

Multiple Scattering of Electrons

S. GOUDSMIT AND J. L. SAUNDERSON*
University of Michigan, Ann Arbor, Michigan

(Received November 8, 1939)

A theoretical treatment of multiple Rutherford scattering is given which is exact if one considers electrons with the same total path length in the scatterer. The approximations necessary for the actual solution of the problem for a thin scatterer can be shown to have little effect on the results for small angle scattering. The distribution of scattering is expressed as a series in Legendre polynomials which has been evaluated numerically; the final result for thin scattering is approximately a Gaussian curve. It is shown that this distribution depends in a sensitive way upon the deviation from the Rutherford law for small angle scattering due to screening by electrons. This may perhaps explain the discrepancies between experiments and theory.

1. INTRODUCTION

EXPERIMENTS on the scattering of electrons have, to some extent, lacked importance because of the uncertainties concerning their interpretation. It is very difficult to investigate any theory of single scattering since observation must be made at large angles where the intensity of scattered electrons is very small. Moreover, for somewhat smaller angles, it is difficult to determine exactly how multiple scattering influences the observed scattering. The Wentzel criterion, which is at best only a qualitative statement and can even be shown to be too lenient, is of very little help. A general theory of scattering would therefore be an immense help, inasmuch as it could be compared with observations made on small angle scattering. In principle, this paper uses a method which should yield the complete distribution of scattering. So far, however, calculations have been made which give information concerning only the multiple scattering regions; to extend the theory in order to find the distribution at larger angles requires, as will be seen, additional refinements and calculations.

The most recent theoretical treatment of this problem has been given by E. J. Williams.¹ Comparison of Williams' results with the experimental data² indicates a rather serious and consistent discrepancy. Though somewhat more

involved mathematically, the present paper uses a different method of approach and one which is to a certain extent exact. To facilitate the calculations, certain approximations are made, but the theory shows clearly at which points the use of approximations may influence the results.

2. THE USE OF LEGENDRE POLYNOMIALS

In the following, we think of a particle being scattered a certain number of times, n , after which it has a certain direction, θ . It is a property of the Legendre polynomials that the average value of any polynomial after n impacts is equal to the n th power of the average value of the polynomials after one impact, provided that the law of scattering is cylindrically symmetrical.³ To show this, let θ_1 be the angle of deflection after the first collision, and θ_2 that caused by the second. If the two azimuthal angles are ϕ_1 and ϕ_2 , respectively, and the total deviation from the original direction, θ , the addition theorem for spherical harmonics gives the relation

$$P_l(\cos \theta) = P_l(\cos \theta_1)P_l(\cos \theta_2) + \sum_m P_l^m(\cos \theta_1)P_l^m(\cos \theta_2) \sin m(\phi_2 - \phi_1). \quad (1)$$

If the elementary law of scattering has cylindrical symmetry, the average over $\sin m(\phi_2 - \phi_1)$ gives zero, and only the first term remains. A repetition of this averaging process for n collisions yields

* Now at the Dow Chemical Company.

¹ E. J. Williams, Proc. Roy. Soc. **169**, 531 (1939).

² W. A. Fowler, Phys. Rev. **54**, 773 (1938); N. L. Oleson, K. T. Chao, J. Halpern, H. R. Crane, Phys. Rev. **56**, 482 (1939); C. W. Sheppard and W. A. Fowler, Phys. Rev. **56**, 849 (1939).

³ Compare L. S. Ornstein, Proc. Amsterdam Acad. **40**, 464 (1939).

the result stated above, that⁴

$$\langle P_l(\cos \theta) \rangle_{Av} = \langle P_l(\cos \theta_1) \rangle_{Av}^n. \quad (2)$$

The final total average of any Legendre polynomial will be the average considering all possible values of n , the number of collisions suffered by the electron. Denoting the average by G_l , and by $W(n)$, the probability that an electron makes n collisions, we have

$$G_l = \sum_0^{\infty} W(n) \langle P_l(\cos \theta_1) \rangle_{Av}^n. \quad (3)$$

These averages completely determine the angular distribution of the emerging electrons. The intensity of scattering per unit solid angle in the direction θ is given by

$$f(\theta) = \frac{1}{4\pi} \sum (2l+1) G_l P_l(\cos \theta). \quad (4)$$

This function can be evaluated numerically, if necessary, for various values of θ , when the averages G_l are known. It is well to notice that this formula is normalized such that

$$2\pi \int_0^{\pi} f(\theta) \sin \theta d\theta = 1 \quad (5)$$

and that all averages are meant to be taken over the spherical distribution. For example, G_l would be given by the relation

$$G_l = 2\pi \int_0^{\pi} P_l(\cos \theta) f(\theta) \sin \theta d\theta. \quad (6)$$

With the G_l known, various other averages can easily be obtained from the explicit expressions

⁴ If we restrict ourselves to small angles, we can use the approximation

$$P_l(\cos \theta) \sim J_0(l\theta).$$

The average value after n trials is again the n th power of the average after one trial

$$\langle J_0(l\theta) \rangle_{Av} = \langle J_0(l\theta_1) \rangle_{Av}^n.$$

This can be proved strictly for a centrally symmetrical probability distribution of a variable θ in a plane using the addition theorem of Bessel functions. The coefficient l can then have arbitrary values, but should be chosen equal to the roots of J_0 for use in expansions in series. For a symmetrical probability distribution of the variable θ on a line it is easy to show that $\cos l\theta$ is the function which possesses this multiplicative property of the averages.

for the polynomials, such as,

$$\langle \cos \theta \rangle_{Av} = G_1, \quad (7)$$

$$\langle \cos^2 \theta \rangle_{Av} = \frac{1}{3}(1 + G_2), \quad (8)$$

$$\langle \sin^2 \theta/2 \rangle_{Av} = \frac{1}{2}(1 - G_1), \quad (9)$$

$$\langle \sin^4 \theta/2 \rangle_{Av} = \frac{1}{6}(2 - 3G_1 + G_2), \text{ etc.} \quad (10)$$

The main problem is thus reduced to the evaluation of the average values of the Legendre polynomials. In order to do this, we must know the probability distribution of the number of impacts suffered by an electron traversing a scattering material and, also, the average values of the Legendre polynomials for a single collision according to the chosen law of scattering.

3. THE NUMBER OF COLLISIONS

We shall denote the total collision cross section of an atom by $\pi\rho^2$, the number of atoms per cc by N and the thickness of the scatterer by t . The average number of collisions ν which one electron makes in traversing the scatterer is

$$\nu = \pi\rho^2 Nt. \quad (11)$$

The probability that an electron has n collisions is expressed by a Poisson distribution

$$W(n) = e^{-\nu} \nu^n / n! \quad (12)$$

This formula would be strictly valid if t were the true path length of the electron in passing through the foil, and if this path length were the same for all electrons. It will be pointed out below that this uncertainty in actual length of path in the foil can have very little effect upon the accuracy of the results when foils of such thickness are used such that the distribution of the emerging electrons does not approach complete diffusion.

Using the Poisson formula (12) in (3), we obtain

$$G_l = \sum e^{-\nu} \nu^n \langle P_l(\cos \theta_1) \rangle_{Av}^n / n! = e^{-\nu} Q_l, \quad (13)$$

where

$$Q_l = 1 - \langle P_l(\cos \theta_1) \rangle_{Av}. \quad (14)$$

In most cases we shall find that $Q_l \ll 1$ for a large range of values of l . Therefore, any probability distribution $W(n)$ which has a rather sharp maximum at $n = \nu$ will yield the result given in

Eq. (13). This can, for instance, be verified easily when $W(n)$ is assumed to be Gaussian. Unless the distribution $W(n)$ is very wide and has a half-width of the same order as the average value, the result will still be almost identical with Eq. (13). It is understood that $\nu \gg 1$, if we are to have multiple scattering.

It is fortunate that the result given in Eq. (13) does not require an exact knowledge of $W(n)$ for thin scatterers. When the average path length is taken as the actual thickness t of the foil, the value obtained for ν will be slightly too small. This can be corrected to a first approximation by taking for the path length $t/\langle \cos \theta \rangle_{Av} = t/G_1$. Important modifications may be necessary only when the scatterer is very thick, in which case $W(n)$ becomes very wide, and also, perhaps, when the atoms of the scatterer are distributed in some regular way as in single crystals.

4. THE SCATTERING LAW

The following formulas can be expressed conveniently in terms of

$$y = \sin \theta/2. \quad (15)$$

The distribution of scattering for a single collision, normalized for one incident electron per cm^2 , can be written

$$2\pi I(\theta) \sin \theta d\theta = 2\pi\kappa^2 g(y) dy/y^3, \quad (16)$$

where

$$\kappa = (Ze^2/mc^2)(1 - \beta^2)^{1/2}/\beta^2, \quad (17)$$

in which the symbols have their usual meaning. The factor $g(y)$ is applied to the Rutherford formula in order to correct for the screening of the nucleus by the orbital electrons. For a bare nucleus $g(y) = 1$. For neutral atoms $g(y)$ approaches unity rapidly for larger angles of scattering, that is, as y approaches unity. For values of y near zero, $g(y)$ is very small and keeps the various integrals over y from diverging. The total cross section per atom is

$$\pi\rho^2 = 2\pi\kappa^2 \int_0^1 g(y) dy/y^3. \quad (18)$$

We consider three cases for $g(y)$:

A. *Rutherford scattering with a sharp cut-off of the Coulomb field at the radius a .* From classical

calculations we obtain

$$\begin{aligned} g(y) &= 1 & \text{for } y > y_0 \\ g(y) &= 0 & \text{for } y < y_0, \end{aligned} \quad (19)$$

$$\text{where } y_0 = (1 + a^2/\kappa^2)^{-1/2} \sim \kappa/a. \quad (20)$$

B. *The Born approximation for a field of the form*

$$V(r) = (Ze^2/r)e^{-r/a}. \quad (21)$$

This gives⁵

$$g(y) = y^4/(y^2 + y_0'^2), \quad (22)$$

with

$$y_0' = \lambda/2a, \quad (23)$$

in which $2\pi\lambda$ is the wave-length of the incident electron. The exponential factor is supposed to represent approximately the screening effect. In both cases A and B, it is usual to take

$$a = a_0/Z^{1/2} = \hbar^2/mc^2 Z^{1/2}, \quad (24)$$

where a_0 is the Bohr radius of hydrogen. Hence,

$$y_0' = \frac{1}{2}(e^2/\hbar c)Z^{1/2}/(w^2 - 1)^{1/2}, \quad (25)$$

in which w is the total energy of the particle in units mc^2 .

C. *The field as given by the Thomas-Fermi atom.* The function $g(y)$ can be obtained easily from the tables of Bullard and Massey.⁶

5. THE AVERAGE VALUES OF THE LEGENDRE POLYNOMIALS AFTER ONE COLLISION

We use Murphy's expansion expressing the Legendre polynomials in powers of $\sin \theta/2$,

$$P_l(\cos \theta) = \sum_0^l (-1)^k \frac{(l+k)!}{(l-k)!k!^2} y^{2k}. \quad (26)$$

The average value after one collision is

$$\begin{aligned} \langle P_l(\cos \theta_1) \rangle_{Av} &= 2\pi\kappa^2 \sum_0^l (-1)^k \frac{(l+k)!}{(l-k)!k!^2} \\ &\quad \times \int_0^1 g(y) y^{2k-3} dy / \pi\rho^2. \end{aligned} \quad (27)$$

Substituting for $\pi\rho^2$ as given in Eq. (18), and separating from the summation the first two

⁵ G. Wentzél, Zeits. f. Physik **40**, 590 (1927).

⁶ E. C. Bullard and H. S. W. Massey, Proc. Camb. Phil. Soc. **26**, 556 (1930) or N. F. Mott and H. S. W. Massey, *Atomic Collisions* (Oxford, 1933), p. 126.

terms, we obtain

$$\begin{aligned} \langle P_l(\cos \theta_1) \rangle_{Av} &= 1 - Q_l = 1 - 2(\kappa^2/\rho^2) \\ &\times \left[l(l+1) \int_0^1 g(y) dy/y \right. \\ &\left. - \sum_2^l (-1)^k \frac{(l+k)!}{(l-k)!k!^2} \int_0^1 g(y) y^{2k-3} dy \right]. \quad (28) \end{aligned}$$

6. THE AVERAGE VALUES AFTER MULTIPLE SCATTERING

The average number of collisions in a path length t is

$$\nu = \pi \rho^2 N t.$$

If the conditions mentioned in Section 3 are valid, we find for the final averages of the Legendre polynomials

$$G_l = e^{-\nu Q_l}$$

with

$$\begin{aligned} \nu Q_l &= 2\pi \kappa^2 N l \left[l(l+1) \int_0^1 g(y) dy/y \right. \\ &\left. - \sum_2^l (-1)^k \frac{(l+k)!}{(l-k)!k!^2} \int_0^1 g(y) y^{2k-3} dy \right]. \quad (29) \end{aligned}$$

It is important to note that the factor in front is always the same, independent of $g(y)$.

7. EVALUATION OF THE INTEGRALS

We shall denote the sum in the bracket expression of Eq. (29) by S_l and consider it first. For the higher powers of y , the main contribution to the integrals comes from the region near $y=1$ where $g(y)=1$ and thus we can approximate the result by omitting the screening factor. This gives

$$\begin{aligned} S_l &= \sum_2^l (-1)^k \frac{(l+k)!}{(l-k)!k!^2} \frac{1}{(2k-2)} \\ &= l(l+1) \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + 1/l \right). \quad (30) \end{aligned}$$

The proof we have for this last identity is somewhat too lengthy to be included in this paper.⁷

⁷ The sum is also equal to

$$2(l+1) \sum_2^l (-1)^k c(l,k)/(2k-2),$$

where $c(l,k)$ denotes binomial coefficients.

The omission of $g(y)$ is not permissible for very large values of l . This can be shown most easily by taking the example of case A, which gives for the sum

$$S_l = \sum_2^l (-1)^k \frac{(l+k)!}{(l-k)!k!^2} \frac{1-y_0^{2k-2}}{(2k-2)} \quad (31)$$

as long as

$$ly_0 \ll 1. \quad (32)$$

The approximation (30) is sufficient, but for larger l values the terms containing y_0 do not decrease at first and may, therefore, give appreciable contributions. The relation $ly=1$ gives the approximate value of y where the Legendre polynomial has its first root.

Writing for the first integral in Eq. (29)

$$\int_0^1 g(y) dy/y = \log \xi, \quad (33)$$

and using Eqs. (19) and (20), we find by direct integration for case A,

$$\log \xi_A = \log (a/\kappa). \quad (34)$$

In a similar manner for case B, with the use of Eqs. (22) and (23), integration gives

$$\log \xi_B = \log (2a/\lambda) - \frac{1}{2} = \log (1.21a/\lambda). \quad (35)$$

Finally, from an approximate numerical integration of Bullard and Massey's data for the Thomas-Fermi field,⁸ the result is

$$\log \xi_C = \log (1.10a/\lambda). \quad (36)$$

Assuming in all cases that $a = a_0/Z^{\frac{1}{3}}$ we see that cases B and C give practically the same result. Case A, however, usually differs very much from the other two, since

$$\xi_B/\xi_A \sim \kappa/\lambda = (Z/137)w^{\frac{1}{2}}/(w^2-1)^{\frac{1}{2}}.$$

For the radius of the total scattering cross section in each case, we find

$$\begin{aligned} \rho_A &= a \\ \rho_B &= 2\kappa a/\lambda \\ \rho_C &= 2.9\kappa a/\lambda. \end{aligned}$$

It is possible, of course, to choose a different value for a in each case so as to yield equal

⁸ In the formula following formula (13) on page 125 in Mott and Massey, reference 6, the exponent $17/3$ should read $7/3$.

values for ρ or ξ . It is clear that the results are quite sensitive to the behavior of $g(y)$ near $y=0$.

In view of Eqs. (30) and (33), we may now write for the average value of the Legendre polynomials after multiple scattering

$$G_l = e^{-2\pi\kappa^2 N l(l+1) [\log \xi - (\frac{1}{2} + \frac{1}{3} + \dots + 1/l)]}, \quad (37)$$

for not too large values of l . The restriction on l is that the exponent must remain negative (compare Eq. (32)).

8. THE PROBABILITY DISTRIBUTION FOR MULTIPLE SCATTERING

As was mentioned above, it is possible to write the scattering probability as a series in Legendre polynomials. This was given by Eq. (4)

$$f(\theta) = \frac{1}{4\pi} \sum (2l+1) G_l P_l(\cos \theta).$$

The summation does not need to be extended to too large values of l , the restriction being given in this case by Eq. (32).

An approximate Gaussian behavior of $f(\theta)$ for small angles would have the form

$$4\pi f(\theta) = e^{-y^2/\alpha^2}/\alpha^2 \sim e^{-\theta^2/4\alpha^2}/\alpha^2. \quad (38)$$

Note that this Gaussian curve is again normalized to unity on the sphere, that is,

$$2\pi \int_0^\pi f(\theta) \sin \theta d\theta \sim \int_0^\infty e^{-y^2/\alpha^2} 2y dy / \alpha^2 = 1, \quad (39)$$

provided we may extend the integration over y to infinity.

It is possible to transform the sum for $f(\theta)$ into a power series in y by again using Murphy's expansion for Legendre polynomials. The coefficients of the various powers are then sums themselves, which may be approximated by replacing with integrals. It is, however, not difficult to use numerical values directly in the expansion (4). In this way the numerical examples given in Table I were obtained, by using from 10 to 20 terms in the series.⁹ The numerical results closely follow a Gaussian curve in y over

TABLE I. Numerical calculations.

l	w	$\pi\kappa^2 N l$	$\log \xi$	ν	θ_{\max}^*		
					calc.	obs.	
Al	0.0025 cm	2.8	0.00103	5.22	96	6°	6°
	0.010	2.8	0.00411	5.22	380	14°	11°
	0.025	2.8	0.0103	5.22	960	23°	19°
Pb	0.0038	2.8	0.035	4.60	940	43°	22°
	0.013	20	0.0018	6.63	2800	11°	6°
	0.0063	9	0.0045	5.82	1400	16°	11°
C	0.3	9	0.0045	6.70	6900	15°	11°

* θ_{\max} actually means the angle corresponding to the most probable value of $\sin \theta/2$ and is simply a measure for the width of the Gaussian curve, $\sin (\theta_{\max}/2) = \alpha/\sqrt{2}$.

a range extending from $y=0$ to well beyond the half-value, with

$$1/\alpha^2 \sim \sum (2l+1) G_l. \quad (40)$$

The calculations are made for case B of Section 4.

The experimental results on the first four foils of the table were obtained by using electrons of approximately uniform energy,¹⁰ while a range of energies was used in the experiments on the last three.² For the first lead foil the calculated angle is so large that it is unlikely that the conditions mentioned in Section 3 are valid. The discrepancy with the last three examples is unexplained.

9. COMPARISON WITH WILLIAMS' THEORY

The theoretical treatment of the problem by Williams is based on the assumption that small angle multiple scattering will yield a Gaussian distribution. The width of the Gaussian curve is calculated by multiplying the mean square deflection for a single collision by the average number of collisions. In this calculation large deflections, occurring on the average less than once for each electron, are at first omitted and afterwards added as a small correction, giving the so-called "single scattering tail." We shall not go into a detailed analysis of Williams' method, but his result can be obtained with the present method by proceeding as follows. We integrate Eq. (29) up to a value $y=y_1$ instead of $y=1$ and omit the partial harmonic series. This gives for

⁹ Tables up to P_{32} by Hj. Tallqvist, Soc. Scient. Fennica or Finska Vet. Soc., Comm. Phys.-Math. Vol. 6, No. 3 and No. 6 (1932).

¹⁰ M. M. Slawsky and H. R. Crane, Phys. Rev., to be published.

the exponent of G_l

$$\nu Q_l = 2\pi\kappa^2 Nt(l+1) \log(\xi y_1), \quad (41)$$

which, substituted in Eq. (4), yields approximately a Gaussian curve with

$$\alpha^2 = 2\pi\kappa^2 Nt \log(\xi y_1). \quad (42)$$

The upper limit y_1 is defined by¹¹

$$2\pi\kappa^2 Nt \int_{y_1}^1 dy/y^3 = 2 \sim \pi\kappa^2 Nt/y_1^2, \quad (43)$$

which means that the two largest deflections have been omitted. In our examples, however, these deflections are almost as large as the average angle due to multiple scattering. The omission of the partial harmonic series has a more subtle meaning and is connected with the way in which the distribution approaches a Gaussian curve. Neither for a very small nor for a very large number of impacts do we expect a Gaussian distribution. It will be a good approximation only in some intermediate cases.

10. DISCUSSION AND CONCLUSIONS

The approximations frequently encountered in the several solutions of the scattering problem may be classified as follows:

I. Assumption of a Gaussian distribution

In this case it is assumed that the single scattering intensity decreases rapidly enough with increasing angle to insure the validity of a Gaussian distribution for multiple scattering. In the more general case, this assumption leads to the treatment of the problem by means of a diffusion equation.¹²

IIa. Assumption of equal path lengths

Here it is assumed that all angles involved are very small so that the path length of the electrons can be taken equal to the thickness of the scatterer.¹³ The situation may be somewhat improved by a simple first-order correction (compare Section 3).

¹¹ Setting this equation equal to 2 instead of 1 is not very essential and is done because Williams considers the projection on a plane of the scattered beam as observed in a cloud chamber.

¹² W. W. Bothe, *Zeits. f. Physik* **54**, 161 (1929); H. A. Bethe, M. E. Rose and L. P. Smith, *Proc. Am. Phil. Soc.* **78**, 573 (1938).

¹³ This would mean neglecting the factor $\cos \theta$ in Eq. (11) of the paper of Bethe *et al.*

IIb. Disregard of back scattering

It is frequently assumed that all electrons are scattered only in the forward direction, and that all electrons which enter the scatterer on one side leave it at the other.

III. Disregard of inelastic scattering

The effect of energy loss by the electrons upon the distributions of scattering is very difficult to determine, hence the assumption is made that all scattering is elastic.

Inasmuch as no one has satisfactorily treated the relation between energy loss and scattering, all theories contain assumption III. The theory of Williams makes the other assumptions listed above as well. The present paper makes all except the first approximation, and is valid for thin scatterers only. It is an exact solution (except for III) for the ideal case of electrons with equal total path lengths. The theory of Bethe, Rose and Smith does not involve assumptions IIa and IIb, but their solution is, according to them, valid only for thick scatterers. It is not easy to see what effect assumption I has upon their results.

In regard to this paper, several more remarks may be made concerning the reliability of the results. An approximation of a different nature arises from the circumstance that we have no exact knowledge of the single scattering law. We believe that more precise information about $g(y)$ is likely to improve considerably the agreement between theory and experiment, which is at present quite bad even for thin foils. The sensitivity of the results to $g(y)$ is clearly seen from Eq. (33).

For high velocities and small Z the relativity corrections add a factor $(1-y^2)$ to $g(y)$ for large angles. It is also possible that higher terms in the series of Eq. (4) give more important contributions than expected, but preliminary investigations make this appear unlikely.

We wish to express our appreciation for discussions about the scattering problem with Professors Crane, Fermi and Williams during the Summer Symposium at the University of Michigan, and more recently with Professor Uhlenbeck. This investigation was undertaken in connection with work supported by the Horace H. Rackham Fund.