_k(x)\beta\cdot\pi_k(x') = -\Sigma_k\epsilon_k^{-1}\alpha\cdot\pi_k^*(x)\beta\cdot\psi_k^*(x') = i(\alpha\cdot\beta)\delta(x-x'). \quad (69)$$

We may prove (68) and (69) by returning to the classical equations (52) and (53). If they exist, the following vector functions form a system of Green's functions for (52) and (53):

$$\begin{aligned} G(x,t;x',t') &= -i\Sigma_k\epsilon_k^{-1}e^{iE_k(t'-t)}\alpha\cdot\psi_k(x)\pi_k(x'),\\ L(x,t;x',t') &= i\Sigma_k\epsilon_k^{-1}e^{iE_k(t'-t)}\alpha\cdot\psi_k(x)\psi_k^*(x'),\\ \Gamma(x,t;x',t') &= -i\Sigma_k\epsilon_k^{-1}e^{iE_k(t'-t)}\alpha\cdot\pi_k^*(x)\pi_k(x'),\\ \Lambda(x,t;x',t') &= i\Sigma_k\epsilon_k^{-1}e^{iE_k(t'-t)}\alpha\cdot\pi_k^*(x)\psi_k^*(x'), \end{aligned} \quad (70)$$

in the sense that

$$\begin{aligned} \alpha\cdot\psi(x,t) &= \int G(x,t;x',t')\cdot\psi(x',t')dv' + \int L(x,t;x',t')\cdot\pi^*(x',t')dv',\\ \alpha\cdot\pi^*(x,t) &= \int\Gamma(x,t;x',t')\cdot\psi(x',t')dv' + \int\Lambda(x,t;x',t')\cdot\pi^*(x',t')dv'. \end{aligned} \quad (71)$$

(71) may be verified by applying (54) and (60) to the general solution of (52) and (53).

Because the latter form a linear system containing only first derivatives in the time, it is assumed that we may assign initial values independently to $\psi(x,t)$, $\pi^*(x,t)$ for all values of x at some time $t=t'$, when ψ, π^* are expressible in the form (54). This is equivalent to the assumption in Section IV that ψ and $\dot{\psi}$ may be given arbitrary, independent, initial values. Choosing this pair of initial values as 0, $\beta\delta(x-x')$ at $t=t'$ in (71):

$$L(x,t';x',t')\cdot\beta = \Gamma(x,t';x',t')\cdot\beta = 0, \quad (72)$$

$$G(x,t';x',t')\cdot\beta = \Lambda(x,t';x',t')\cdot\beta = (\alpha\cdot\beta)\delta(x-x'). \quad (73)$$

Making use of the definitions (70) in (72) and (73), shows the latter to be identical with the desired relations (68) and (69); and hence the validity of (66) is proved.

In this way we have shown how to bring the energy of the mesotron field into diagonal form in the presence of static electromagnetic fields, subject to the restriction of (47).

VI.

In this section we deal with the problem of finding a transformation from canonical field coordinates $\psi(x)$, $\pi(x)$ to new field quantities $\phi_+(x)$, $\phi_-(x)$ in terms of which the charge q and energy H take their diagonal form:

$$q = e\int\{\phi_+^*\phi_+ - \phi_-^*\phi_-\}dv, \quad (74)$$

$$H = \int\{\phi_+^*\Omega_+\phi_+ + \phi_-^*\Omega_-\phi_-\}dv, \quad (75)$$

where the products of ϕ's are to be taken symmetrized, and where $\Omega_\pm$ are Hermitian, linear, functional operators on the space-coordinates. We shall also require the commutation rules corresponding to the Einstein-Bose quantization to hold:

$$[\phi_+(x),\phi_+^*(x')] = [\phi_-(x),\phi_-^*(x')] = \delta(x-x'), \quad (76)$$

all other commutators zero.

If the ϕ's are expanded in terms of the orthonormal set of eigenfunctions of the corresponding $\Omega_\pm$, one obtains the canonical forms of Section III, (25), (27), (28). This means that $\Omega_\pm$ may be interpreted as a Hamiltonian for a single particle of charge $\pm e$; and the field may then (insofar as the total charge and energy are concerned) be interpreted as an Einstein-Bose assembly of such particles.

Pauli and Weisskopf[2] have shown how to construct a "particle Hamiltonian" of the scalar mesotron field when no external fields are present, by means of the transformation[7]

$$\psi = 2^{-\frac{1}{2}}(1-\Delta)^{-\frac{1}{4}}(\phi_+ + \phi_-^*); \quad \pi = i2^{-\frac{1}{2}}(1-\Delta)^{\frac{1}{4}}(\phi_+^* - \phi_-), \quad (77)$$

[7] The "square root of a Hermitian operator C", $C^{\frac{1}{2}}$, operating on any function f may be defined by expanding f in eigenfunctions γ_k of C.

$$C\gamma_k = c_k\gamma_k, \qquad f = \Sigma_k f_k\gamma_k$$

for then we may set

$$C^{\frac{1}{2}}f = \Sigma_k C_k^{\frac{1}{2}} f_k\gamma_k.$$

with $$\Omega_+=\Omega_-=(1-\Delta)^{\frac{1}{2}}. \quad (78)$$

One sees clearly how to modify the transformation when magnetostatic fields $\nabla\times A(x)$ or world-scalar fields $-\nabla K(x)$ are present:

$$\psi=(2\mathbf{O})^{-\frac{1}{2}}(\phi_++\phi_-^*),\ \pi=i2^{-\frac{1}{2}}\mathbf{O}^{\frac{1}{2}}(\phi_+^*-\phi_-), \quad (79)$$

with $$\Omega_+=\Omega_-=\mathbf{O},\ \ \mathbf{O}^2=K(x)-(\nabla-ieA)^2. \quad (80)$$

When electrostatic fields, $\mathcal{E}=-\nabla V$, are present, however, it is not at all easy to obtain such a transformation. The behavior of positive and negative charges in this field is essentially different; and $\Omega_\pm$ must be different operators which are interchanged by changing the sign of e. The difficulty also may be understood formally from the fact that the solutions of the K-G equation, in this case, are not properly orthogonal.

It may be shown that when it is possible to obtain $\phi_\pm$ by linear functional transformation on π and ψ, this may be done in the form:

$$\begin{aligned}\psi&=(\mathbf{O}+\mathbf{O}^\dagger)^{-\frac{1}{2}}(\phi_++\phi_-^*),\\ \pi&=i(\mathbf{O}+\mathbf{O}^\dagger)^{-\frac{1}{2}}(\mathbf{O}^\dagger\phi_+^*-\mathbf{O}\phi_-),\end{aligned} \quad (81)$$

where $\mathbf{O}$ is defined as a solution of

$$\mathbf{O}^2+e[\mathbf{O},V]=1-\Delta; \quad (82)$$

and $\mathbf{O}^\dagger$ is the operator complex-adjoint to $\mathbf{O}$, satisfying

$$(\mathbf{O}^\dagger)^2-e[\mathbf{O}^\dagger,V]=1-\Delta. \quad (83)$$

The energy operators are given by

$$\begin{aligned}\Omega_+&=(\mathbf{O}+\mathbf{O}^\dagger)^{\frac{1}{2}}(\mathbf{O}+eV)(\mathbf{O}+\mathbf{O}^\dagger)^{-\frac{1}{2}},\\ \Omega_-&=(\mathbf{O}+\mathbf{O}^\dagger)^{\frac{1}{2}}(\mathbf{O}^\dagger-eV)(\mathbf{O}+\mathbf{O}^\dagger)^{-\frac{1}{2}}.\end{aligned} \quad (84)$$

From the work of the preceding sections, it is not surprising that the characteristic values of $\Omega_\pm$ turn out to be $\pm E_k$, where E_k are the frequencies corresponding to the solutions ψ_k with $\epsilon_k/|\epsilon_k|=\pm1$ of the Klein-Gordon equation with the given potential V. The connection between the eigenfunctions $\phi_\pm^{(k)}$ of $\Omega_\pm$ and the corresponding solutions ψ_k of the Klein-Gordon equation is:

$$\phi_\pm^{(k)}=(\mathbf{O}+\mathbf{O}^\dagger)^{\frac{1}{2}}\psi_k, \qquad \epsilon_k/|\epsilon_k|=\pm1, \quad (85)$$

with $$\Omega_\pm\phi_\pm^{(k)}=\pm E_k\phi_\pm^{(k)}. \quad (86)$$

These last remarks make clear the limitations under which it turns out that one may construct a "particle-Hamiltonian" of the field with the aid of a linear functional transformation:

The Klein-Gordon equation must satisfy the restrictions of (47); and, in particular, the fields must not be so strong that there occur any Klein-Gordon solutions with $\epsilon_k=0$. In this formulation of the problem, complex frequencies E_k may not occur because of the Hermiticity of $\Omega_\pm$. These cases are discussed separately in the following paper. Thus, when the total Hamiltonian and charge may be diagonalized at all, the "particle-Hamiltonian" discussed here may be constructed.

When high Fourier-components of V are not present, one may neglect $|[\Delta,V]|\ll1$; and assuming that all momenta $p\ll1$, the equation defining $\mathbf{O}$ becomes

$$\mathbf{O}^2\cong1-\Delta, \qquad \mathbf{O}^\dagger\cong\mathbf{O}\cong1-\tfrac{1}{2}\Delta. \quad (87)$$

In this nonrelativistic limit, we obtain the ordinary Hamiltonians for particles of charge $\pm e$.

$$\Omega_+\cong1-\tfrac{1}{2}\Delta+eV, \quad \Omega_-\cong1-\tfrac{1}{2}\Delta-eV,$$

$$\phi_+\cong2^{-\frac{1}{2}}(\psi+i\pi^*), \quad (88)$$

$$\phi_-\cong2^{-\frac{1}{2}}(\psi^*+i\pi), \quad (89)$$

where $$\pi^*=\dot{\psi}+ieV\psi.$$

The $\mathbf{O}$ equation may be solved exactly in the simple case of a homogeneous electric field of infinite extent $eV=x/a$, $a>0$. Although the analogue of the Klein paradox is certainly present in this case, there are no solutions of the Klein-Gordon equation:

$$(1-\Delta)\psi_k=(E_k-x/a)^2\psi_k \quad (90)$$

with $\epsilon_k=0$. If we set $p=-i(d/dx)$, $[\mathbf{O},V]=-(i/a)\cdot(d\mathbf{O}/dp)$, and $\mathbf{O}$ is determined by

$$\mathbf{O}^2-(i/a)\cdot(d\mathbf{O}/dp)=1+p^2. \quad (91)$$

We may define $\omega(p)$ by

$$(-i/a)(d/dp\ \ln\omega)=\mathbf{O} \quad (92)$$

and then ω must satisfy the simple equation

$$(d^2/dp^2)\omega+a^2(1+p^2)\omega=0. \quad (93)$$

In this case, where $\mathbf{O}$ is a function of p alone, $[\mathbf{O},\mathbf{O}^\dagger]=0$, and we may write

$$(\mathbf{O}+\mathbf{O}^\dagger)^{-1}=\omega\omega^\dagger/W. \quad (94)$$

From (84) and (92),

$$\Omega_+ = (-i/a)[d/dp \ln (\omega/\omega^\dagger)^{\frac{1}{2}}] + (i/a)d/dp,$$
$$\Omega_- = (-i/a)[d/dp \ln (\omega/\omega^\dagger)^{\frac{1}{2}}] - (i/a)d/dp, \quad (95)$$

where $$W = (-i/a)\begin{vmatrix} d\omega/dp & \omega \\ d\omega^\dagger/dp & \omega^\dagger \end{vmatrix} \quad (96)$$

is a real, constant number, which should be taken positive to make $(\mathbf{O}+\mathbf{O}^\dagger)^{\frac{1}{2}}$ Hermitian.

The orthonormal eigenfunctions $\phi_\pm^{(k)}$ defined by (86) are given explicitly by (95) in the form

$$\phi_\pm^{(k)}(x) = a^{\frac{1}{2}}/2\pi \int_{-\infty}^{\infty} (\omega/\omega^\dagger)^{\pm\frac{1}{2}} e^{ip(x-aE_k)}dp. \quad (97)$$

One can easily verify that $\phi_\pm^{(k)}$ are orthonormal in the sense:

$$\int_{-\infty}^{\infty} \phi_\pm^{*(k)}\phi_\pm^{(l)}dx = \delta(E_k - E_l).$$

From (85), (94), (97) one obtains:

$$\psi_k(x) = (1/2\pi)(a/W)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \omega(p)e^{ip(x-aE_k)}dp \quad (98)$$

for a state k corresponding to *positive* charge. By a brief calculation, one may verify that $\psi_k(x)$ actually satisfies the Klein-Gordon equation and the required relation (18):

$$\int_{-\infty}^{+\infty} \psi_k^*(E_k + E_l - 2x/a)\psi_l dx = \delta(E_k - E_l).$$

Because of the covariance of the wave functions under translation of the origin of coordinates, there is some freedom of choice of ω, the only restrictions on solutions of (93) being that $W>0$ and that the integrals defining ψ_k and $\phi_\pm^{(k)}$, (97) and (98) must exist. Here we choose ω so that the asymptotic behavior of $\phi_\pm^{(k)}(x)$ for large values of $|x|$ is simply obtainable. For large values of $|p|$, we take

$$\omega(p)/\omega^\dagger(p) \sim e^{-(ia/2)(\ln p + p^2 + \vartheta)}, \quad (99)$$

where ϑ is a real constant depending only on a.

Then:

$$\begin{aligned} \phi_+^{(k)}(x) &\sim \phi_-^{(k)}(-x) \\ &\sim (2\pi)^{\frac{1}{2}} e^{-a\pi/2 + (ia/2)(\ln \xi_k - \xi_k^2 + \vartheta + \pi/2a)}, \\ \phi_+^{(k)}(-x) &\sim \phi_-^{(k)}(x) \\ &\sim (2\pi)^{\frac{1}{2}} e^{-(ia/2)(\ln \xi_k - \xi_k^2 + \vartheta + \pi/2a)} \end{aligned} \quad (100)$$

for large positive values of x and $\xi_k = (x - aE_k)/a$. The transmission coefficient $e^{-a\pi/2}$, analogous to that of Dirac's electron theory, appears in these results.

In a recent paper, Kemmer[8] has carried through a linearization of the Proca[9] and K-G equations with the aid of Duffin's[10] 5- and 10-rowed matrices. Although the energy appears in a form analogous to (75),

$$H = -i\int \Psi^\dagger \beta_4 \mathcal{H} \Psi dv,$$

where $i\dot{\Psi} = \mathcal{H}\Psi$, one should observe that, in our units, $(\hbar = c = m_0 = 1)$,

$$\mathcal{H} = eV + \beta_4 \times \Sigma_k[\beta_k, \beta_4]\frac{\partial}{\partial x_k} + (e/2)\Sigma_k \mathcal{E}_k\{[\beta_k, \beta_4^2] + \beta_k\},$$

where β_k, β_4 are Hermitian matrices, and $\mathcal{E}_k = -\partial V/\partial x_k$, $(k=1, 2, 3)$ are the field-strengths of the external electrostatic field. Thus $\mathcal{H}$ is not Hermitian, but contains noncommuting Hermitian and skew Hermitian parts proportional to $\mathcal{E}_k$. This is intimately connected with the fact that the eigenfunctions of $\mathcal{H}$ (essentially the ψ_k's) are in general not orthogonal and that characteristic values of $\mathcal{H}$ may be complex. These limitations must be kept in mind in applying Kemmer's particle-Hamiltonian.

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