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Note on p-Wave Anomalies in Proton-Proton Scattering

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The p-wave effects on Bethe's neutral form of meson theory are calculated for proton energies of 2.0 Mev, 2.4 Mev and 3.0 Mev. The range of nuclear force corresponding to a meson mass of 180 electron masses gives effects of the order of a few percent of the total scattering. These effects do not vanish at the scattering angle of 45°. If present, they call for a slight increase in the range of force derived from s scattering. The comparison of the theoretical and experimental angular distributions indicates, however, that p-wave effects predicted by the above meson theory with a mass of 180 m are too large since they give too much small angle scattering.

IN the analysis of proton-proton scattering data[1-3] it has been assumed that the p-wave anomaly can be represented sufficiently well on the assumption of a central field. In the present note the noncentral forces characteristic of the meson theory are used and it is found that the effects on the angular distribution are not so clearly and unambiguously separable from the s-wave anomaly as on the hypothesis of central fields. In particular, at a scattering angle of 45° there is an effect of the p wave so that observations at this angle cannot be taken as a measure of the s-wave anomaly as before. Up to 2400 kev, however, the effects on the s-wave phase shift, which is derived from experiment, are small and do not affect the main conclusions arrived at on the central field theory.

The phase shifts for $^3P_0$, $^3P_1$, $^3P_2$ are in general different from each other, and the $^3P_2$ state interacts with $^3F_2$. The latter interaction will be neglected, being presumably of little importance at the lower energies. In the notation of BCP a plane wave modified by the Coulomb field has the asymptotic form

$$\psi^c=[1+\cdots]\exp[ikz+i\eta\ln k(r-z)]-\frac{\eta}{k(r-z)}\exp[ikr-i\eta\ln k(r-z)+2i\sigma_0]+\cdots$$

valid at large distances. The normalization is such as to give unit density for the incident wave. The second term in the above expression represents the scattered wave. For fields which are Coulom-

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[1] G. Breit, E. U. Condon and R. D. Present, Phys. Rev. **50**, 825 (1936). This is referred to as BCP.
[2] G. Breit, H. M. Thaxton and L. Eisenbud, Phys. Rev. **55**, 1018 (1939). Referred to as BTE. The notation in the present note is the same as in BTE and BCP.
[3] E. Creutz, Phys. Rev. **56**, 893 (1939).

bian at a large distance but are central and non-Coulombian at small distances one has for the scattered part

$$(\rho\psi^c)_{sc}=\sum(2L+1)P_L\sin K_L\exp[i\rho-i\eta\ln 2\rho+2i\sigma_L+iK_L]$$
$$-(\eta/2s^2)\exp[i\rho-i\eta\ln 2\rho-i\eta\ln s^2+2i\sigma_0].$$

Here the K_L are the usual phase shifts in a central field. The last formula can be generalized so as to apply to the collision of two particles with spin. Nonidentical particles will be considered first, and the wave function representing relative motion in the system of the center of mass will be dealt with. The wave function will then be antisymmetrized so as to apply to the collision of two protons. The collision may be considered as consisting statistically of a mixture of four relative spin orientations described by the four two-particle spin functions[4]

$$\check{S}_0,\ S_1,\ S_0,\ S_{-1}.$$

The first of these corresponds to a total spin $=0$, and is antisymmetric; the last three describe states with total spin 1, spin projections 1, 0, -1, and are symmetric. After the wave function is antisymmetrized the s anomaly contains only $\check{S}_0$ and the p anomaly contains only linear combinations of S_1, S_0, S_{-1}. The parts of the s anomaly containing S_1, S_0, S_{-1} will be omitted, therefore, and similarly the part of the p anomaly in $\check{S}_0$ will be dropped. Neglecting all but s- and p-wave phase shifts one obtains [for notation see reference 1],

$$(\rho\psi^c\check{S}_0)_{sc}=[e^{iK_0}\sin K_0-(\eta/2s^2)\exp[-i\eta\ln s^2]]\check{S}_0\exp[i\rho-i\eta\ln 2\rho+2i\sigma_0],$$

$$(\rho\psi^cS_1)_{sc}=\{(12\pi)^{\frac{1}{2}}[2^{-\frac{1}{2}}({}^3P_2)_1e^{i\delta_2}\sin\delta_2+2^{-\frac{1}{2}}({}^3P_1)_1e^{i\delta_1}\sin\delta_1]e^{2i(\sigma_1-\sigma_0)}$$
$$-(\eta/2s^2)\exp[-i\eta\ln s^2]S_1\}\exp[i\rho-i\eta\ln 2\rho+2i\sigma_0],$$

$$(\rho\psi^cS_0)_{sc}=\{(12\pi)^{\frac{1}{2}}[(\tfrac{2}{3})^{\frac{1}{2}}({}^3P_2)_0e^{i\delta_2}\sin\delta_2-3^{-\frac{1}{2}}({}^3P_0)_0e^{i\delta_0}\sin\delta_0]e^{2i(\sigma_1-\sigma_0)}$$
$$-(\eta/2s^2)\exp[-i\eta\ln s^2]S_0\}\exp[i\rho-i\eta\ln 2\rho+2i\sigma_0],$$

$$(\rho\psi^cS_{-1})_{sc}=\{(12\pi)^{\frac{1}{2}}[2^{-\frac{1}{2}}({}^3P_2)_{-1}e^{i\delta_2}\sin\delta_2-2^{-\frac{1}{2}}({}^3P_1)_{-1}e^{i\delta_1}\sin\delta_1]e^{2i(\sigma_1-\sigma_0)}$$
$$-(\eta/2s^2)\exp[-i\eta\ln s^2]S_{-1}\}\exp[i\rho-i\eta\ln 2\rho+2i\sigma_0].$$

The phase shifts of ${}^3P_0$, ${}^3P_1$, ${}^3P_2$ are here denoted by δ_0, δ_1, δ_2 and the phase shift of ${}^1S_0$ is written as K_0. The functions $({}^3P_j)_m$ are linear combinations of angular Y_L^M and spin functions corresponding to angular momentum j and magnetic quantum number m. They are normalized to unity for integrations over angles and summation over spin coordinates. The signs of the linear combinations are such as to correspond to standard forms[5] of angular momentum matrices. The orbital function

$$Y_1^0=\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta;\ ({}^3P_2)_2=Y_1^1S_1;\ ({}^3P_1)_1=2^{-\frac{1}{2}}(Y_1^0S_1-Y_1^1S_0);\ ({}^3P_0)_0=3^{-\frac{1}{2}}(Y_1^{-1}S_1-Y_1^0S_0+Y_1^1S_{-1}),$$

and the remaining functions are determined by the above conventions. To obtain the expressions $(\rho\psi^c\check{S}_0)_{sc}$, $(\rho\psi^cS_1)_{sc}$, etc. the waves $(\rho\psi^c\check{S}_0)$, $(\rho\psi^cS_1)$, etc., are first expressed as linear combinations of $\check{S}_0$, $({}^3P_j)_m$. The radial factors are then modified so as to correspond to the phase shifts K_0, δ_0, δ_1, δ_2; and so as to have the modified wave asymptotically of the form of incident+scattered wave. These conditions determine the wave function. The asymptotic form of the difference between the wave function and $(\rho\psi^c\check{S}_0)$, $(\rho\psi^cS_1)$, etc. is $(\rho\psi^c\check{S}_0)_{sc}$, $(\rho\psi^cS_1)_{sc}$, etc. Antisymmetrizing the wave function in

[4] The problem is similar to that in C. Kittel and G. Breit, Phys. Rev. **56**, 744 (1939). For proton-proton scattering the ${}^1P$ state drops out and there is interference with the Coulomb wave.

[5] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, 1935).

the two protons and taking the mean of the scattered intensity of the four cases $\check{S}_0$, S_1, S_0, S_{-1} one obtains

$$\mathfrak{M}(\Delta\mathfrak{R})_1=\sum_i g_i\left\{-\frac{18}{\eta}P_1\left(\frac{\cos\alpha_1}{\mathbf{s}^2}-\frac{\cos\beta_1}{\mathbf{c}^2}\right)\sin\delta_i\cos\delta_i+\left[\frac{108}{\eta^2}P_1{}^2+\frac{18}{\eta}P_1\left(\frac{\sin\alpha_1}{\mathbf{s}^2}-\frac{\sin\beta_1}{\mathbf{c}^2}\right)\right]\sin^2\delta_i\right\}$$
$$+\frac{12}{\eta^2}\left[-\frac{3}{4}(\sin\delta_1-\sin\delta_2)^2-\frac{1}{3}(\sin\delta_0-\sin\delta_2)^2-3\sin\delta_1\sin\delta_2\sin^2\frac{\delta_1-\delta_2}{2}\right.$$
$$\left.-\frac{4}{3}\sin\delta_0\sin\delta_2\sin^2\frac{\delta_0-\delta_2}{2}\right](3\cos^2\theta-1). \quad (1)$$

Here $(\Delta\mathfrak{R})_1$ is the change in the ratio to Mott's scattering due to the phase shifts δ_0, δ_1, δ_2, while

$$\mathfrak{M}=\mathbf{s}^{-4}+\mathbf{c}^{-4}-\mathbf{s}^{-2}\mathbf{c}^{-2}\cos(\eta\ln\mathbf{s}^2\mathbf{c}^{-2}),$$

which is proportional to the scattering expected on Mott's theory. The statistical weights g_i are $g_0=1/9$, $g_1=1/3$, $g_2=5/9$. The sum containing the g_i in Eq. (1) represents the interference of the p-wave anomaly with the Coulomb wave together with a part in $P_1{}^2$ which is part of the direct scattering due to p waves. For $\delta_0=\delta_1=\delta_2$ this sum represents the whole effect, and is in agreement then with the formula for a central field. The part of $\mathfrak{M}(\Delta\mathfrak{R})_1$ with the angular dependence $3\cos^2\theta-1$ is absent on the central field theory and complicates the analysis of experimental data if it is appreciable.

Using the form of neutral meson theory proposed by Bethe,[6] the effects of $(\Delta\mathfrak{R})_1$ have been estimated. Numerical calculations made for the case of a "cut-off" in which the specific nuclear potential is zero inside the cut-off gave values of δ_0, δ_1, δ_2 as in Table I. The cut-off distance was such that the exponential factor at the cut-off radius had the value

$$e^{-0.32}$$

This corresponds to a cut-off radius of $0.32\ \hbar/\mu c=0.70\times 10^{-13}$ cm with $\mu\sim 180$ m. Substitution into Eq. (1) gives values of $(\Delta\mathfrak{R})_1$ as in Table II. Using values for $\mathfrak{R}$ computed for a square potential well with radius e^2/mc^2 and depth 10.5 Mev as an approximation to the experimental data one obtains values of 100 percent $(\Delta\mathfrak{R})_1/\mathfrak{R}$ as in Table III.

The evaluation of the s-wave phase shift K_0 from experimental data made by BTE from the observations[7] of Herb, Kerst, Parkinson and Plain consisted in determining the mean K_0 from

TABLE I. *Values of p-wave phase shift.*

E IN MEV	δ_0	δ_1	δ_2
2.0	−2.07°	1.21°	−0.61°
2.4	−2.66°	1.53°	−0.79°
3.0	−3.57°	2.10°	−1.06°

TABLE II. *Contribution to the ratio of scattering to Mott's value due to p scattering.*

E IN MEV	Θ=15°	20°	25°	30°	35°	40°	45°
2.0	0.032	0.058	0.091	0.132	0.175	0.217	0.234
2.4	0.050	0.094	0.156	0.232	0.326	0.416	0.455
3.0	0.080	0.161	0.286	0.466	0.701	0.941	1.051

TABLE III. *Percentage change in scattering due to p-wave anomaly.*

E IN MEV	Θ=15°	20°	25°	30°	35°	40°	45°
2.0	3.4%	3.1%	2.0%	1.4%	1.0%	0.81%	0.74%
2.4	4.6	3.8	2.5	1.8	1.3	1.13	1.06
3.0	6.0	4.6	3.3	2.5	2.1	1.82	1.74

TABLE IV. *Comparison of observed and computed scattering. Percentage of observed minus computed scattering.*

E IN MEV	Θ=	15°	20°	25°	30°	35°	40°	45°
2.392	Pure s	4.2%	−0.5%	0.4%	0.4%	−0.4%	0.7%	−0.7%
	Meson p	0.5%	−2.9%	−1.0%	−0.1%	−0.3%	0.9%	−0.6%
2.105	Pure s	3.3%	1.7%	0.3%	−0.6%	2.0%	−2.3%	0.9%
	Meson p	0.2%	−0.3%	−0.5%	−0.9%	2.1%	−2.0%	0.9%
1.830	Pure s	2.3%	0 2%	0.6%	0.4%	−1.0%	−0.2%	0.9%
	Meson p	−0.2%	−1.7%	−0.2%	0.2%	−0.9%	0.2%	1.3%

[6] H. A. Bethe, Phys. Rev. 55, 1261 (1939).

[7] R. G. Herb, D. W. Kerst, D. B. Parkinson and G. J. Plain, Phys. Rev. 55, 998 (1939).

$\Theta=30°$, $35°$, $40°$, $45°$. At 2.4 Mev the s-wave scattering is too high at these angles by the average amount of 1.3 percent. According to Table VI of BTE the value of K_0 is too high by $1.3\times 0.26°=0.3(4)°$. According to Table XIX of BTE a decrease of K_0 at 2.6 Mev by 0.4° without changing K_0 at 0.8 Mev gives a decrease in the constant α of the potential $Ae^{-\alpha r^2}$ by 1.8 nuclear units. The above effects correspond, therefore, to a decrease of α from 21.6 to about 20. By itself this effect is not serious since the previously adopted value was $\alpha=16$. If, however, the observations or their interpretation contain another error of about the same amount and in the same direction the range of force arrived at by BTE would be definitely too small. The data of Herb, Kerst, Parkinson, and Plain will now be examined for the presence of the p wave. It will be seen that the calculated p anomaly is too large.

Assuming the above p-wave effects one obtains a change in the comparison of observed and expected angular distribution as in Table IV. In the table there is listed the difference between the observed and computed scattering expressed as a percentage of the observed scattering. The first line for each energy is taken from Table XI of BTE in which the details of the treatment of the data of Herb, Kerst, Parkinson and Plain are explained. The second line for each energy has been obtained by subtracting from the observed ratio to Mott's value the contribution to that ratio corresponding to Table II, attributing the result to s scattering, computing the average K_0 for $\Theta=30°$, $35°$, $40°$, $45°$ and then computing the expected ratio to Mott using this K_0. These p-wave corrections are more accurate for 2.39 Mev. For 2.10 Mev the corrections were interpolated, and for 1.83 Mev they were extrapolated graphically.

The apparent agreement between theory and experiment is improved at $\Theta=15°$ as a result of making the p-wave correction. For $\Theta=20°$, however, the p-wave correction spoils the agreement, giving too high values for the theoretically expected scattering. The apparent improvement for $\Theta=15°$ is probably accidental because the geometrical corrections have not been applied to the experimental values in the calculations for the above Table IV.[8] These corrections cannot be made with certainty on account of the uncertain structure of the incident proton beam. The probable value of the error introduced by imperfect geometry is $+2.(5)$ percent at $\Theta=15°$, while for $\Theta=20°$ the error is probably $\sim+0.2$ percent at 2.4 Mev and perhaps $+0.6$ percent at 1.83 Mev. Applying these corrections to the experimental values the numbers in the column $\Theta=15°$ for "Meson p" in Table IV become consistently negative. The application of the probable geometrical corrections gives therefore approximately the same amount of disagreement between theory and experiment at $\Theta=15°$ and $\Theta=20°$. The meson theory in the above form predicts in both cases ~ 2 percent more scattering than is observed.

The above interpretation is also indicated by the data at 860 kev and 1200 kev. For these energies the expected scattering at $\Theta=15°$ is insensitive to the s-wave phase shift K_0 while the effect of p-wave scattering is considerably smaller than at higher energies. The experimental values (without geometrical corrections) are too high by 4 percent as compared with s-wave calculations in both cases indicating, therefore, that the geometrical correction should be made at $\Theta=15°$. In more detail this situation is as follows. The phase shifts δ for 860 kev can be estimated from the approximate rule of proportionality to $E^{\frac{3}{2}}$. This estimate gives

$$\delta_0=-0.60°;\ \delta_1=+0.36°;\ \delta_2=-0.18°.\ (860\ \text{kev}).$$

The contribution to $\mathcal{R}$ due to these phase shifts is $\sim 10^{-4}$ at $\Theta=45°$ and is negligible in a total of 4. The value of K_0 determined in Table XI of BTE may be left unchanged, therefore, at this energy. The effect of δ_0, δ_1, δ_2 at $\Theta=15°$ is to contribute ~ 0.005 to the theoretical $\mathcal{R}$ so that the theoretically expected $\mathcal{R}$ becomes $0.723+0.005=0.728$ which should be compared with the experimental 0.752. The theoretical value is thus 3 percent lower than the experimental.

Since the p-wave phase shift does not remove the discrepancy between theory and experiment for $\Theta=15°$ at 860 kev and 1200 kev it is necessary to assume that the geometrical corrections are real and it is therefore very probable that the above calculations with the meson potential give too much small angle scattering. The dependence of the s-wave phase shift on energy also does not

[8] These corrections are discussed on pp. 1036–41 of BTE.

agree with the meson potential corresponding to a meson mass of ~ 180 m. It indicates a smaller range and a larger meson mass of ~ 300 m. The p-wave phase shifts are roughly proportional to the cube of the range of force. Using the smaller range of force suggested by the s wave, the p-wave phase shifts are decreased by roughly a factor 4, and their effects become practically negligible. With such a view the interpretation of the experiments is self-consistent, but objections could be raised to the use of the large meson mass. It is also impossible to exclude by means of the experimental material p-wave effects of the order of $\frac{1}{3}$ of those dealt with in Table IV. The only definite conclusion is that the p-wave effects for $\mu \sim 180$ m are too large and that the K_0, E curve is not affected by as large amounts as correspond to Table III.

Appendix

Approximate formulae and computational details

The potentials used have been derived by Bethe on the neutral meson type of theory. They are

$$V(^3P_0)=Ce^{-x}(2x^{-3}+2x^{-2}+x^{-1}),$$
$$V(^3P_1)=-Ce^{-x}(x^{-3}+x^{-2}),$$
$$V(^3P_2)=\tfrac{1}{5}Ce^{-x}(x^{-3}+x^{-2}+2x^{-1}),$$

with $C=14.5$ Mev, $x=r\mu c/\hbar$, $\mu=$meson mass. The potentials $V(^3P_0)$, $V(^3P_1)$ correspond exactly to using Bethe's form of the interaction energy while $V(^3P_2)$ is obtained as a mean over spin coordinates. Using Taylor's first-order approximation one obtains for the phase shifts

$$\delta_0=-(2CC_1{}^2/E)a^5[12\theta_0+40B_1a+180B_2a^2+1008B_3a^3+\cdots],$$
$$\delta_1=(2CC_1{}^2/E)a^5[3\theta_1+8B_1a+30B_2a^2+144B_3a^3+\cdots],$$
$$\delta_2=-(2CC_1{}^2\text{-}5E)a^5[15\theta_2+56B_1a+270B_2a^2+1584B_3a^3+\cdots].$$

Here

$$\theta_0=e^{-x_0}(1+x_0+5x_0{}^2/12+x_0{}^3/12)\cong 1,$$
$$\theta_1=e^{-x_0}(1+x_0+x_0{}^2/3)\cong 1,$$
$$\theta_2=e^{-x_0}(1+x_0+7x_0{}^2/15+2x_0{}^3/15)\cong 1,$$
$$a=\frac{\rho}{x}=(ME/2)^{\frac{1}{2}}/\mu c;\quad \rho=Mv/\hbar,$$

$E=$proton energy, $M=$proton mass, $v=$proton velocity,
$x_0=$value of x for the cut-off,
$F_1=C_1\rho^2\Phi_1=$regular Coulomb function.

The coefficients B_1, B_2, $\cdots$ are defined by

$$\Phi_1{}^2=1+B_1\rho+B_2\rho^2+\cdots.$$

In the above formulae the coefficients of B_1a, B_2a^2 come out to be products of e^{-x_0} and long polynomials in x_0. For the small x_0 used here these products can be replaced by unity. The contributions to $\int F_1{}^2 Vdr$ due to the region inside the cut-off are small and are omitted in the above formulae. For order of magnitude estimates the terms in B_1a may be omitted.

The numerical work has been done by the method previously described.[4] Some of it was checked also by direct numerical integration of the differential equation. The sensitivity to the method of cut-off ("straight" or "zero") was tested by using integrations up to $r=4e^2/mc^2$ for $E=2.4$ Mev. The results checked to 0.01° for δ_0 and δ_2. For δ_1 the "straight" cut-off gave 1.40° and the "zero" cut-off gave 1.50°. An accuracy of 0.01° was not obtained until the integrations were extended to $r=5e^2/mc^2$. The values used correspond to the "zero" cut-off with $x_0=0.32$.

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